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Duality and Equivalence of Module Categories in Noncommutative Geometry

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In memory of Raoul

Abstract. We develop a general framework to describe dualities from algebraic, differential, and noncommutative geometry, as well as physics. We pursue a relationship between the Baum – Connes conjecture in operator K-theory and derived equivalence statements in algebraic geometry and physics. We associate to certain data, reminiscent of spectral triple data, a differential graded category in such a way that we can recover the derived category of coherent sheaves on a complex manifold.

Introduction

In various geometric contexts, there are duality statements that are expressed in terms of appropriate categories of modules. We have in mind, for example, the Baum – Connes conjecture from noncommutative geometry, T-duality and Mirror symmetry from complex geometry and mathematical physics. This is the first in a series of papers that sets up a framework to study and unify these dualities from a noncommutative geometric point of view. We also view this project as an attempt to connect the noncommutative geometry of Connes, [6] with the categorical approach to noncommutative geometry, represented for example by Manin and Kontsevich.

Traditionally, the complex structure is encoded in the sheaf of holomorphic functions. However, for situations we have in mind coming from noncommutative geometry, one can not use local types of constructions, and we are left only with global differential geometric ones. A convenient setting to talk about integrability of geometric structures and the integrability of geometric structures on their modules is that of a differential graded algebra and more generally, a curved differential graded algebra. Thus, for example, a complex structure on a manifold is encoded in its Dolbeault algebra $\mathsf{A} = (\mathcal{A}^{0,\bullet}(X), \bar{\partial})$, and a holomorphic vector bundle can be viewed as the data of a finitely generated projective module over $\mathcal{A}^{0,0}$ together with

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This is the final form of the paper.

a flat $\bar{\partial}$-connection. Similarly, holomorphic gerbes can be encoded in terms of a curved differential graded algebra with non-trivial curvature. Curved dgas appear naturally in the context of matrix factorizations and Laundau – Ginzburg models, [11, 21]. Indeed, these fit very easily into our framework.

Of course, one is interested in more modules than just the finitely generated projective ones. In algebraic geometry, the notion of coherent module is fundamental. In contrast to projective algebraic geometry however, not every coherent sheaf has a resolution by vector bundles; they only locally have such resolutions. Toledo and Tong, [28, 29], handled this issue by introducing twisted complexes. Our construction is a global differential geometric version of theirs.

We have found the language of differential graded categories to be useful, [5, 10, 18]. In particular, for a curved dga A we construct a very natural differential graded category $\mathcal{P}_{\mathsf{A}}$ which can then be derived. The desiderata of such a category are

- it should be large enough to contain in a natural way the coherent holomorphic sheaves (in the case of the Dolbeault algebra), and
- it should be flexible enough to allow for some of Grothendieck's six operations, so that we can prove Mukai type duality statements.

The reason for introducing $\mathcal{P}_{\mathsf{A}}$ is that the ordinary category of dg-modules over the Dolbeault dga has the wrong homological algebra; it has the wrong notion of quasi-isomorphism. A morphism between complexes of holomorphic vector bundles considered as dg-modules over the Dolbeault algebra is a quasi-isomorphism if it induces an isomorphism on the total complex formed by the gloabal sections of the Dolbeault algebra with values in the complexes of holomorphic vector bundles, which is isomorphic to their hypercohomology. On the other hand, $\mathcal{P}_{\mathsf{A}}$ and the modules over it have the correct notion of quasi-isomorphism. In particular, $\mathcal{P}_{\mathsf{A}}$ is not an invariant of quasi-isomorphism of dga's. To be sure, we would not want this. For example, the dga which is $\mathbb{C}$ in degree 0 and 0 otherwise is quasi-isomorphic to the Dolbeault algebra of $\mathbb{CP}^n$. But $\mathbb{CP}^n$ has a much richer module category than anything $\mathbb{C}$ could provide. We show that the homotopy category of $\mathcal{P}_{\mathsf{A}}$ where A is the Dolbeault algebra of a compact complex manifold X is equivalent to the derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Our description of the coherent derived category has recently been used by Bergman, [2] as models for B-model D-branes.

To some extent, what we do is a synthesis of Kasparov's KK-theory, [17] and of Toledo and Tong's twisted complexes, [20, 28, 29].

In appreciation of Raoul Bott. I am always amazed by the profound impact that he had, and still has, on my life. During the time I was his student, I learned much more from him than mere mathematics. It was his huge personality, his magnanimous heart, his joy in life and his keen aesthetic that has had such a lasting effect. I miss him.

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1. Baum – Connes and Fourier – Mukai

There are two major motivations for our project. The first is to have a general framework that will be useful in dealing with categories of modules that arise in geometry and physics. For example, we will apply our framework to construct categories of modules over symplectic manifolds. Second, as mentioned earlier in the introduction, this series of papers is meant to pursue a relationship between

(1) the Baum – Connes conjecture in operator K-theory and
(2) derived equivalence statements in algebraic geometry and physics.

In particular, we plan to refine, in certain cases, the Baum – Connes conjecture from a statement about isomorphism of two topological K-groups to a derived equivalence of categories consisting of modules with geometric structures, for example, coherent sheaves on complex manifolds. We will see that there are natural noncommutative geometric spaces that are derived equivalent to classical algebraic geometric objects.

Let us explain the obvious formal analogies between (1) and (2). For simplicity let Γ be a discrete torsion-free group with compact $B\Gamma$. In this situation, the Baum – Connes conjecture says that an explicit map, called the assembly map,

$$\mu\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma) \tag{1.1}$$

is an isomorphism. Here $C_r^*\Gamma$ denotes the reduced group C^*-algebra of Γ. The assembly map can be described in the following way. On $C(B\Gamma) \otimes C_r^*\Gamma$ there is a finitely generated projective right module $\mathscr{P}$ which can be defined as the sections of the bundle of $C_r^*\Gamma$-modules

$$E\Gamma \times_\Gamma C_r^*\Gamma.$$

This projective module is a "line bundle" over $C(B\Gamma) \otimes C_r^*\Gamma$. Here, $C(X)$ denote the complex-valued continuous functions on a compact space X. The assembly map is the map defined by taking the Kasparov product with $\mathscr{P}$ over $C(B\Gamma)$. This is some sort of index map.

$$\mu\colon x \in KK(C(B\Gamma), \mathbb{C}) \mapsto x \cup \mathscr{P} \in KK(\mathbb{C}, C_r^*\Gamma)$$

where $\mathscr{P} \in KK(\mathbb{C}, C(B\Gamma) \otimes C_r^*\Gamma)$

We now describe Mukai duality in a way that makes it clear that it refines Baum – Connes. Now let X be a complex torus. Thus $X = V/\Lambda$ where V is a g-dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2g}$ is a lattice in V. Let $X^\vee$ denote the dual complex torus. This can be described in a number of ways:

- as $\mathrm{Pic}^0(X)$, the manifold of holomorphic line bundles on X with first Chern class 0 (i.e., they are topologically trivial);
- as the moduli space of flat unitary line bundles on X. This is the same as the space of irreducible unitary representations of $\pi_1(X)$, but it has a complex structure that depends on that of X;
- and most explicitly as $\overline{V}^\vee/\Lambda^\vee$ where $\Lambda^\vee$ is the dual lattice,

  $$\Lambda^\vee = \{v \in \overline{V}^\vee \mid \mathrm{Im}\langle v, \lambda\rangle \in \mathbb{Z}\ \forall \lambda \in \Lambda\}.$$

  Here $\overline{V}^\vee$ consists of conjugate linear homomorphisms from V to $\mathbb{C}$.

We note that $X = B\Lambda$ and that $C(X^\vee)$ is canonically $C_r^*\Lambda$. Hence Baum – Connes predicts (and in fact it is classical in this case) that $K_*(X) \cong K_*(C_r^*\Lambda) \cong K^*(X^\vee)$.

On $X \times X^\vee$ there is a canonical line bundle, $\mathscr{P}$, the Poincaré bundle, which is uniquely determined by the following universal properties:

- $\mathscr{P}|X \times \{p\} \cong p$ where $p \in X^\vee$ and is therefore a line bundle on X.
- $\mathscr{P}|\{0\} \times X^\vee$ is trivial.

Now Mukai duality says that there is an equivalence of derived categories of coherent sheaves

$$D^b(X) \to D^b(X^\vee)$$

induced by the functor

$$\mathcal{F} \mapsto p_{2*}(p_1^*\mathcal{F} \otimes \mathscr{P})$$

where p_i are the two obvious projections. The induced map at the level of K_0 is an isomorphism and is clearly a holomorphic version of the Baum – Connes Conjecture for the group Λ.

2. The dg-category $\mathcal{P}_{\mathsf{A}}$ of a curved dga

2.1. dg-categories.

Definition 2.1. For complete definitions and facts regarding dg-categories, see [5, 10, 18, 19]. Fix a field k. A *differential graded category* (*dg-category*) is a category enriched over $\mathbb{Z}$-graded complexes (over k) with differentials increasing degree. That is, a category $\mathcal{C}$ is a dg-category if for x and y in $\mathrm{Ob}\,\mathcal{C}$ the hom set

$$\mathcal{C}(x, y)$$

forms a $\mathbb{Z}$-graded complex of k-vector spaces. Write $(\mathcal{C}^\bullet(x, y), d)$ for this complex, if we need to reference the degree or differential in the complex. In addition, the composition, for $x, y, z \in \mathrm{Ob}\,\mathcal{C}$

$$\mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z)$$

is a morphism of complexes. Furthermore, there are obvious associativity and unit axioms.

2.2. Curved dgas. In many situations the integrability conditions are not expressed in terms of flatness but are defined in terms of other curvature conditions. This leads us to set up everything in the more general setting of curved dga's. These are dga's where d^2 is not necessarily zero.

Definition 2.2. A curved dga [23] (Schwarz [25] calls them Q-algebras) is a triple

$$\mathsf{A} = (\mathcal{A}^\bullet, d, c)$$

where $\mathcal{A}^\bullet$ is a (nonnegatively) graded algebra over a field k of characteristic 0, with a derivation

$$d\colon \mathcal{A}^\bullet \to \mathcal{A}^{\bullet+1}$$

which satisfies the usual graded Leibniz relation but

$$d^2(a) = [c, a]$$

where $c \in \mathcal{A}^2$ is a fixed element (the curvature). Furthermore we require the Bianchi identity $dc = 0$. Let us write $\mathcal{A}$ for the degree 0 part of $\mathcal{A}^\bullet$, the “functions” of A.

A dga is the special case where $c = 0$. Note that c is part of the data and even if $d^2 = 0$, that c might not be 0, and gives a non-dga example of a curved dga. The prototypical example of a curved dga is $\big(\mathcal{A}^\bullet\big(M, \mathrm{End}(E)\big), \mathrm{Ad}\,\nabla, F\big)$ of differential forms on a manifold with values in the endomorphisms of a vector bundle E with connection ∇ and curvature F.

2.3. The dg-category $\mathcal{P}_{\mathsf{A}}$. Our category $\mathcal{P}_{\mathsf{A}}$ consists of special types of A-modules. We start with a $\mathbb{Z}$-graded right module $E^\bullet$ over $\mathcal{A}$.

Definition 2.3. A $\mathbb{Z}$-connection $\mathbb{E}$ is a k-linear map

$$\mathbb{E}\colon E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

of total degree one, which satisfies the usual Leibniz condition

$$\mathbb{E}(e\omega) = \big(\mathbb{E}(e \otimes 1)\big)\omega + (-1)^e e d\omega$$

Such a connection is determined by its value on $E^\bullet$. Let $\mathbb{E}^k$ be the component of $\mathbb{E}$ such that $\mathbb{E}^k\colon E^\bullet \to E^{\bullet-k+1} \otimes_{\mathcal{A}} \mathcal{A}^k$; thus $\mathbb{E} = \mathbb{E}^0 + \mathbb{E}^1 + \mathbb{E}^2 + \cdots$. It is clear that $\mathbb{E}^1$ is a connection on each component E^n in the ordinary sense (or the negative of a connection if n is odd) and that $\mathbb{E}^k$ is $\mathcal{A}$-linear for $k \neq 1$.

Note that for a $\mathbb{Z}$-connection $\mathbb{E}$ on $E^\bullet$ over a curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$, the usual curvature $\mathbb{E} \circ \mathbb{E}$ is not $\mathcal{A}$-linear. Rather, we define the *relative curvature* to be the operator

$$F_{\mathbb{E}}(e) = \mathbb{E} \circ \mathbb{E}(e) + e \cdot c$$

and this is $\mathcal{A}$-linear.

Definition 2.4. For a curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$, we define the dg-category $\mathcal{P}_{\mathsf{A}}$:

(1) An object $E = (E^\bullet, \mathbb{E})$ in $\mathcal{P}_{\mathsf{A}}$, which we call a *cohesive module*, is a $\mathbb{Z}$-graded (but bounded in both directions) right module $E^\bullet$ over $\mathcal{A}$ which is finitely generated and projective, together with a $\mathbb{Z}$-connection

$$\mathbb{E}\colon E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

that satisfies the integrability condition that the relative curvature vanishes

$$F_{\mathbb{E}}(e) = \mathbb{E} \circ \mathbb{E}(e) + e \cdot c = 0$$

for all $e \in E^\bullet$.

(2) The morphisms of degree k, $\mathcal{P}^k_{\mathsf{A}}(E_1, E_2)$ between two cohesive modules $E_1 = (E_1^\bullet, \mathbb{E}_1)$ and $E_2 = (E_2^\bullet, \mathbb{E}_2)$ of degree k are

$$\{\phi\colon E_1^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to E_2^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \mid \text{of degree } k \text{ and } \phi(ea) = \phi(e)a \ \forall a \in \mathcal{A}^\bullet\}$$

with differential defined in the standard way

$$d(\phi)(e) = \mathbb{E}_2\big(\phi(e)\big) - (-1)^{|\phi|}\phi\big(\mathbb{E}_1(e)\big)$$

Again, such a ϕ is determined by its restriction to $E_1^\bullet$ and if necessary we denote the component of ϕ that maps

$$E_1^\bullet \to E_2^{\bullet+k-j} \otimes_{\mathcal{A}} \mathcal{A}^j \tag{2.1}$$

by ϕ^j.

