Some Remarks Concerning the Baum-Connes Conjecture

JONATHAN BLOCK
University of Pennsylvania

P. Baum and A. Connes have made a deep conjecture about the calculation of the K-theory of certain types of C^*-algebras [1, 2]. In particular, for a discrete group Γ they have conjectured the calculation of $K_*(C_r^*(\Gamma))$, the K-theory of the reduced C^*-algebra of Γ. So far, there is quite little evidence for this conjecture. For example, there is not a single property T group for which it is known to be true. In this note we show that, in some sense, the homological algebra of their conjecture is correct. In many cases, the periodic cyclic homology of certain dense subalgebras suggests what the K-theory should be. In the case of a discrete group Γ, the periodic cyclic homology of the algebraic group algebra $\mathbb{C}\Gamma$ is quite easy to calculate. Let $\langle\Gamma\rangle_f$ denote the set of conjugacy classes of elements of finite order, and let $\langle\Gamma\rangle_i$ denote the set of conjugacy classes of infinite order. For $\gamma \in \Gamma$, let Γ_γ denote the centralizer of γ in Γ. Let Γ_γ/γ be the quotient of Γ_γ by the cyclic subgroup generated by γ. If γ is of finite order, then $H_i(B\Gamma_\gamma/\gamma;\mathbb{C}) \cong H_i(B\Gamma;\mathbb{C})$. (Note that for a discrete group $B\Gamma = K(\Gamma,1)$; we will freely use both notations.) If γ is of infinite order, then

$$1 \to \{\gamma\} \to \Gamma_\gamma \to \Gamma_\gamma/\gamma \to 1$$

induces an S^1 bundle:

$$\begin{array}{ccc} S^1 & \to & B\Gamma_\gamma \\ & & \downarrow \\ & & B\Gamma_\gamma/\gamma \end{array}$$

and so there is a Gysin map $s : H_i(B\Gamma_\gamma/\gamma;\mathbb{C}) \to H_{i-2}(B\Gamma_\gamma/\gamma;\mathbb{C})$. Let

$$\widehat{\widehat{K}}_i(B\Gamma_\gamma/\gamma) = \lim_{\leftarrow_s} H_i(B\Gamma_\gamma/\gamma;\mathbb{C}) \quad \text{for } i = 0, 1,$$

and

$$\widehat{K}_i(B\Gamma_\gamma) = \bigoplus_{j\equiv i(2)} H_j(B\Gamma_\gamma;\mathbb{C})\,.$$

Communications on Pure and Applied Mathematics, Vol. L, 0813–0820 (1997)
 CCC 0010–3640/97/090813-08

THEOREM 0.1 (Burghelea [4]) *For a discrete group* Γ

$$HP_i(\mathbb{C}\Gamma) \cong \bigoplus_{\gamma \in \langle\Gamma\rangle_f} \widehat{K}_i(B\Gamma_\gamma) \oplus \bigoplus_{\gamma \in \langle\Gamma\rangle_i} \widehat{\widehat{K}}_i(B\Gamma_\gamma/\gamma)$$

.

The conjecture of Baum and Connes predicts terms in the K-theory of $C_r^*\Gamma$ corresponding to the first summand but not the second.

In this note we show that there exist groups Γ for which the second summand above is nonzero, yet for which the Baum-Connes conjecture is correct as stated. Thus we believe the conjecture is correct as stated, or at least if it fails, it fails for analytical reasons, not algebraic ones. In fact, we construct groups of any given homological type for which the Baum-Connes conjecture is true. In particular, Kan and Thurston [7] prove that given any path-connected space X there exists a space TX and a map $t : TX \to X$ such that

1. TX is a $K(\pi, 1)$ and
2. $t : TX \to X$ is a homology equivalence.

Second, Pimsner's result on the K-theory of groups acting on trees [8] essentially proves that the Baum-Connes conjecture holds for any group in a rather large class of groups, a class large enough to contain all homological types.

1 Pimsner's Theorem and the Baum-Connes Conjecture

In this section we recall the Baum-Connes conjecture and show that Pimsner's theorem implies that if a group Γ acts on a tree in such a way that the BC conjecture holds for all isotropy groups, then the BC conjecture is true for Γ itself.

Throughout this section let G be a (not necessarily discrete) locally compact, second countable Hausdorff topological group. Let X be a proper G-space in the sense of [2]. If $G\backslash X$ is compact, then $KK_G^*(C_0(X), \mathbb{C})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group defined in terms of homotopy classes of G-equivariant abstract elliptic operators on X. Recall that an even G-equivariant abstract elliptic operator on X is a pair (H, F) where

1. H is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space with a unitary G-representation and a G-equivariant (even) representation of $C_0(X)$.

2. F is a bounded G-equivariant operator on H that is odd with respect to the grading and such that

   (a) $[f, F]$ is compact $\forall f \in C_0(X)$, and

   (b) $\exists$ a G-equivariant (odd) bounded operator $Q : H \to H$ so that $f(FQ-1)$ and $f(QF-1)$ are compact $\forall f \in C_0(X)$.

An odd abstract elliptic operator is the same type of object but without any sort of grading.

If (H, F) is an abstract elliptic G-operator on X, its G-index is defined as follows [2]:

$$\mathrm{Ind}(H, F) \in K_i(C_r^*(G))$$

by defining $\mathcal{H}_0 = C_0(X)H$. $\mathcal{H}_0$ is a module for $C_c(G) \subseteq C_r^*(G)$ and has the $C_c(G)$-valued inner product

$$\langle \xi_1, \xi_2 \rangle (g) = (\xi_1, g \cdot \xi_2)$$

Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ in the Hilbert module norm. F induces an operator $\mathcal{F}$ on H that is $C_r^*(G)$ Fredholm. This Fredholm module over $C_r^*(G)$ defines the index above.

DEFINITION 1.1 If X is any proper G-space, define the *equivariant K-homology with G-compact supports* by

$$K_j^{G,c}(X) = \varinjlim_{\{Y \subseteq X | Y\ G\text{-compact}\}} KK_j^G(C_0(Y), \mathbb{C}) .$$

Define $\mu : K^{j\,G,c}(X) \to K_j(C_r^*(G))$ for X G-compact by taking $\mu(H, F) = \mathrm{Ind}(H, F)$ and then noting that μ is compatible with the direct limit defining $K_j^{G,c}(X)$ where X is not necessarily G-compact. For any group G there is a space, called $\underline{EG}$ in [2], that is universal for proper actions.

CONJECTURE BC (Baum-Connes [2]) For any group G as above,

$$\mu : K_j^{G,c}(\underline{EG}) \to K_j(C_r^*(G))$$

is an isomorphism.

Now recall the following theorem of Pimsner:

THEOREM 1.2 (Pimsner [8]) *Let G be a group acting on an oriented tree T. Let Σ be the associated graph of groups, Σ_0 its vertices, and Σ_1 its edges.*

There is a long exact sequence

$$\cdots \to \bigoplus_{e\in\Sigma_1} K_j(C_r^*(G_e)) \to \bigoplus_{v\in\Sigma_0} K_j(C_r^*(G_v)) \to K_j(C_r^*(G)) \to \cdots$$

From the preceding theorem we draw the following corollary:

COROLLARY 1.3 *If G is a group acting on an oriented tree T and* BC *is true for the isotropy groups of all vertices and edges, then* BC *is true for G.*

PROOF: Let $\underline{EG}$ be the universal space for proper G-spaces. Clearly $\underline{EG