Thus $\mathcal{P}^k_{\mathsf{A}}(E_1, E_2) = \mathrm{Hom}^k_{\mathcal{A}}(E_1^\bullet, E_2^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet)$.

Proposition 2.5. *For $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ a curved dga, the category $\mathcal{P}_{\mathsf{A}}$ is a dg-category.*

This is clear from the following lemma.

Lemma 2.6. *Let E_1, E_2 be cohesive modules over the curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$. Then the differential defined above*

$$d\colon \mathcal{P}_{\mathsf{A}}^\bullet(E_1, E_2) \to \mathcal{P}_{\mathsf{A}}^{\bullet+1}(E_1, E_2)$$

satisfies $d^2 = 0$.

2.4. The homotopy category and triangulated structure. Given a dg-category $\mathcal{C}$, one can form the subcategory $Z^0\mathcal{C}$ which has the same objects as $\mathcal{C}$ and whose morphisms from an object $x \in \mathcal{C}$ to an object $y \in \mathcal{C}$ are the degree 0 closed morphisms in $\mathcal{C}(x, y)$. We also form the homotopy category $\operatorname{Ho}\mathcal{C}$ which has the same objects as $\mathcal{C}$ and whose morphisms are the 0th cohomology

$$\operatorname{Ho}\mathcal{C}(x, y) = H^0\big(\mathcal{C}(x, y)\big).$$

We define a shift functor on the category $\mathcal{P}_{\mathsf{A}}$. For $E = (E^\bullet, \mathbb{E})$ set $E[1] = (E[1]^\bullet, \mathbb{E}[1])$ where $E[1]^\bullet = E^{\bullet+1}$ and $\mathbb{E}[1] = -\mathbb{E}$. It is easy to verify that $E[1] \in \mathcal{P}_{\mathsf{A}}$. Next for $E, F \in \mathcal{P}_{\mathsf{A}}$ and $\phi \in Z^0\mathcal{P}_{\mathsf{A}}(E, F)$, define the cone of ϕ, $\operatorname{Cone}(\phi) = (\operatorname{Cone}(\phi)^\bullet, \mathbb{C}_\phi)$ by

$$\operatorname{Cone}(\phi)^\bullet = \begin{pmatrix} F^\bullet \\ \oplus \\ E[1]^\bullet \end{pmatrix}$$

and

$$\mathbb{C}_\phi = \begin{pmatrix} \mathbb{F} & \phi \\ 0 & \mathbb{E}[1] \end{pmatrix}$$

We then have a triangle of degree 0 closed morphisms

$$E \xrightarrow{\phi} F \to \operatorname{Cone}(\phi) \to E[1] \tag{2.2}$$

Proposition 2.7. *Let A be a curved dga. Then the dg-category $\mathcal{P}_{\mathsf{A}}$ is pretriangulated in the sense of Bondal and Kapranov,* [5]. *Therefore, the category $\operatorname{Ho}\mathcal{P}_{\mathsf{A}}$ is triangulated with the collection of distinguished triangles being isomorphic to those of the form* (2.2).

Proof. The proof of this is the same as that of Propositions 1 and 2 of [5]. □

2.5. Homotopy equivalences. As described above, a degree 0 closed morphism ϕ between cohesive modules $E_i = (E_i^\bullet, \mathbb{E}_i)$, $i = 1, 2$, over A is a *homotopy equivalence* if it induces an isomorphism in $\operatorname{Ho}\mathcal{P}_{\mathsf{A}}$. We want to give a simple criterion for ϕ to define such a homotopy equivalence. On the complex $\mathcal{P}_{\mathsf{A}}(E_1, E_2)$ define a decreasing filtration by

$$F^k\mathcal{P}_{\mathsf{A}}^j(E_1, E_2) = \{\phi \in \mathcal{P}_{\mathsf{A}}^j(E_1, E_2) \mid \phi^i = 0 \text{ for } i < k\}$$

where ϕ^i is defined as in (2.1).

Proposition 2.8. *There is a spectral sequence*

$$E_0^{pq} \implies H^{p+q}\big(\mathcal{P}_{\mathsf{A}}^\bullet(E_1, E_2)\big) \tag{2.3}$$

where

$$E_0^{pq} = \operatorname{gr}\mathcal{P}_{\mathsf{A}}^\bullet(E_1, E_2) = \{\phi^p \in \mathcal{P}_{\mathsf{A}}^{p+q}(E_1, E_2)\colon E_1^\bullet \to E_2^{\bullet+q} \otimes_{\mathcal{A}} \mathcal{A}^p\}$$

with differential

$$d_0(\phi^p) = \mathbb{E}_2^0 \circ \phi^p - (-1)^{p+q}\phi^p \circ \mathbb{E}_1^0$$

Proposition 2.9. *A closed morphism $\phi \in \mathcal{P}^0_{\mathsf{A}}(E_1, E_2)$ is a homotopy equivalence if and only if $\phi^0 \colon (E_1^\bullet, \mathbb{E}_1^0) \to (E_2^\bullet, \mathbb{E}_2^0)$ is a quasi-isomorphism of complexes of $\mathcal{A}$-modules.*

Proof. Let $E = (E^\bullet, \mathbb{E})$ be any object in $\mathcal{P}_{\mathsf{A}}$. Then ϕ induces a map of complexes

$$\phi \colon \mathcal{P}^\bullet_{\mathsf{A}}(E, E_1) \to \mathcal{P}^\bullet_{\mathsf{A}}(E, E_2) \tag{2.4}$$

We show that the induced map on E_1-terms of the spectral sequences are isomorphisms. Indeed, the quasi-isomorphism of $(E_1^\bullet, \mathbb{E}_1^0) \to (E_2^\bullet, \mathbb{E}_2^0)$ implies that they are actually chain homotopy equivalent since $E_1^\bullet$ and $E_2^\bullet$ are projective, hence for each p that

$$\phi^0 \otimes \mathbb{I} \colon (E_1^\bullet \otimes_{\mathcal{A}} \mathcal{A}^p, \mathbb{E}_1^0 \otimes \mathbb{I}) \to (E_2^\bullet \otimes_{\mathcal{A}} \mathcal{A}^p, \mathbb{E}_2^0 \otimes \mathbb{I})$$

is a quasi-isomorphism and then

$$\mathrm{gr}(\phi) = \phi^0 \colon E_0^{pq} \cong \mathrm{Hom}_{\mathcal{A}}(E^\bullet, E_1^{\bullet+q} \otimes_{\mathcal{A}} \mathcal{A}^p) \to E_0^{pq} \cong \mathrm{Hom}_{\mathcal{A}}(E^\bullet, E_2^{\bullet+q} \otimes_{\mathcal{A}} \mathcal{A}^p)$$

is a quasi-isomorphism after one last double complex argument since the modules $E^\bullet$ are projective over $\mathcal{A}$. Thus (2.4) is a quasi-isomorphism for all E and this implies ϕ is an isomorphism in $\mathrm{Ho}\,\mathcal{P}_{\mathsf{A}}$.

The other direction follows easily. □

2.6. The dual of a cohesive module. We define a duality functor which will be of use in future sections. Let $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ be a curved dga. Its opposite is $\mathsf{A}^\circ = (\mathcal{A}^{\circ\bullet}, d, -c)$ where $\mathcal{A}^{\circ\bullet}$ is the graded algebra whose product is given by

$$a \cdot_\circ b = (-1)^{|a||b|} ba$$

We will not use the notation $\cdot_\circ$ for the product any longer. We can now define the category of left cohesive modules over A as $\mathcal{P}_{\mathsf{A}^\circ}$.

We define the duality dg functor

$$^\vee \colon \mathcal{P}_{\mathsf{A}} \to \mathcal{P}_{\mathsf{A}^\circ}$$

by

$$E = (E^\bullet, \mathbb{E}) \mapsto E^\vee = (E^{\vee\bullet}, \mathbb{E}^\vee)$$

where $E^{\vee k} = \mathrm{Hom}_{\mathcal{A}}(E^{-k}, \mathcal{A})$ and for $\phi \in E^{\vee\bullet}$

$$(\mathbb{E}^\vee \phi)(e) = d\big(\phi(e)\big) - (-1)^{|\phi|} \phi\big(\mathbb{E}(e)\big)$$

It is straightforward that $E^\vee$ is indeed cohesive over A°. There is a natural pairing between E and $E^\vee$. And moreover the connection was defined so that the relation

$$\langle \mathbb{E}^\vee(\phi), e\rangle + (-1)^{|\phi|} \langle \phi, \mathbb{E}(e)\rangle = d\langle \phi, e\rangle$$

holds. Note that the complex of morphisms $\mathcal{P}^\bullet_{\mathsf{A}}(E_1, E_2)$ between cohesive modules can be identified with

$$(E_2^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \otimes_{\mathcal{A}} E_1^{\vee\bullet}, 1 \otimes 1 \otimes \mathbb{E}_1^\vee + 1 \otimes d \otimes 1 + \mathbb{E}_2 \otimes 1 \otimes 1)$$

2.7. Functoriality. We now discuss a construction of functors between categories of the form $\mathcal{P}_{\mathsf{A}}$. Given two curved dgas, $\mathsf{A}_1 = (\mathcal{A}_1^\bullet, d_1, c_1)$ and $\mathsf{A}_2 = (\mathcal{A}_2^\bullet, d_2, c_2)$ a homomorphism from A_1 to A_2 is a pair (f, ω) where $f\colon \mathcal{A}_1^\bullet \to \mathcal{A}_2^\bullet$ is a morphism of graded algebras, $\omega \in \mathcal{A}_2^1$ and they satisfy

(1) $f(d_1 a_1) = d_2 f(a_1) + [\omega, f(a_1)]$ and
(2) $f(c_1) = c_2 + d_2\omega + \omega^2$.

Given a homomorphism of curved dgas (f, ω) we define a dg functor

$$f_*\colon \mathcal{P}_{\mathsf{A}_1} \to \mathcal{P}_{\mathsf{A}_2}$$

as follows. Given $E = (E^\bullet, \mathbb{E})$ a cohesive module over A_1, set $f^*(E)$ to be the cohesive module over A_2

$$(E^\bullet \otimes_{\mathcal{A}_1} \mathcal{A}_2, \mathbb{E}_2)$$

where $\mathbb{E}_2(e \otimes b) = \mathbb{E}(e)b + e \otimes (d_2 b + \omega b)$. One checks that $\mathbb{E}_2$ is still an $\mathbb{E}$-connection and satisfies

$$(\mathbb{E}_2)^2(e \otimes b) = -(e \otimes b)c_2.$$

This is a special case of the following construction. Consider the following data, $X = (X^\bullet, \mathbb{X})$ where

(1) $X^\bullet$ is a graded finitely generated projective right-$\mathcal{A}_2$-module,
(2) $\mathbb{X}\colon X^\bullet \to X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ is a $\mathbb{Z}$-connection,
(3) $\mathcal{A}_1^\bullet$ acts on the left of $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$ satisfying

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b$$

and

$$\mathbb{X}\big(a \cdot (x \otimes b)\big) = da \cdot (x \otimes b) + a \cdot \mathbb{X}(x \otimes b)$$

for $a \in \mathcal{A}_1^\bullet$, $x \in X^\bullet$ and $b \in \mathcal{A}_2^\bullet$,
(4) $\mathbb{X}$ satisfies the following condition:

$$\mathbb{X} \circ \mathbb{X}(x \otimes b) = c_1 \cdot (x \otimes b) - (x \otimes b) \cdot c_2$$

on the complex $X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$.

Let us call such a pair $X = (X^\bullet, \mathbb{X})$ an A_1-A_2-cohesive bimodule.

Given an A_1-A_2-cohesive bimodule $X = (X^\bullet, \mathbb{X})$, we can then define a dg-functor (see the next section for the definition)

$$X^*\colon \mathcal{P}_{\mathsf{A}_1} \to \mathcal{P}_{\mathsf{A}_2}$$

by

$$X^*(E^\bullet, \mathbb{E}) = (E^\bullet \otimes_{\mathcal{A}_1} X^\bullet, \mathbb{E}_2)$$

where $\mathbb{E}_2(e \otimes x) = \mathbb{E}(e) \cdot x + e \otimes \mathbb{X}(x)$, where the $\cdot$ denotes the action of $\mathcal{A}_1^\bullet$ on $X^\bullet \otimes_{\mathcal{A}_1} \mathcal{A}_2^\bullet$. One easily checks that $X^*(E)$ is an object of $\mathcal{P}_{\mathsf{A}_2}$. We will write $\mathbb{E} \otimes \mathbb{X}$ for $\mathbb{E}_2$.

Remark 2.10. (1) The previous case of a homomorphism between curved dgas occurs by setting $X^\bullet = \mathcal{A}_2$ in degree 0. $\mathcal{A}_1^\bullet$ acts by the morphism f and the $\mathbb{Z}$-connection is

$$\mathbb{X}(a_2) = d_2(a_2) + \omega \cdot a_2.$$

(2) To give another example of an A_1-A_2-cohesive bimodule, consider a manifold M with two vector bundles with connection (E_1, ∇_2) and (E_2, ∇_2). Let c_i be the curvature of ∇_i. Set $\mathsf{A}_i = (\mathcal{A}_i^\bullet, d_i, c_i) = (\mathcal{A}^\bullet(M; \mathrm{End}(E_i), \mathrm{Ad}\,\nabla_i)$. Then we define a cohesive bimodule between them by setting

$$X^\bullet = \Gamma\big(M; \mathrm{Hom}(E_2, E_1)\big)$$

in degree 0. $X^\bullet$ has a $\mathbb{Z}$-connection

$$\mathbb{X}(\phi)(e_2) = \nabla_1\big(\phi(e_2)\big) - \phi(\nabla_2 e_2)$$

and maps $X^\bullet \to X^\bullet \otimes_{\mathcal{A}_2} \mathcal{A}_2^\bullet$. Then $(\mathbb{X})^2(\phi) = c_1 \cdot \phi - \phi \cdot c_2$ as is required. This cohesive bimodule implements a dg-quasi-equivalence between $\mathcal{P}_{\mathsf{A}_1}$ and $\mathcal{P}_{\mathsf{A}_2}$. (See the next section for the definition of a dg-quasi-equivalence.)

3. Modules over $\mathcal{P}_{\mathsf{A}}$

It will be important for us to work with modules over $\mathcal{P}_{\mathsf{A}}$ and not just with the objects of $\mathcal{P}_{\mathsf{A}}$ itself.

3.1. Modules over a dg-category.

We first collect some general definitions, see [18] for more details.

Definition 3.1. A functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ between two dg-categories is a *dg-functor* if the map on hom sets

$$F\colon \mathcal{C}_1(x, y) \to \mathcal{C}_2(Fx, Fy) \tag{3.1}$$

is a chain map of complexes. A dg-functor F as above is a *quasi-equivalence* if the maps in (3.1) are quasi-isomorphisms and $\mathrm{Ho}(F)\colon \mathrm{Ho}\,\mathcal{C}_1 \to \mathrm{Ho}\,\mathcal{C}_2$ is an equivalence of categories.

Given a dg-category $\mathcal{C}$, one can define the category of (right) dg-modules over $\mathcal{C}$, Mod-$\mathcal{C}$. This consists of dg-functors from the opposite dg-category $\mathcal{C}^\circ$ to the dg-category $\mathcal{C}(k)$ of complexes over k. More explicitly, a right $\mathcal{C}$-module M is an assignment, to each $x \in \mathcal{C}$, a complex $M(x)$ and chain maps for any $x, y \in \mathcal{C}$

$$M(x) \otimes \mathcal{C}(y, x) \to M(y) \tag{3.2}$$

satisfying the obvious associativity and unit conditions. A morphism $f \in$ Mod-$\mathcal{C}(M, N)$ between right $\mathcal{C}$-modules M and N is an assignment of a map of complexes

$$f_x\colon M(x) \to N(x) \tag{3.3}$$

for each object $x \in \mathcal{C}$ compatible with the maps in (3.2). Such a map is called a *quasi-isomorphism* if f_x in (3.3) is a quasi-isomorphism of complexes for each $x \in \mathcal{C}$. One can make modules over a dg-category into a dg-category itself. The morphisms we have defined in Mod-$\mathcal{C}$ are the degree 0 closed morphisms of this dg-category. The category of left modules $\mathcal{C}$-Mod is defined in an analogous way.

The category Mod-$\mathcal{C}$ has a model structure used by Keller to define its derived category, [18,19]. The quasi-isomorphisms in Mod-$\mathcal{C}$ are those we just defined. The fibrations are the componentwise surjections and the cofibrations are defined by the usual lifting property. Using this model structure we may form the homotopy category of Mod-$\mathcal{C}$, obtained by inverting all the quasi-isomorphisms in Mod-$\mathcal{C}$. This is what Keller calls the derived category of $\mathcal{C}$, and we will denote it by $D(\text{Mod-}\mathcal{C})$.

There is the standard fully faithful Yoneda embedding

$$Z^0\mathcal{C} \to \text{Mod-}\mathcal{C} \qquad \text{where } x \in \mathcal{C} \mapsto h_x = \mathcal{C}(\cdot, x).$$

Moreover, the Yoneda embedding induces a fully faithful functor

$$\operatorname{Ho}\mathcal{C} \to D(\text{Mod-}\mathcal{C})$$

This is simply because for an object $x \in \mathcal{C}$, the module h_x is trivially cofibrant.

Definition 3.2. (1) A module $M \in \text{Mod-}\mathcal{C}$ is called *representable* if it is isomorphic in Mod-$\mathcal{C}$ to an object of the form h_x for some $x \in \mathcal{C}$.
(2) A module $M \in \text{Mod-}\mathcal{C}$ is called *quasi-representable* if it is isomorphic in $D(\text{Mod-}\mathcal{C})$ to an object of the form h_x for some $x \in \mathcal{C}$.

Definition 3.3. Let $M \in \text{Mod-}\mathcal{C}$ and $N \in \mathcal{C}\text{-Mod}$. Their tensor product is defined to be the complex

$$M \otimes_{\mathcal{C}} N = \operatorname{cok}\Big\{ \coprod_{c,c' \in \mathcal{C}} M(c) \otimes \mathcal{C}(c', c) \otimes N(c') \xrightarrow{\alpha} \coprod_{c \in \mathcal{C}} M(c) \otimes N(c) \Big\}$$

where for $m \in M(c)$, $\phi \in \mathcal{C}(c', c)$ and $n \in N(c')$

$$\alpha(m \otimes \phi \otimes n) = m\phi \otimes n - m \otimes \phi n$$

Bimodules are the main mechanism to construct functors between module categories over rings. They play the same role for modules over dg-categories.

Definition 3.4. Let $\mathcal{C}$ and $\mathfrak{D}$ denote two dg-categories. A bimodule $X \in \mathfrak{D}\text{-Mod-}\mathcal{C}$ is a dg-functor

$$X \colon \mathcal{C}^\circ \otimes \mathfrak{D} \to \mathcal{C}(k)$$

More explicitly, for objects $c, c' \in \mathcal{C}$ and $d, d' \in \mathfrak{D}$ there are maps of complexes

$$\mathfrak{D}(d, d') \otimes X(c, d) \otimes \mathcal{C}(c', c) \to X(c', d')$$

satisfying the obvious conditions.

Definition 3.5. For a bimodule $X \in \mathfrak{D}\text{-Mod-}\mathcal{C}$ and $d \in \mathfrak{D}$, we get an object

$$X^d \in \text{Mod-}\mathcal{C} \qquad \text{where } X^d(c) = X(c, d).$$

Similarly, for $c \in \mathcal{C}$, we get an object

$${}^cX \in \mathfrak{D}\text{-Mod} \qquad \text{where } {}^cX(d) = X(c, d).$$

Therefore, we may define for $M \in \text{Mod-}\mathfrak{D}$ the complex

$$M \otimes_{\mathfrak{D}} {}^cX$$

Furthermore the assignment $c \mapsto {}^cX$ defines a functor $\mathcal{C}^\circ \to \mathfrak{D}\text{-Mod}$ and so $c \mapsto M \otimes_{\mathfrak{D}} {}^cX$ defines an object in Mod-$\mathcal{C}$. Thus $\cdot \otimes_{\mathfrak{D}} X$ defines a functor from Mod-$\mathfrak{D} \to$ Mod-$\mathcal{C}$. Moreover, by deriving this functor, we get a functor

$$M \mapsto M \overset{\mathbb{L}}{\otimes}_{\mathfrak{D}} X$$

from $D(\text{Mod-}\mathfrak{D}) \to D(\text{Mod-}\mathcal{C})$.

Definition 3.6 (Keller, [18]). A bimodule $X \in \mathfrak{D}\text{-Mod-}\mathcal{C}$ is called a *quasi-functor* if for all $d \in \mathfrak{D}$, the object $X^d \in \text{Mod-}\mathcal{C}$ is quasi-representable. Such a bimodule therefore defines a functor

$$\operatorname{Ho}\mathfrak{D} \to \operatorname{Ho}\mathcal{C}.$$

Toen [27] calls quasi-functors *right quasi-representable bimodules* and it is a deep theorem of his that they form the correct morphisms in the localization of the category of dg-categories by inverting dg-quasi-equivalences.

3.2. Construction and properties of modules over $\mathcal{P}_{\mathsf{A}}$. We now define a class of modules over the curved dga A that will define modules over the dg-category $\mathcal{P}_{\mathsf{A}}$.

Definition 3.7. For a curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$, we define a *quasi-cohesive module* to be the data of $X = (X^\bullet, \mathbb{X})$ where $X^\bullet$ is a $\mathbb{Z}$-graded right module $X^\bullet$ over $\mathcal{A}$ together with a $\mathbb{Z}$-connection

$$\mathbb{X}\colon X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

that satisfies the integrability condition that the relative curvature

$$F_{\mathbb{X}}(x) = \mathbb{X} \circ \mathbb{X}(x) + x \cdot c = 0$$

for all $x \in X^\bullet$. Thus, they differ from cohesive modules by having possibly infinitely many nonzero graded components as well as not being projective or finitely generated over $\mathcal{A}$.

Definition 3.8. To a quasi-cohesive A-module $X = (X^\bullet, \mathbb{X})$ we associate the $\mathcal{P}_{\mathsf{A}}$-module, denoted $\tilde{h}_X$, by

$$\tilde{h}_X(E) = \{\phi\colon E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \mid \text{of degree } k \text{ and } \phi(xa) = \phi(x)a \ \forall a \in \mathcal{A}^\bullet\}$$

with differential defined in the standard way

$$d(\phi)(ex) = \mathbb{X}\big(\phi(x)\big) - (-1)^{|\phi|}\phi\big(\mathbb{E}(x)\big)$$

for all $E = (E^\bullet, \mathbb{E}) \in \mathcal{P}_{\mathsf{A}}$. We use $\tilde{h}_X$ because of its similarity to the Yoneda embedding h, but beware that X is not an object in $\mathcal{P}_{\mathsf{A}}$. However, in the same way as $\mathcal{P}_{\mathsf{A}}$ is shown to be a dg-category, $\tilde{h}_X$ is shown to be a module over $\mathcal{P}_{\mathsf{A}}$. For two quasi-cohesive A-modules X and Y, and

$$f\colon X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to Y^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

of degree 0 and satisfying $f\mathbb{X} = \mathbb{Y}f$, we get a morphism of $\mathcal{P}_{\mathsf{A}}$-modules

$$\tilde{h}_f\colon \tilde{h}_X \to \tilde{h}_Y$$

The point of a quasi-cohesive A-module $X = (X^\bullet, \mathbb{X})$ is that the differential and morphisms decompose just the same as they do for cohesive modules. For example, $\mathbb{X} = \sum_k \mathbb{X}^k$ where $\mathbb{X}^k\colon E^\bullet \to X^{\bullet-k+1} \otimes_{\mathcal{A}} \mathcal{A}^k$ and similarly for morphisms.

Proposition 3.9. *Let X and Y be quasi-cohesive A-modules and f a morphism. Suppose $f^0\colon (X^\bullet, \mathbb{X}^0) \to (Y^\bullet, \mathbb{Y}^0)$ is a quasi-isomorphism of complexes. Then $\tilde{h}_f$ is a quasi-isomorphism in* Mod-$\mathcal{P}_{\mathsf{A}}$. *The converse is not true.*

It will be important for us to have a criterion for when a quasi-cohesive A-module X induces a quasi-representable $\mathcal{P}_{\mathsf{A}}$-module.

Definition 3.10. Define a map $\phi\colon C \to D$ between $\mathcal{A}$-modules to be *algebraically $\mathcal{A}$-nuclear*, [24], if there are finite sets of elements $\phi_k \in \operatorname{Hom}_{\mathcal{A}}(C, \mathcal{A})$ and $y_k \in D$, $k = 1, \ldots, N$ such that

$$\phi(x) = \sum_k y_k \cdot \phi_k(x)$$

Proposition 3.11 (See Quillen, [24, Proposition 1.1]**).** *For $C^\bullet$ a complex of $\mathcal{A}$-modules, the following are equivalent:*

(1) *$C^\bullet$ is homotopy equivalent to a bounded complex of finitely generated projective $\mathcal{A}$-modules.*
(2) *For any other complex of $\mathcal{A}$-modules $D^\bullet$, the homomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(C^\bullet, \mathcal{A}) \otimes_{\mathcal{A}} D^\bullet \to \mathrm{Hom}_{\mathcal{A}}(C^\bullet, D^\bullet)$$

is a homotopy equivalence of complexes (over k).
(3) *The endomorphism 1_C of $C^\bullet$ is homotopic to an algebraically nuclear endomorphism.*

Definition 3.12. Suppose $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ is a curved dga. Let $X = (X^\bullet, \mathbb{X})$ be a quasi-cohesive module over A. Suppose there exist $\mathcal{A}$-linear morphisms $h^0 \colon X^\bullet \to X^{\bullet-1}$ of degree -1 and $T^0 \colon X^\bullet \to X^\bullet$ of degree 0 satisfying

(1) T^0 is algebraically $\mathcal{A}$-nuclear,
(2) $[\mathbb{X}^0, h^0] = 1 - T^0$

Then we will call X a quasi-finite quasi-cohesive module.

Our criterion is the following.

Theorem 3.13. *Suppose $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ is a curved dga. Let $X = (X^\bullet, \mathbb{X})$ be a quasi-cohesive module over A. Then there is an object $E = (E^\bullet, \mathbb{E}) \in \mathcal{P}_{\mathsf{A}}$ such that $\tilde{h}_X$ is quasi-isomorphic to h_E; that is, $\tilde{h}_X$ is quasi-representable, under either of the two following conditions:*

(1) *X is a quasi-finite quasi-cohesive module.*
(2) *$\mathcal{A}^\bullet$ is flat over $\mathcal{A}$ and there is a bounded complex $(E^\bullet, \mathbb{E}^0)$ of finitely generated projective right $\mathcal{A}$-modules and an $\mathcal{A}$-linear quasi-isomorphism $e^0 \colon (E^\bullet, \mathbb{E}^0) \to (X^\bullet, \mathbb{X}^0)$.*

Proof. In either case (1) or (2) of the theorem, there exists a bounded complex of finitely generated projective right $\mathcal{A}$-modules $(E^\bullet, \mathbb{E}^0)$ and a quasi-isomorphism $e^0 \colon (E^\bullet, \mathbb{E}^0) \to (X^\bullet, \mathbb{X}^0)$. In case (1), X is quasi-finite-cohesive, and Proposition 3.11 implies that e^0 is in fact a homotopy equivalence. In case (2) it is simply the hypothesis.

In particular, $e^0\mathbb{E}^0 - \mathbb{X}^0 e^0 = 0$. Now we construct a $\mathbb{Z}$-connection term by term. The $\mathbb{Z}$-connection $\mathbb{X}$ on $X^\bullet$ induces a connection

$$\mathbb{H} \colon H^k(X^\bullet, \mathbb{X}^0) \to H^k(X^\bullet, \mathbb{X}^0) \otimes_{\mathcal{A}} \mathcal{A}^1$$

for each k. We use the quasi-morphism e^0 to transport this connection to a connection on $H^k(E^\bullet; \mathbb{E}^0)$

$$\begin{tikzcd} H^k(E^\bullet; \mathbb{E}^0) \arrow[r] \arrow[d, "e^0"] & H^k(E^\bullet, \mathbb{E}^0) \otimes_{\mathcal{A}} \mathcal{A}^1 \arrow[d, "e^0 \otimes 1"] \\ H^k(X^\bullet, \mathbb{X}^0) \arrow[r, "\mathbb{H}"] & H^k(X^\bullet, \mathbb{X}^0) \otimes_{\mathcal{A}} \mathcal{A}^1 \end{tikzcd} \tag{3.4}$$

The right vertical arrow above $e^0 \otimes 1$ is a quasi-isomorphism; in case (1) this is because e^0 is a homotopy equivalence and in case (2) because $\mathcal{A}^\bullet$ is flat. The first step is handled by the following lemma.

Lemma 3.14. *Given a bounded complex of finitely generated projective $\mathcal{A}$ modules $(E^\bullet, \mathbb{E}^0)$ with connections $\mathbb{H}\colon H^k(E^\bullet;\mathbb{E}^0) \to H^k(E^\bullet,\mathbb{E}^0)\otimes_{\mathcal{A}}\mathcal{A}^1$, for each k, there exist connections*

$$\widetilde{\mathbb{H}}\colon E^k \to E^k \otimes_{\mathcal{A}} \mathcal{A}^1$$

lifting $\mathbb{H}$. That is,

$$\widetilde{\mathbb{H}}\mathbb{E}^0 = (\mathbb{E}^0 \otimes 1)\widetilde{\mathbb{H}}$$

and the connection induced on the cohomology is $\mathbb{H}$.

Proof (of lemma). Since $E^\bullet$ is a bounded complex of $\mathcal{A}$-modules it lives in some bounded range of degrees $k \in [N, M]$. Pick an arbitrary connection on E^M, ∇. Consider the diagram with exact rows

$$\begin{tikzcd} E^M \arrow[r, "j"] \arrow[d, "\nabla"'] \arrow[dr, "\theta"] & H^M(E^\bullet,\mathbb{E}^0) \arrow[r] \arrow[d, "\mathbb{H}"] & 0 \\ E^M \otimes_{\mathcal{A}} \mathcal{A}^1 \arrow[r, "j\otimes 1"] & H^M(E^\bullet,\mathbb{E}^0)\otimes_{\mathcal{A}}\mathcal{A}^1 \arrow[r] & 0. \end{tikzcd} \tag{3.5}$$

In the diagram, $\theta = \mathbb{H}\circ j - (j\otimes 1)\circ\nabla$ is easily checked to be $\mathcal{A}$-linear and $j\otimes 1$ is surjective by the right exactness of tensor product. By the projectivity of E^M, θ lifts to

$$\tilde{\theta}\colon E^M \to E^M \otimes_{\mathcal{A}} \mathcal{A}^1$$

so that $(j\otimes 1)\tilde{\theta} = \theta$. Set $\widetilde{\mathbb{H}} = \nabla + \tilde{\theta}$. With $\widetilde{\mathbb{H}}$ in place of ∇, the diagram above commutes.

Now choose on E^{M-1} any connection ∇_{M-1}. But ∇_{M-1} does not necessarily satisfy $\mathbb{E}^0\nabla_{M-1} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. So we correct it as follows. Set $\mu = \widetilde{\mathbb{H}}\mathbb{E}^0 - (\mathbb{E}^0\otimes 1)\nabla_{M-1}$. Then μ is $\mathcal{A}$-linear. Furthermore, $\operatorname{Im}\mu \subset \operatorname{Im}\mathbb{E}^0\otimes 1$; this is because $\widetilde{\mathbb{H}}\mathbb{E} \in \operatorname{Im}\mathbb{E}\otimes 1$ since $\widetilde{\mathbb{H}}$ lifts $\mathbb{H}$. So by projectivity it lifts to $\widetilde{\theta}\colon E^{M-1} \to E^{M-1}\otimes_{\mathcal{A}}\mathcal{A}^1$ such that $(\mathbb{E}^0\otimes 1)\circ\widetilde{\theta} = \theta$. Set $\widetilde{\mathbb{H}}\colon E^{M-1} \to E^{M-1}\otimes_{\mathcal{A}}\mathcal{A}^1$ to be $\nabla_{M-1} + \widetilde{\theta}$. Then $\mathbb{E}^0\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0$ in the right most square below.

$$\begin{tikzcd} E^N \arrow[r, "\mathbb{E}^0"] & E^{N+1} \arrow[r, "\mathbb{E}^0"] & \cdots \arrow[r, "\mathbb{E}^0"] & E^{M-1} \arrow[r, "\mathbb{E}^0"] \arrow[d, "\nabla_{M-1}"'] \arrow[dr, "\mu"] & E^M \arrow[r] \arrow[d, "\widetilde{\mathbb{H}}"] & 0 \\ E^N\otimes_{\mathcal{A}}\mathcal{A}^1 \arrow[r, "\mathbb{E}^0\otimes 1"] & E^{N+1}\otimes_{\mathcal{A}}\mathcal{A}^1 \arrow[r, "\mathbb{E}^0\otimes 1"] & \cdots \arrow[r, "\mathbb{E}^0\otimes 1"] & E^{M-1}\otimes_{\mathcal{A}}\mathcal{A}^1 \arrow[r, "\mathbb{E}^0\otimes 1"] & E^M\otimes_{\mathcal{A}}\mathcal{A}^1 \arrow[r] & 0. \end{tikzcd} \tag{3.6}$$

Now we continue backwards to construct all $\widetilde{\mathbb{H}}\colon E^\bullet \to E^\bullet\otimes_{\mathcal{A}}\mathcal{A}^1$ satisfying $(\mathbb{E}^0\otimes 1)\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. This completes the proof of the lemma. □

(Proof of the theorem, continued). Set $\widetilde{\mathbb{E}}^1 = (-1)^k\widetilde{\mathbb{H}}$ on E^k. Then

$$\mathbb{E}^0\widetilde{\mathbb{E}}^1 + \widetilde{\mathbb{E}}^1\mathbb{E}^0 = 0$$

but it is not necessarily true that $e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0 = 0$. We correct this as follows. Consider $\psi = e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0\colon E^\bullet \to X^\bullet\otimes_{\mathcal{A}}\mathcal{A}^1$. Check that ψ is $\mathcal{A}$-linear and a map of complexes.

$$\begin{tikzcd} & (E^\bullet\otimes_{\mathcal{A}}\mathcal{A}^1, \mathbb{E}^0\otimes 1) \arrow[d, "e^0\otimes 1"] \\ E^\bullet \arrow[ur, "\tilde{\psi}"] \arrow[r, "\psi"] & (X^\bullet\otimes_{\mathcal{A}}\mathcal{A}^1, \mathbb{X}^0\otimes 1). \end{tikzcd} \tag{3.7}$$

In the above diagram, $e^0 \otimes 1$ is a quasi-isomorphism since e^0 is a homotopy equivalence. So by [20, Lemma 1.2.5] there is a lift $\tilde{\psi}$ of ψ and a homotopy $e^1 \colon E^\bullet \to X^{\bullet-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ between $(e^0 \otimes 1)\tilde{\psi}$ and ψ,

$$\psi - (e^0 \otimes 1)\widetilde{\psi} = (e^1 \mathbb{E}^0 + \mathbb{X}^0 e^1)$$

So let $\mathbb{E}^1 = \widetilde{\mathbb{E}}^1 - \widetilde{\psi}$. Then

$$\mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = 0 \quad \text{and} \quad e^0\mathbb{E}^1 - \mathbb{X}^1 e^0 = e^1\mathbb{E}^0 + \mathbb{X}^0 e^1. \tag{3.8}$$

So we have constructed the first two components $\mathbb{E}^0$ and $\mathbb{E}^1$ of the $\mathbb{Z}$-connection and the first components e^0 and e^1 of the quasi-isomorphism $E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$.

To construct the rest, consider the mapping cone $L^\bullet$ of e^0. Thus

$$L^\bullet = E[1]^\bullet \oplus X^\bullet$$

Let $\mathbb{L}^0$ be defined as the matrix

$$\mathbb{L}^0 = \begin{pmatrix} \mathbb{E}^0[1] & 0 \\ e^0[1] & \mathbb{X}^0 \end{pmatrix} \tag{3.9}$$

Define $\mathbb{L}^1$ as the matrix

$$\mathbb{L}^1 = \begin{pmatrix} \mathbb{E}^1[1] & 0 \\ e^1[1] & \mathbb{X}^1 \end{pmatrix} \tag{3.10}$$

Now $\mathbb{L}^0\mathbb{L}^0 = 0$ and $[\mathbb{L}^0, \mathbb{L}^1] = 0$ express the identities (3.8). Let

$$D = \mathbb{L}^1\mathbb{L}^1 + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^2 e^0 & [\mathbb{X}^0, \mathbb{X}^2] \end{pmatrix} + r_c \tag{3.11}$$

where r_c denotes right multiplication by c. Then, as is easily checked, D is $\mathcal{A}$-linear and

(1) $[\mathbb{L}^0, D] = 0$ and
(2) $D|_{0 \oplus X^\bullet} = 0$.

Since $(L^\bullet, \mathbb{L}^0)$ is the mapping cone of a quasi-isomorphism, it is acyclic and since $\mathcal{A}^\bullet$ is flat over $\mathcal{A}$, $(L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0 \otimes 1)$ is acyclic too. Since $E^\bullet$ is projective, we have that

$$\mathrm{Hom}^\bullet_{\mathcal{A}}\big((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0)\big)$$

is acyclic. Moreover

$$\mathrm{Hom}^\bullet_{\mathcal{A}}\big((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0)\big) \subset \mathrm{Hom}^\bullet_{\mathcal{A}}\big(L^\bullet, (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, [\mathbb{L}^0, \cdot])\big)$$

is a subcomplex. Now we have that $D \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ is a cycle and so there is $\widetilde{\mathbb{L}}^2 \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ such that $-D = [\mathbb{L}^0, \widetilde{\mathbb{L}}^2]$. Define $\mathbb{L}^2$ on $L^\bullet$ by

$$\mathbb{L}^2 = \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix}. \tag{3.12}$$

Then

$$[\mathbb{L}^0, \mathbb{L}^2] = \left[\mathbb{L}^0, \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix}\right] = -D + \left[\mathbb{L}^0, \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix}\right] = -\mathbb{L}^1\mathbb{L}^1 - r_c. \tag{3.13}$$

So

$$\mathbb{L}^0\mathbb{L}^2 + \mathbb{L}^1\mathbb{L}^1 + \mathbb{L}^2\mathbb{L}^0 + r_c = 0.$$

We continue by setting

$$D = \mathbb{L}^1\mathbb{L}^2 + \mathbb{L}^2\mathbb{L}^1 + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^3 e^0 & [\mathbb{X}^0, \mathbb{X}^3] \end{pmatrix}. \tag{3.14}$$

Then $D\colon L^\bullet \to L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^3$ is $\mathcal{A}$-linear, $D|_{0\oplus X^\bullet} = 0$ and

$$[\mathbb{L}^0, D] = \mathbb{L}^1 \circ r_c - r_c \circ \mathbb{L}^1 = 0$$

by the Bianchi identity $d(c) = 0$. Hence, by the same reasoning as above, there is $\widetilde{\mathbb{L}}^3 \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^3)$ such that $-D = [\mathbb{L}^0, \widetilde{\mathbb{L}}^3]$. Define

$$\mathbb{L}^3 = \widetilde{\mathbb{L}}^3 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^3 \end{pmatrix}. \tag{3.15}$$

Then one can compute that $\sum_{i=0}^3 \mathbb{L}^i \mathbb{L}^{3-i} = 0$.

Now suppose we have defined $\mathbb{L}^0, \dots, \mathbb{L}^n$ satisfying for $k = 0, 1, \dots, n$

$$\sum_{i=0}^{k} \mathbb{L}^i \mathbb{L}^{k-i} = 0 \qquad \text{for } k \neq 2$$

and

$$\sum_{i=0}^{2} \mathbb{L}^i \mathbb{L}^{2-i} + r_c = 0 \qquad \text{for } k = 2.$$

Then define

$$D = \sum_{i=1}^{n} \mathbb{L}^i \mathbb{L}^{n+1-i} + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^{n+1} e^0 & [\mathbb{X}^0, \mathbb{X}^{n+1}] \end{pmatrix}. \tag{3.16}$$

$D|_{0\oplus X^\bullet} = 0$ and we may continue the inductive construction of $\mathbb{L}$ to finally arrive at a $\mathbb{Z}$-connection satisfying $\mathbb{L}\mathbb{L} + r_c = 0$. The components of $\mathbb{L}$ construct both the $\mathbb{Z}$-connection on $E^\bullet$ as well as the morphism from $(E^\bullet, \mathbb{E})$ to $(X^\bullet, \mathbb{X})$. □

4. Complex manifolds

We justify our framework in this section by showing that, for a complex manifold, the derived category of sheaves on X with coherent cohomology is equivalent to the homotopy category $\mathcal{P}_{\mathsf{A}}$ for the Dolbeault algebra. Throughout this section let X be a compact complex manifold and $\mathsf{A} = (\mathcal{A}^\bullet, d, 0) = (\mathcal{A}^{0,\bullet}(X), \bar\partial, 0)$ the Dolbeault dga. This is the global sections of the sheaf of dgas $(\mathcal{A}^\bullet_X, d, 0) = (\mathcal{A}^{0,\bullet}_X, \bar\partial, 0)$. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Koszul and Malgrange have shown that a holomorphic vector bundle ξ on a complex manifold X is the same thing as a C^∞ vector bundle with a flat $\bar\partial$-connection, i.e., an operator

$$\bar\partial_\xi\colon E_\xi \to E_\xi \otimes_{\mathcal{A}} \mathcal{A}^1$$

such that $\bar\partial_\xi(f\phi) = \bar\partial(f)\phi + f\bar\partial_\xi(\phi)$ for $f \in \mathcal{A}$, $\phi \in \Gamma(X;\xi)$ and satisfying the integrability condition that $\bar\partial_\xi \circ \bar\partial_\xi = 0$. Here E_ξ denotes the global C^∞ sections of ξ. The notion of a cohesive module over A clearly generalizes this notion but in fact will also include coherent analytic sheaves on X and even more generally, bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology as well.

For example, if $(\xi^\bullet, \delta)$ denotes a complex of holomorphic vector bundles, with corresponding global C^∞-sections $E^\bullet$ and $\bar\partial$-operator $\bar\partial_\xi\colon E^i \to E^i \otimes_{\mathcal{A}} \mathcal{A}^1$ then the holomorphic condition on δ is that $\delta\bar\partial_\xi = \bar\partial_\xi\delta$. Thus $E = (E^\bullet, \mathbb{E})$, where $\mathbb{E}^0 = \delta$ and $\mathbb{E}^1 = (-1)^\bullet\bar\partial_\xi$ defines the cohesive module corresponding to $(\xi^\bullet, \delta)$. So we see that, for coherent sheaves with locally free resolutions, there is nothing new here.

4.1. The derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Pali, [22] was the first to give a characterization of general coherent analytic sheaves in terms of sheaves over $(\mathcal{A}_X^\bullet, d)$ equipped with flat $\bar{\partial}$-connections. He defines a *$\bar{\partial}$-coherent analytic sheaf* $\mathcal{F}$ to be a sheaf of modules over the sheaf of C^∞-functions $\mathcal{A}_X$ satisfying two conditions:

Finiteness: locally on X, $\mathcal{F}$ has a finite resolution by finitely generated free modules, and

Holomorphic: $\mathcal{F}$ is equipped with a $\bar{\partial}$-connection, i.e., an operator (at the level of sheaves)

$$\bar{\partial}\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{A}_X} \mathcal{A}_X^1$$

and satisfying $\bar{\partial}^2 = 0$.

Theorem 4.1 (Pali, [22]). *The category of coherent analytic sheaves on X is equivalent to the category of $\bar{\partial}$-coherent sheaves.*

We prove our theorem independently of his proof. We use the following proposition of Illusie, [3].

Proposition 4.2. *Suppose $(X, \mathcal{A}_X)$ is a ringed space, where X is compact and $\mathcal{A}_X$ is a soft sheaf of rings. Then*

(1) *The global sections functor*

$$\Gamma\colon \text{Mod-}\mathcal{A}_X \to \text{Mod-}\mathcal{A}_X(X)$$

is exact and establishes an equivalence of categories between the category of sheaves of right $\mathcal{A}_X$-modules and the category of right modules over the global sections $\mathcal{A}_X(X)$.

(2) *If $M \in \text{Mod-}\mathcal{A}_X$ locally has finite resolutions by finitely generated free $\mathcal{A}_X$-modules, then $\Gamma(X; M)$ has a finite resolution by finitely generated projectives.*

(3) *The derived category of perfect complexes of sheaves $D_{\text{perf}}(\text{Mod-}\mathcal{A}_X)$ is equivalent to the derived category of perfect complexes of modules $D_{\text{perf}}\big(\text{Mod-}\mathcal{A}_X(X)\big)$.*

Proof. See [3, Proposition 2.3.2, Exposé II]. □

Our goal is to derive the following description of the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology on a complex manifold. Note that this is equivalent to the category of perfect complexes, since we are on a smooth manifold. Recall that $\mathsf{A} = (\mathcal{A}^\bullet, d, 0) = (\mathcal{A}^{0,\bullet}(X), \bar{\partial}, 0)$, the Dolbeault dga, is the global sections of the sheaf of dgas $(\mathcal{A}_X^\bullet, d, 0) = (\mathcal{A}_X^{0,\bullet}, \bar{\partial}, 0)$

Theorem 4.3. *Let X be a compact complex manifold and $\mathsf{A} = (\mathcal{A}^\bullet, d, 0) = (\mathcal{A}^{0,\bullet}(X), \bar{\partial}, 0)$ the Dolbeault dga. Then the category $\operatorname{Ho}\mathcal{P}_{\mathsf{A}}$ is equivalent to the bounded derived category of complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology $D^b_{coh}(X)$.*

Remark 4.4. This theorem is stated only for X compact. This is because Proposition 4.2 is stated only for X compact. A version of Theorem 4.3 will be true once one is able to characterize the perfect $\mathcal{A}_X$-modules in terms of modules over the global sections for X which are not compact.

A module M over $\mathcal{A}$ naturally localizes to a sheaf M_X of $\mathcal{A}_X$-modules, where

$$M_X(U) = M \otimes_{\mathcal{A}} \mathcal{A}_X(U)$$

For an object $E = (E^\bullet, \mathbb{E})$ of $\mathcal{P}_{\mathsf{A}}$, define the sheaves $\mathcal{E}_X^{p,q}$ by

$$\mathcal{E}_X^{p,q}(U) = E^p \otimes_{\mathcal{A}} \mathcal{A}_X^q(U).$$

We define a complex of sheaves by $(\mathcal{E}_X^\bullet, \mathbb{E}) = (\sum_{p+q=\bullet} \mathcal{E}_X^{p,q}, \mathbb{E})$. This is a complex of soft sheaves of $\mathcal{O}_X$-modules, since $\mathbb{E}$ is a $\bar\partial$-connection. The theorem above will be broken up into several lemmas.

Lemma 4.5. *The complex $\mathcal{E}_X^\bullet$ has coherent cohomology and*

$$E = (E^\bullet, \mathbb{E}) \mapsto \alpha(E) = (\mathcal{E}_X^\bullet, \mathbb{E})$$

defines a fully faithful functor $\alpha\colon \operatorname{Ho}\mathcal{P}_{\mathsf{A}} \to \mathcal{D}_{\mathrm{perf}}(X) \simeq \mathcal{D}^b_{\mathrm{coh}}(X)$.

Proof. Let U be a polydisc in X. We show that on a possibly smaller polydisc V, there is gauge transformation $\phi\colon \mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet|_V$ of degree zero such that $\phi \circ \mathbb{E} \circ \phi^{-1} = \mathbb{F}^0 + \bar\partial$. Thus $\mathcal{E}^\bullet|_V$ is gauge equivalent to a complex of holomorphic vector bundles. Or, in other words, for each p the sheaf $H^p(\mathcal{E}^{\bullet,0}, \mathbb{E}^0)$ is $\bar\partial$-coherent, with $\bar\partial$-connection $\mathbb{E}^1$. Since U is Stein there is no higher cohomology (with respect to $\mathbb{E}^1$) and we are left with the holomorphic sections over U of each of these $\bar\partial$-coherent sheaves, which are thus coherent.

The construction of the gauge transformation follows the proof of the integrability theorem for complex structures on vector bundles, [9, Section 2.2.2, p. 50]. Thus we may assume we are in a polydisc $U = \{(z_1, \dots, z_n) \mid |z|_i < r_i\}$. In these coordinates we may write the $\mathbb{Z}$-connection $\mathbb{E}$ as $\mathbb{E} = \mathbb{E}^0 + \bar\partial + J$, where

$$J\colon \mathcal{E}^{p,q}(U) \to \bigoplus_{i \le p} \mathcal{E}^{i,q+(p-i)+1}(U)$$

is $\mathcal{A}_X(U)$-linear. Now write $J = J' \wedge d\bar z_1 + J''$ where $\iota_{\partial/\partial \bar z_1} J' = \iota_{\partial/\partial \bar z_1} J'' = 0$. Write $\bar\partial_i$ for $d\bar z_i \wedge (\partial/\partial \bar z_i)$. As in [9, p. 51], we find a ϕ_1 such that $\phi_1(\bar\partial_1 + J' \wedge d\bar z_1)\phi_1^{-1} = \bar\partial_1$, by solving $\phi_1^{-1}\bar\partial_1(\phi_1) = J' \wedge d\bar z_1$ for ϕ_1, possibly having to shrink the polydisc. Here, we are treating the variables $z_2, \dots, z_n$ as parameters. Then we set $\mathbb{E}_1 = \phi_1(\mathbb{E}^0 + \bar\partial + J' + J'')\phi_1^{-1}$. Then $\mathbb{E}_1 \circ \mathbb{E}_1 = 0$ and we can write

$$\mathbb{E}_1 = \mathbb{E}_1^0 + \bar\partial_1 + \bar\partial_{\ge 2} + J_1$$

where $\iota_{\partial/\partial \bar z_1} J_1 = 0$ and we can check that both $\mathbb{E}_1^0$ and J_1 are holomorphic in z_1. For $0 = \mathbb{E}_1 \circ \mathbb{E}_1$ and therefore

$$\begin{aligned} 0 &= \iota_{\partial/\partial \bar z_1}(\mathbb{E}_1 \circ \mathbb{E}_1) \\ &= \iota_{\partial/\partial \bar z_1}(\mathbb{E}_1^0 \circ \bar\partial_1 + \bar\partial_1 \circ E_1^0 + J_1 \circ \bar\partial_1 + \bar\partial_1 \circ J_1) \\ &= \iota_{\partial/\partial \bar z_1}\big(\bar\partial_1(\mathbb{E}_1^0) + \bar\partial_1(J_1)\big) \end{aligned} \tag{4.1}$$

Now each of the two summands in the last line must individually be zero since $\iota_{\partial/\partial \bar z_1}\big(\bar\partial_1(\mathbb{E}_1^0)\big)$ increases the p-degree by one and $\iota_{\partial/\partial \bar z_1}\big(\bar\partial_1(J_1)\big)$ preserves or decreases the p-degree by one. So we have arrived at the following situation:

(1) $\mathbb{E}_1^0 \circ \mathbb{E}_1^0 = 0$,
(2) $\mathbb{E}_1^0$ and J_1 are holomorphic in z_1, and
(3) $\iota_{\partial/\partial \bar z_1} J_1 = 0$.

We now iterate this procedure. Write $J_1 = J_1' \wedge d\bar z_2 + J_1''$ where $\iota_{\partial/\partial \bar z_1} J_1' = \iota_{\partial/\partial \bar z_2} J_1' = \iota_{\partial/\partial \bar z_1} J_1'' = \iota_{\partial/\partial \bar z_2} J_1'' = 0$. Now solve

$$\phi_2^{-1}\bar\partial_2(\phi_2) = J_1' \wedge d\bar z_2$$

for ϕ_2. Since J_1' is holomorphic in z_1 and smooth in $z_2, \dots, z_n$, so will ϕ_2. Then as before we have

$$\phi_2(\bar\partial_2 + J_1' \wedge d\bar z_2)\phi_2^{-1} = \bar\partial_2$$

as well as

$$\phi_2(\bar\partial_1)\phi_2^{-1} = \bar\partial_1$$

since ϕ_2 is holomorphic in z_1. Setting $\mathbb{E}_2 = \phi_2 \circ \mathbb{E}_1 \circ \phi_2^{-1}$, we see that

$$\mathbb{E}_2 = \mathbb{E}_2^0 + \bar\partial_1 + \bar\partial_2 + \bar\partial_{\geq 3} + J_2$$

where $\iota_{\partial/\partial \bar z_1} J_2 = \iota_{\partial/\partial \bar z_2} J_2 = 0$ and we can check as before that both $\mathbb{E}_2^0$ and J_2 are holomorphic in z_1 and z_2. We continue until we arrive at $\mathbb{F} = \mathbb{E}_n = \mathbb{E}_n^0 + \bar\partial$. $\square$

Lemma 4.6. *To any complex of sheaves of $\mathcal{O}_X$-modules $(\mathcal{E}_X^\bullet, d)$ on X with coherent cohomology there corresponds a cohesive A-module $E = (E^\bullet, \mathbb{E})$, unique up to quasi-isomorphism in $\mathcal{P}_{\mathsf{A}}$ and a quasi-isomorphism*

$$\alpha(E) \to (\mathcal{E}^\bullet, d)$$

This correspondence has the property that, for any two such complexes $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$, the corresponding twisted complexes $(E_1^\bullet, \mathbb{E}_1)$ and $(E_2^\bullet, \mathbb{E}_2)$ satisfy

$$\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet) \cong H^k\big(\mathcal{P}_{\mathsf{A}}(E_1, E_2)\big)$$

PROOF. Since we are on a manifold we may assume that $(\mathcal{E}^\bullet, d)$ is a perfect complex. Set $\mathcal{E}_\infty^\bullet = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{A}_X$. Now the map $(\mathcal{E}^\bullet, d) \to (\mathcal{E}_\infty^\bullet \otimes_{\mathcal{A}} \mathcal{A}_X^\bullet, d \otimes 1 + 1 \otimes \bar\partial)$ is a quasi-isomorphism of sheaves of $\mathcal{O}_X$-modules by the flatness of $\mathcal{A}_X$ over $\mathcal{O}_X$. Again, by the flatness of $\mathcal{A}_X$ over $\mathcal{O}_X$, it follows that $(\mathcal{E}_\infty^\bullet, d)$ is a perfect complex of $\mathcal{A}_X$-modules. By Proposition 4.2, there is a (strictly) perfect complex $(E^\bullet, \mathbb{E}^0)$ of $\mathcal{A}$-modules and quasi-isomorphism $e^0 \colon (E^\bullet, \mathbb{E}^0) \to (\Gamma(X, \mathcal{E}_\infty^\bullet), d)$. Moreover $(\Gamma(X, \mathcal{E}_\infty^\bullet), d \otimes 1 + 1 \otimes \bar\partial)$ defines a quasi-cohesive module over A. So the hypotheses of Theorem 3.13(2) are satisfied. The lemma is proved. $\square$

4.2. Gerbes on complex manifolds. The theorem above has an analogue for gerbes over compact manifolds. X is still a compact complex manifold. A class $b \in H^2(X, \mathcal{O}_X^\times)$ defines an $\mathcal{O}_X^\times$-gerbe on X. From the exponential sequence of sheaves

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp 2\pi\imath\,\cdot} \mathcal{O}_X^\times \to 0$$

there is a long exact sequence

$$\cdots \to H^2(X; \mathcal{O}_X) \to H^2(X; \mathcal{O}_X^\times) \to H^3(X; \mathbb{Z}_X) \to \cdots$$

If b maps to $0 \in H^3(X;\mathbb{Z})$ (that is, the gerbe is topologically trivializable) then b pulls back to a class represented by a $(0,2)$-form $B \in \mathcal{A}^{0,2}(X)$. Consider the curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, B) = (\mathcal{A}^{0,\bullet}(X), \bar\partial, B)$ — the same Dolbeault algebra as before but with a curvature. Then we have a theorem [4], corresponding to (4.3),

Theorem 4.7. *The category $\mathrm{Ho}\,\mathcal{P}_{\mathsf{A}}$ is equivalent to the bounded derived category of complexes of sheaves on the gerbe b over X of $\mathcal{O}_X$-modules with coherent cohomology and weight one $D^b_{coh}(X)_{(1)}$.*

Sheaves on a gerbe are often called twisted sheaves. One can deal with gerbes which are not necessarily topologically trivial, but the curved dga is slightly more complicated, [4].

5. Examples

5.1. Elliptic curved dgas. In this section we define a class of curved dga's A such that the corresponding dg-category $\mathcal{P}_{\mathsf{A}}$ is *proper*, that is, the cohomology of the hom sets are finite dimensional. It is often useful to equip a manifold with a Riemannian metric so that one can use Hilbert space methods. We introduce a relative of the notion spectral triple in the sense of Connes, [6], so that we can use Hilbert space methods to guarantee the properness of the dg-category.

Again, our basic data is a curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$.

Definition 5.1. We say that A is equipped with a *Hilbert structure* if there is a positive definite Hermitian inner product on $\mathcal{A}^\bullet$

$$\langle \cdot, \cdot \rangle \colon \mathcal{A}^k \times \mathcal{A}^k \to \mathbb{C}$$

satisfying the following conditions: Let $\mathcal{H}^\bullet$ be the completion of $\mathcal{A}^\bullet$.

(1) For $a \in \mathcal{A}^\bullet$, the operator l_a (respectively r_a) of left (respectively, right) multiplication by a extends to $\mathcal{H}^\bullet$ as a bounded operator. Furthermore, the operators l_a^* and r_a^* map $\mathcal{A}^\bullet \subset \mathcal{H}^\bullet$ to itself.
(2) $\mathcal{A}$ has an anti-linear involution $*\colon \mathcal{A} \to \mathcal{A}$ such that for $a \in \mathcal{A}$, there is $(l_a)^* = l_{a^*}$ and $(r_a)^* = r_{a^*}$.
(3) The differential d is required to be closable in $\mathcal{H}^\bullet$. Its adjoint satisfies $d^*(\mathcal{A}^\bullet) \subset \mathcal{A}^\bullet$ and the operator $D = d + d^*$ is essentially self-adjoint with core $\mathcal{A}^\bullet$.
(4) For $a \in \mathcal{A}^\bullet$, $[D, l_a]$, $[D, r_a]$, $[D, l_a^*]$ and $[D, r_a^*]$ are bounded operators on $\mathcal{H}^\bullet$.

Definition 5.2. An elliptic curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ is a curved dga with a Hilbert structure which in addition satisfies

(1) The operator e^{-tD^2} is trace class for all $t > 0$.
(2) $\mathcal{A}^\bullet = \bigcap_n \mathrm{Dom}(D^n)$

The following proposition follows from very standard arguments.

Proposition 5.3. *Given an elliptic curved dga A then for $E = (E^\bullet, \mathbb{E})$ and $F = (F^\bullet, \mathbb{F})$ in $\mathcal{P}_{\mathsf{A}}$ one has that the cohomology of $\mathcal{P}_{\mathsf{A}}(E, F)$ is finite dimensional.*

Bondal and Kapranov have given a very beautiful formulation of Serre duality purely in the derived category. We adapt their definitions to our situation of dg-categories.

Definition 5.4. For a dg-category $\mathcal{C}$, such that all Hom complexes have finite dimensional cohomology, a Serre functor is a dg-functor

$$S \colon \mathcal{C} \to \mathcal{C}$$

which is a dg-equivalence and so that there are pairings of degree zero, functorial in both E and F

$$\langle \cdot, \cdot \rangle \colon \mathcal{C}^\bullet(E, F) \times \mathcal{C}^\bullet(F, SE) \to \mathbb{C}[0]$$

satisfying

$$\langle d\phi, \psi \rangle + (-1)^{|\phi|} \langle \phi, d\psi \rangle = 0$$

which are perfect on cohomology for any E and F in $\mathcal{C}$.

Motivated by the case of Lie algebroids below, we make the following definition, which will guarantee the existence of a Serre functor.

Definition 5.5. Let $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ be an elliptic curved dga. A dualizing module (of dimension g) is a triple $((D, \mathbb{D}), \bar{*}, \int)$ where

(1) $(D, \mathbb{D})$ is an A-A cohesive bimodule,

(2) $\bar{*}\colon \mathcal{A}^k \to D \otimes_{\mathcal{A}} \mathcal{A}^{g-k}$ is a conjugate linear isomorphism and satisfies
$$\bar{*}(a\omega) = \bar{*}(\omega)a^* \quad \text{and} \quad \bar{*}(\omega a) = a^* \bar{*}(\omega)$$
for $a \in \mathcal{A}$ and $\omega \in \mathcal{A}^\bullet$.

(3) There is a $\mathbb{C}$-linear map $\int\colon D \otimes_{\mathcal{A}} \mathcal{A}^g \to \mathbb{C}$ such that $\int \mathbb{D}(x) = 0$ for all $x \in D \otimes_{\mathcal{A}} \mathcal{A}^\bullet$ and
$$\int \omega \cdot x = (-1)^{|\omega||x|} \int x \cdot \omega$$
for all $\omega \in \mathcal{A}^\bullet$ and $x \in D \otimes_{\mathcal{A}} \mathcal{A}^\bullet$, and
$$\langle \omega, \eta \rangle = \int \bar{*}(\omega)\eta$$

Proposition 5.6. *Given an elliptic curved dga $\mathsf{A} = (\mathcal{A}^\bullet, d, c)$ with a dualizing module $((D, \mathbb{D}), \bar{*}, \int)$, the category $\mathcal{P}_{\mathsf{A}}$ has a Serre functor given by the cohesive bimodule $(D[g], \mathbb{D})$. That is,*
$$S(E^\bullet, \mathbb{E}) = (E \otimes_{\mathcal{A}} D[g], \mathbb{E} \# \mathbb{D})$$
is a dg-equivalence for which there are functorial pairings
$$\langle \cdot, \cdot \rangle\colon \mathcal{P}^\bullet_{\mathsf{A}}(E, F) \times \mathcal{P}^\bullet_{\mathsf{A}}(F, SE) \to \mathbb{C}$$
satisfying
$$\langle d\phi, \psi \rangle + (-1)^{|\phi|} \langle \phi, d\psi \rangle = 0$$
is perfect on cohomology for any E and F in $\mathcal{P}_{\mathsf{A}}$.

5.2. Lie algebroids. Lie algebroids provide a natural source of dga's and thus, by passing to their cohesive modules, interesting dg-categories.

Let X be a C^∞-manifold and let $\mathfrak{a}$ be a complex Lie algebroid over X. Thus $\mathfrak{a}$ is a C^∞ vector bundle on X with a bracket operation on $\Gamma(X; \mathfrak{a})$ making $\Gamma(X; \mathfrak{a})$ into a Lie algebra and such that the induced map into vector fields $\rho\colon \Gamma(X; \mathfrak{a}) \to \mathcal{V}(X)$ is a Lie algebra homomorphism and for $f \in C^\infty(X)$ and $x, y \in \Gamma(X; \mathfrak{a})$ we have
$$[x, fy] = f[x, y] + (\rho(x)f)y.$$
Let g be the rank of $\mathfrak{a}$ and n for the dimension of X.

There is a dga corresponding to any Lie algebroid $\mathfrak{a}$ over X as follows. Let
$$\mathcal{A}^\bullet_{\mathfrak{a}} = \Gamma(X; \textstyle\bigwedge^\bullet \mathfrak{a}^\vee)$$
denote the space of smooth $\mathfrak{a}$-differential forms. It has a differential d of degree one, with $d = 0$ given by the usual formula,

$$(5.1) \quad (d\eta)(x_1, \dots, x_k) = \sum_i (-1)^{i+1} \rho(x_i)\big(\eta(x_1, \dots, \hat{x}_i, \dots, x_k)\big) + \sum_{i<j} (-1)^{i+j} \eta([x_i, x_j], \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k).$$

turning it into a differential graded algebra. Note that $\mathcal{A}_{\mathfrak{a}} = \mathcal{A}^0_{\mathfrak{a}}$ is just the C^∞-functions on X. Then $\mathsf{A}_{\mathfrak{a}} = (\mathcal{A}^\bullet_{\mathfrak{a}}, d, 0)$ is a curved dga.

5.2.1. *The dualizing $\mathfrak{a}$-module $\mathcal{D}_{\mathfrak{a}}$.* We recall the definition of the "dualizing module" of a Lie algebroid. This was first defined in [12] where they used it to define the modular class of the Lie algebroid.

Let $\mathfrak{a}$ be a Lie algebroid over X with anchor map ρ. Consider the line bundle

$$\mathcal{D}_{\mathfrak{a}} = \bigwedge^g \mathfrak{a} \otimes \bigwedge^n T^\vee_{\mathbb{C}} X. \tag{5.2}$$

Write $D_{\mathfrak{a}} = \Gamma(X; \mathcal{D}_{\mathfrak{a}})$. Define

$$\mathbb{D}\colon D_{\mathfrak{a}} \to D_{\mathfrak{a}} \otimes_{\mathcal{A}_{\mathfrak{a}}} \mathcal{A}^1_{\mathfrak{a}}$$

by

$$\mathbb{D}(X \otimes \mu)(x) = L_x(X) \otimes \mu + X \otimes L_{\rho(x)}\mu \tag{5.3}$$

where $x \in \Gamma(X; \mathfrak{a})$, $X \in \Gamma(X; \bigwedge^g \mathfrak{a})$, $\mu \in \Gamma(\bigwedge^n T^\vee_{\mathbb{C}} X)$, and $L_{\rho(x)}\mu$ denotes the Lie derivative of μ in the direction of $\rho(x)$. See [12] for more details.

Now we note that $\mathsf{A}_{\mathfrak{a}}$ acts on the left of $D_{\mathfrak{a}} \otimes_{\mathcal{A}_{\mathfrak{a}}} \mathcal{A}^\bullet_{\mathfrak{a}}$ and $\mathbb{D}\colon D_{\mathfrak{a}} \to D_{\mathfrak{a}} \otimes_{\mathcal{A}_{\mathfrak{a}}} \mathcal{A}^1_{\mathfrak{a}}$ defines a flat $\mathcal{A}^\bullet_{\mathfrak{a}}$-connection [12]. Therefore $(D_{\mathfrak{a}}, \mathbb{D})$ denote a cohesive $\mathsf{A}_{\mathfrak{a}}$-$\mathsf{A}_{\mathfrak{a}}$-bimodule, and thus a dg-functor from $\mathcal{P}_{\mathsf{A}_{\mathfrak{a}}}$ to itself.

We have the pairing

$$\mathcal{D}_{\mathfrak{a}} \otimes \bigwedge^g \mathfrak{a}^\vee \to \bigwedge^n T^*_{\mathbb{C}} X.$$

Which allows us to define $\int\colon D_{\mathfrak{a}} \otimes_{\mathcal{A}} \mathcal{A}^g \to \mathbb{C}$ for $(X \otimes \mu) \otimes \nu \in \mathcal{D}_{\mathfrak{a}} \otimes \bigwedge^g \mathfrak{a}^\vee$

$$\int (X, \nu)\mu$$

Then we have

Theorem 5.7 (Stokes' theorem, [12]**).** *Identify $D_{\mathfrak{a}} \otimes_{\mathcal{A}_{\mathfrak{a}}} \mathcal{A}^g_{\mathfrak{a}}(X) = \Gamma(\bigwedge^g \mathfrak{a} \otimes \bigwedge^g \mathfrak{a}^\vee \otimes \bigwedge^n T^\vee_{\mathbb{C}} X)$ with the space of top-degree forms on X by pairing the factors in $\bigwedge^g \mathfrak{a}^*$ and $\bigwedge^g \mathfrak{a}$ pointwise. We have, for every $c = (X \otimes \mu) \otimes \nu \in D_{\mathcal{A}_{\mathfrak{a}}} \otimes_{\mathcal{A}_{\mathfrak{a}}} \mathcal{A}^{g-1}_{\mathfrak{a}}(X)$,*

$$\mathbb{D}(c) = (-1)^{g-1} d(\rho(\mu \lrcorner X) \lrcorner \nu). \tag{5.4}$$

Consequently,

$$\int_X \mathbb{D}(c) = 0. \tag{5.5}$$

5.2.2. *Hermitian structures and the $\bar{*}$-operator.* Let us equip the algebroid $\mathfrak{a}$ with a Hermitian inner product $\langle\ ,\ \rangle$. Then $\mathfrak{a}^\vee$ and $\bigwedge^\bullet \mathfrak{a}^\vee$ all inherit Hermitian inner products according to the rule

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det(\langle \alpha_i, \beta_j \rangle).$$

Also let us put on X a Riemannian structure and let ν_X be the volume form. Then there is a Hermitian inner product on $\mathcal{A}^\bullet_{\mathfrak{a}}(X)$ defined by

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha, \beta \rangle \nu_X.$$

Recall that there is a canonical identification of $\mathcal{D}_{\mathfrak{a}} \otimes \bigwedge^g \mathfrak{a}^\vee$ with $\bigwedge^n T^\vee_{\mathbb{C}} X$. Define the operator $\bar{*}\colon \bigwedge^k \mathfrak{a}^\vee \to \mathcal{D}_{\mathfrak{a}} \otimes \bigwedge^{g-k} \mathfrak{a}^\vee$ by requiring that

$$\alpha \wedge \bar{*}\beta = \langle \alpha, \beta \rangle \nu_X. \tag{5.6}$$

This is well defined because the pairing

$$\bigwedge^k \mathfrak{a}^\vee \times (\mathcal{D}_{\mathfrak{a}} \otimes \bigwedge^{g-k} \mathfrak{a}^\vee) \to \bigwedge^n T^\vee_{\mathbb{C}} X$$

is perfect. Our $\bar{*}$ operator is conjugate linear. This is because we have no conjugation operator on $\mathfrak{a}$, as would be the case when we define the Hodge $*$ operator on the bigraded Dolbeault complex.

As usual, we have the familiar local expressions for the $\bar{*}$-operator. So if $\alpha_1, \ldots, \alpha_g$ is an orthonormal frame of $\mathfrak{a}$ with $\alpha^1, \ldots, \alpha^g$ the dual frame, then for a multi-index $I \subset \{1, \ldots, g\}$ we have

$$\bar{*}(\lambda\alpha^I) = (-1)^{\sigma(I)}(\alpha_{\{1,\ldots,g\}} \otimes \nu_X) \otimes \bar{\lambda}\alpha^{I^c}$$

where I^c is the complement of the multi-index and $\sigma(I)$ is the sign of the permutation $(1, \ldots, g) \mapsto (I, I^c)$. For an object $(E, \mathbb{E})$ of $\mathcal{P}_{\mathsf{A}_\mathfrak{a}}$ we equip E with a Hermitian structure (no condition) and we extend $\bar{*}$ to

$$\bar{*}\colon E \otimes \textstyle\bigwedge^k \mathfrak{a}^\vee \to E^\vee \otimes \mathcal{D}_\mathfrak{a} \otimes \textstyle\bigwedge^{g-k} \mathfrak{a}^\vee$$

by the same formula, (5.6). Locally, we have

$$\bar{*}(e_i \otimes \lambda\alpha^I) = (-1)^{\sigma(I)} e^i \otimes (\alpha_{\{1,\ldots,g\}} \otimes \nu_X) \otimes \bar{\lambda}\alpha^{I^c}$$

where e_i and e^i are dual pairs of orthonormal frames of E and $E^\vee$ respectively.

Now we make a basic assumption on our Lie algebroid $\mathfrak{a}$.

Definition 5.8. A complex Lie algebroid $\rho\colon \mathfrak{a} \to T_\mathbb{C} X$ is called *elliptic* if

$$\rho^\vee\colon T^\vee X \to T^\vee_\mathbb{C} X \to \mathfrak{a}^\vee$$

is injective.

Note that a real Lie algebroid being elliptic means that it is transitive. The point of this definition is the following proposition.

Theorem 5.9. *For an elliptic Lie algebroid $\mathfrak{a}$, the corresponding dga $\mathsf{A}_\mathfrak{a} = (\mathcal{A}^\bullet_\mathfrak{a}, d, 0)$ is an elliptic dga and $((D_\mathfrak{a}, \mathbb{D}), \bar{*}, \int)$ is a dualizing manifold with a representation.*

Proof. Everything follows from basic elliptic theory. $\square$

As an immediate corollary we have

Theorem 5.10. *For an elliptic Lie algebroid $\mathfrak{a}$ with $(E, \mathbb{E}), (F, \mathbb{F}) \in \mathcal{P}_{\mathsf{A}_\mathfrak{a}}$, there is a perfect duality pairing*

$$H^k\big(\mathcal{P}_{\mathsf{A}_\mathfrak{a}}(E, F)\big) \times H^{g-k}\big(\mathcal{P}_{\mathsf{A}_\mathfrak{a}}(F, E \otimes D_\mathfrak{a})\big) \to \mathbb{C}.$$

5.2.3. *The de Rham Lie algebroid and Poincaré duality.* For $\rho = \mathrm{Id}\colon \mathfrak{a} = TM \to TM$ the duality theorem is Poincaré's for local systems. That is, the dualizing module is the trivial one-dimensional vector bundle (we made the blanket assumption that M is orientable) and for a flat vector bundle E over X there is a perfect pairing

$$H^k(X; E) \times H^{n-k}(X, E^\vee) \to \mathbb{C}.$$

5.2.4. *The Dolbeault Lie algebroid and Serre duality.* For X a complex n-dimensional manifold, let $\rho\colon \mathfrak{a} = T^{0,1} \hookrightarrow T_{\mathbb{C}}X$ be the natural inclusion. Thus, a holomorphic vector bundle is the same thing as an $T^{0,1}$-module. Moreover

$$\mathcal{D}_{T^{0,1}} = \bigwedge^n T^{0,1} \otimes \bigwedge^{2n} T^\vee_{\mathbb{C}} X \cong \bigwedge^n T^{0,1} \otimes \bigwedge^n (T^{0,1\,\vee}X \oplus T^{1,0\,\vee}X) \cong \bigwedge^n T^{1,0\,\vee} X$$

is the usual canonical (or dualizing) bundle K in complex geometry. And Theorem 5.10 reduces to Serre's duality theorem that for a holomorphic vector bundle E the sheaf (i.e., Dolbeault) cohomology satisfies

$$H^k_{\bar\partial}(X;E)^\vee \cong H^{n-k}_{\bar\partial}(X;E^\vee \otimes K).$$

from which it follows by letting E be $\bigwedge^p T^{1,0\vee}$ that

$$H^{p,q}(X) \cong H^{n-p,n-q}(X).$$

Stated in terms of Serre functors, we have that with $SE = E \otimes K[n]$, S is a Serre functor on $\mathcal{P}_{T^{0,1}}$.

5.2.5. *The Higgs Lie algebroid.* Again, let X be an n-dimensional complex manifold. We define a new Lie algebroid as follows.

$$\mathfrak{a} = T_{\mathbb{C}}X = T^{0,1} \oplus T^{1,0} \xrightarrow{p''} T_{\mathbb{C}}X$$

where p'' is the projection of the complexified tangent bundle onto $T^{0,1}X$. Let p' be the projection onto $T^{1,0}X$. We need to adjust the bracket by

$$\{X' + X'', Y' + Y''\} = [X'', Y''] + p'([X'', Y'] + [X', Y'']).$$

for $X', Y' \in \Gamma(T^{1,0})$ and $X'', Y'' \in \Gamma(T^{0,1})$ and where the square brackets denote the usual bracket of vector fields.

Proposition 5.11. (1) *$\mathfrak{a}$ is an elliptic Lie algebroid.*

(2) *A module over $\mathfrak{a}$ is comprised of the following data*: (E, Φ) *where E is a holomorphic vector bundle and Φ is a holomorphic section of* $\mathrm{Hom}(E, E \otimes T^{1,0\vee}X)$ *and satisfies the integrability condition* $\Phi \wedge \Phi = 0$, *that is*, (E, Φ) *is a Higgs bundle in the sense of Hitchin* [14], *and Simpson* [26].

(3) *The dualizing module $\mathcal{D}_{\mathfrak{a}}$ is the trivial one-dimensional vector bundle with the Higgs field* $\Phi = 0$.

Proof. That $\{\cdot,\cdot\}$ satisfies Jacobi is a straightforward calculation that only uses the integrability of the complex structure, that is, that $T^{0,1}$ and $T^{1,0}$ are both closed under bracket. To check the algebroid condition we calculate

$$\begin{aligned}
(5.7)\quad & \{X, fY\} \\
&= \{X' + X'', fY' + fY''\} = [X'' + fY''] + p'([X', fY''] + [X'', fY']) \\
&= f[X'', Y''] + X''(f)Y'' + p'(f[X', Y''] + X'(f)Y'' + f[X'', Y'] + X''(f)Y') \\
&= f\big([X'', Y''] + p'([X', Y''] + [X'', Y'])\big) + X''(f)(Y' + Y'') + p'(X'(f)Y'') \\
&= f\{X, Y\} + p''(X)(f)Y.
\end{aligned}$$

To show it is an elliptic Lie algebroid, let $\xi \in TX^\vee$. Since its image in $T^\vee_{\mathbb{C}}X$ is real it can be written as $e + \bar e$ for $e \in T^{0,1\vee}$. The projection to $T^{0,1\vee}$ is simply e and thus $\rho^\vee$ is injective from $TX^\vee \to \mathfrak{a}^\vee$.

For the statement about modules, suppose $(E, \mathbb{E})$ is a module over $\mathfrak{a}$. Then we have the decomposition

$$\mathbb{E}\colon \Gamma(E) \to \Gamma\big(E \otimes (T^{1,0\vee} \oplus T^{0,1\vee})\big) \cong \Gamma(E \otimes T^{1,0\vee}) \oplus \Gamma(E \otimes T^{0,1\vee})$$

in which $\mathbb{E}$ decomposes as $\mathbb{E} = \mathbb{E}' \oplus \mathbb{E}''$. The condition of being an $\mathfrak{a}$ connection means that $\mathbb{E}''$ satisfies Leibniz with respect to the $\bar{\partial}$-operator and $\mathbb{E}'$ is linear over the functions and thus $\Phi = \mathbb{E}' \colon E \to E \otimes T^{1,0\vee}$. The flatness condition $\nabla^2 = 0$ implies

(1) $\mathbb{E}''^2 = 0$ and thus defines a holomorphic structure on E,
(2) $\mathbb{E}'' \circ \Phi + \Phi \circ \mathbb{E}'' = 0$ and so Φ is a holomorphic section,
(3) $\Phi \wedge \Phi = 0$.

The statement about the dualizing module is also clear. The duality theorem in this case is due to Simpson, [26]. □

5.2.6. *Generalized Higgs algebroids.* The example above is a special case of a general construction. Let $\rho \colon \mathfrak{a} \to T_{\mathbb{C}}X$ be a Lie algebroid and $(E, \mathbb{E})$ a module over $\mathfrak{a}$. Then set $\mathfrak{a}_E = \mathfrak{a} \oplus E$ with the anchor map being the composition $\mathfrak{a} \oplus E \to \mathfrak{a} \xrightarrow{\rho} T_{\mathbb{C}}X$. Define the bracket as

$$[X_1 + e_1, X_2 + e_2]_E = [X, Y] + \mathbb{E}_{X_1} e_2 - \mathbb{E}_{X_2} e_1.$$

Proposition 5.12. (1) *$\mathfrak{a}_E$ is a Lie algebroid.*
(2) *If $\mathfrak{a}$ is elliptic, then $\mathfrak{a}_E$ is elliptic as well.*
(3) *A module $(H, \mathbb{H} = \mathbb{H}_0 + \Phi)$ over $\mathfrak{a}_E$ consists of a triple $(H, \mathbb{H}_0, \Phi)$ where $(H, \mathbb{H}_0)$ is an $\mathfrak{a}$-module and $\Phi \colon H \to H \otimes \mathfrak{a}^\vee$ satisfies $[\mathbb{H}_0, \Phi] = 0$ (i.e., Φ is a morphism of $\mathfrak{a}$-modules) and $\Phi \wedge \Phi = 0$.*

Proof. All of these statements follow as in the previous example. □

We call such a triple $(H, \mathbb{H}_0, \Phi)$ a Higgs bundle with coefficients in E.

5.2.7. *The generalized complex Lie algebroid.* Recall from [15] and [13] that an almost generalized complex structure on a manifold X is defined by a subbundle

$$E \subseteq (TX \oplus T^\vee X)_{\mathbb{C}}$$

satisfying that E is a maximal isotropic complex subbundle $E \subset (TX \oplus TX^\vee)_{\mathbb{C}}$ such that $E \cap \overline{E} = \{0\}$. The isotropic condition is with respect to the bilinear form

$$\langle X + \xi, Y + \eta \rangle = \tfrac{1}{2}\big(\xi(Y) + \eta(x)\big)$$

The almost generalized complex structure E is integrable and E is called a generalized complex structure if the sections of E, $\Gamma(E)$, are closed under the Courant bracket. The Courant bracket is a skew-symmetric bracket defined on smooth sections of $(TX \oplus TX^\vee)_{\mathbb{C}}$, given by

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - L_Y \xi - \tfrac{1}{2} d(i_X \eta - i_Y \xi),$$

where $X + \xi, Y + \eta \in \Gamma(TX \oplus TX^\vee)_{\mathbb{C}}$. It is shown in [15] and [13] how symplectic and complex manifolds are examples of generalized complex manifolds.

In the case of a generalized complex structure, the projection map $\rho \colon E \to T_{\mathbb{C}}X$ defines a Lie algebroid, the Lie algebra structure on the sections of E being the Courant bracket. Note that the Courant bracket on the full space $(TX \oplus TX^\vee) \otimes \mathbb{C}$ does not satisfy Jacobi.

Proposition 5.13. *E is an elliptic Lie algebroid.*

Proof. That it is a Lie algebroid is a straightforward calculation, as in [13]. That it is elliptic follows just as in the case of the Higgs Lie algebroid. □

In this case, Gualitieri [13] calls cohesive modules generalized holomorphic vector bundles. There is therefore a duality theorem in this context. In general it cannot be made any more explicit than the general duality theorem (Theorem 5.10). On the other hand, in the special case where the generalized complex manifold is a complex manifold X, $E = T^{0,1}X \oplus T^{1,0\vee}X$, [13]. Then we have

Proposition 5.14. *Let E be the algebroid coming from the generalized complex structure defined by an honest complex structure as defined above. Then*

(1) *a module over E consists of the following data: (E, Φ) where E is a holomorphic vector bundle on X, and $\Phi \in \mathrm{Hom}(E, E \otimes T^{1,0})$ is a holomorphic section and satisfies $\Phi \wedge \Phi = 0$.*
(2) *The dualizing module $\mathcal{D}_E$ is $(K^{\otimes 2}, 0)$, the square of the canonical bundle with the zero Higgs field 0.*

Proof. The proof is the same as for the Higgs algebroid. □

5.3. Noncommutative tori.

5.3.1. *Real noncommutative tori.* We now describe noncommutative tori. We will describe them in terms of twisted group algebras. Let V be a real vector space, and $\Lambda \subset V$ a lattice subgroup. The we can form the group ring $\mathscr{S}^*(\Lambda)$, the Schwartz space of complex valued functions on Λ which decrease faster than any polynomial. Let $B \in \Lambda^2 V^\vee$, and form the biadditive, antisymmetric group cocycle $\sigma \colon \Lambda \times \Lambda \to \mathrm{U}(1)$ by

$$\sigma(\lambda_1, \lambda_2) = \mathrm{e}^{2\pi\imath B(\lambda_1,\lambda_2)}$$

In our computations, we will often implicitly make use of the fact that σ is biadditive and antisymmetric. Now we can form the twisted group algebra $\mathcal{A}(\Lambda; \sigma)$ consisting of the same space of functions as $\mathscr{S}^*(\Lambda)$ but where the multiplication is defined by

$$[\lambda_1] \circ [\lambda_2] = \sigma(\lambda_1, \lambda_2)[\lambda_1 + \lambda_2]$$

This is a $*$-algebra where $f^*(\lambda) = \overline{f(\lambda^{-1})}$. This is one of the standard ways to describe the (smooth version) of the noncommutative torus. Given $\xi \in V^\vee$ it is easy to check that

$$\xi(f)(\lambda) = 2\pi\imath\langle \xi, \lambda\rangle f(\lambda) \tag{5.8}$$

defines a derivation on $\mathcal{A}(\Lambda; \sigma)$. Note that the derivation ξ is "real" in the sense that $\xi(f^*) = -\xi(f)$. Finally define a (de Rham) dga A by

$$\mathcal{A}^\bullet(\Lambda; \sigma) = \mathcal{A}(\Lambda; \sigma) \otimes \Lambda^\bullet V_{\mathbb{C}}$$

where $V_{\mathbb{C}} = V \otimes \mathbb{C}$ and the differential d is defined on functions $\phi \in \mathcal{A}(\Lambda; \sigma)$ by

$$\langle df, \xi\rangle = \xi(f)$$

for $\xi \in V_{\mathbb{C}}^\vee$. In other words, for $\lambda \in \Lambda$ one has $d\lambda = 2\pi\imath\lambda \otimes D(\lambda)$ where $D(\lambda)$ denotes λ as an element of $\Lambda^1 V$. Extend d to the rest of $\mathcal{A}^\bullet(\Lambda; \sigma)$ by Leibniz. Note that $d^2 = 0$.

Remark 5.15. We just want to point out that V appears as the "cotangent" space. This is a manifestation of the fact that there is a duality going on. That is, in the case that $\sigma = 1$, we have that the dga $\mathsf{A} = (\mathcal{A}^\bullet(\Lambda; \sigma), d, 0)$ is naturally isomorphic to $(\mathcal{A}^\bullet(V^\vee/\Lambda^\vee), d, 0)$, the de Rham algebra of the dual torus, and $T_0^\vee(V^\vee/\Lambda^\vee)$ is naturally isomorphic to V. See Proposition 5.16 for the complex version of this.

5.3.2. *Complex noncommutative tori.* We are most interested in the case where our torus has a complex structure and in defining the analogue of the Dolbeault DGA for a noncommutative complex torus. So now let V will be a vector space with a complex structure $J\colon V \to V$, $J^2 = -\mathbb{1}$. Let g be the complex dimension of V. Set $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Then $J \otimes 1\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ still squares to $-\mathbb{1}$ and so $V_{\mathbb{C}}$ decomposes into $\imath$ and $-\imath$ eigenspaces, $V_{1,0} \oplus V_{0,1}$. The dual $V_{\mathbb{C}}^{\vee}$ also decomposes as $V_{\mathbb{C}}^{\vee} = V^{1,0} \oplus V^{0,1}$. Let $D'\colon V_{\mathbb{C}} \otimes \mathbb{C} \to V_{1,0}$ and $D''\colon V \otimes \mathbb{C} \to V_{0,1}$ denote the corresponding projections. Explicitly

$$D' = \frac{J \otimes 1 + 1 \otimes \imath}{2\imath}$$

and

$$D'' = \frac{-J \otimes 1 + 1 \otimes \imath}{2\imath}$$

and $D = D' + D''$ where D denotes the identity. This also establishes a decomposition

$$\Lambda^k V_{\mathbb{C}} = \bigotimes_{p+q=\bullet} \Lambda^{p,q} V$$

where $\Lambda^{p,q}V = \Lambda^p V_{1,0} \otimes \Lambda^q V_{0,1}$. Complex conjugation on $V_{\mathbb{C}}$ defines an involution and identifies V with the $v \in V_{\mathbb{C}}$ such that $\bar{v} = v$.

Now let $X = V/\Lambda$, a complex torus of dimension g, and $X^{\vee} = \overline{V}^{\vee}/\Lambda^{\vee}$ its dual torus. Let $B \in \Lambda^2 V^{\vee}$ be a real (constant) two form on X. Then B will decompose into parts

$$B = B^{2,0} + B^{1,1} + B^{0,2}$$

where $B^{p,q} \in \Lambda^{p,q}V^{\vee}$, $B^{0,2} = \overline{B^{2,0}}$ and $\overline{B^{1,1}} = B^{1,1}$. Now $B^{0,2} \in \Lambda^2 V^{0,1} \cong H^{0,2}(X)$. Then it also represents a class $\Pi \in \Lambda^2 V^{0,1} \cong H^0(X^{\vee}; \Lambda^2 T_{1,0}X)$. Let $\sigma\colon \Lambda \wedge \Lambda \to \mathrm{U}(1)$ denote the group 2-cocycle given by

$$\sigma(\lambda_1, \lambda_2) = \mathrm{e}^{2\pi\imath B(\lambda_1,\lambda_2)}.$$

and form as above $\mathcal{A}(\Lambda;\sigma)$, the twisted group algebra based on rapidly decreasing functions. Define the Dolbeault dga A $\mathcal{A}^{0,\bullet}(\Lambda;\sigma)$ to be

$$\mathcal{A}(\Lambda;\sigma) \otimes \Lambda^{\bullet} V_{1,0}$$

where for $\lambda \in \mathcal{A}(\Lambda;\sigma)$ we define

$$\bar{\partial}\lambda = 2\pi\imath\lambda \otimes D'(\lambda) \in \mathcal{A}(\Lambda;\sigma) \otimes V_{1,0}$$

We can then extend $\bar{\partial}$ to the rest of $\mathcal{A}^{0,\bullet}(\Lambda;\sigma)$ by the Leibniz rule. Let us reiterate the remarks above. Even though we are defining the $\bar{\partial}$ operator, we are using the $(1,0)$ component of $V_{\mathbb{C}}$. This is because of duality. In the case of the trivial cocycle σ, this definition is meant to reconstruct the Dolbeault algebra on $X^{\vee}$. In this case, $\mathcal{A}^{0,\bullet}(X^{\vee}) \cong \mathcal{A}^{\bullet}(X^{\vee}) \otimes \Lambda^{\bullet} T_0^{0,1} X^{\vee}$. But

$$T_0^{0,1} X^{\vee} = \overline{\overline{V}^{\vee}}^{\vee} \cong V_{1,0}.$$

To check the reasonableness of this definition we have

Proposition 5.16. *If $\sigma = 1$ is the trivial cocycle, then the dga $(\mathcal{A}^{0,\bullet}(\Lambda;\sigma), \bar{\partial})$ is isomorphic to the Dolbeault dga $(\mathcal{A}^{0,\bullet}(X^{\vee}), \bar{\partial})$.*

We now show that the dga $\mathsf{A} = (\mathcal{A}^{0,\bullet}(\Lambda;\sigma), \bar\partial, 0)$ is elliptic. Let

$$\tau \colon \mathcal{A}(\Lambda;\sigma) \to \mathbb{C}$$

denote the continuous $\mathbb{C}$-linear functional defined by $\tau(\sum a_\lambda \lambda) = a_0$. This is a trace, that is, $\tau(ab) = \tau(ba)$ and in many cases it is the unique normalized trace on $\mathcal{A}(\Lambda;\sigma)$. (It is unique when σ is "irrational" enough.) We note the following lemma whose proof is straightforward.

Lemma 5.17. *For any $\xi \in V^\vee$, the derivation ξ defined by* (5.8) *has the property*

$$\tau\big(\xi(f)\big) = 0 \tag{5.9}$$

for all $f \in \mathcal{A}(\Lambda;\sigma)$.

Equip $V_\mathbb{C}$ with a Hermitian inner product $\langle \cdot,\cdot\rangle \colon V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$. Let $v_1, \dots, v_g$ and $v^1, \dots, v^g$ be dual orthonormal bases of $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$ respectively. Equip $V_\mathbb{C}$ with a Hermitian structure. Then $V_{1,0}$ and $V_{0,1}$ inherit Hermitian structures as well. Let v_i' and v_i'' $(i = 1, \dots, g)$ be orthonormal bases of $V_{1,0}$ and $V_{0,1}$ respectively. We let $D = \mathcal{A}(\Lambda;\sigma) \otimes \Lambda^g V_{0,1}$ with

$$\mathbb{D} \colon D \to D \otimes_{\mathcal{A}(\Lambda;\sigma)} \mathcal{A}^{0,1}(\Lambda;\sigma)$$

defined by

$$\mathbb{D}(f \otimes v''_{\{1,\dots,g\}}) = \bar\partial(f) \otimes v''_{\{1,\dots,g\}}$$

Recall that $V_{1,0}$ is the *anti-holomorphic cotangent* space of the noncommutative complex torus and $V_{0,1}$ is the *holomorphic cotangent space.*

Define $\bar{*} \colon \mathcal{A}^{0,k}(\Lambda;\sigma) \to D \otimes_{\mathcal{A}(\Lambda;\sigma)} \mathcal{A}^{0,g-k}$ by

$$\bar{*}(f \otimes v_I') = v''_{\{1,\dots,g\}} \otimes f^* \otimes v'_{I^c}$$

Now note that $D \otimes_{\mathcal{A}(\Lambda;\sigma)} \mathcal{A}^{0,g}(\Lambda;\sigma) \cong \mathcal{A}^{2g}(\Lambda;\sigma)$ and so we define

$$\int \colon D \otimes_{\mathcal{A}(\Lambda;\sigma)} \mathcal{A}^{0,g}(\Lambda;\sigma) \to \mathbb{C}$$

Define

$$\int a_\lambda \lambda \otimes v_I' \wedge v_I'' = \begin{cases} 0 & \text{if } I \neq \{1,\dots,g\} \\ \tau(a_\lambda) & \text{if } I = \{1,\dots,g\} \end{cases} \tag{5.10}$$

The following lemma is trivial to verify.

Lemma 5.18. *For all $x \in D \otimes_{\mathcal{A}(\Lambda;\sigma)} \mathcal{A}^{0,g}(\Lambda;\sigma)$ we have*

$$\int \mathbb{D}(x) = 0.$$

Theorem 5.19 (Serre duality for complex noncommutative tori).

(1) *The dga $\mathsf{A} = (\mathcal{A}^{0,\bullet}(\Lambda;\sigma), \bar\partial, 0$ is elliptic with dualizing module $(D, \mathbb{D}), \bar{*}, \int)$.*
(2) *On the category $\mathcal{P}_{\mathcal{A}^{0,\bullet}(\Lambda;\sigma)}$ there is a Serre functor defined by*

$$(E^\bullet, \mathbb{E}) \mapsto (E \otimes_{\mathcal{A}(\Lambda;\sigma)} D, \mathbb{E} \,\#\, \mathbb{D})$$

(3) *In the case when $\sigma = 1$, the Serre functor coincides with the usual Serre functor on $X^\vee$ using the isomorphism described in Proposition* 5.16.

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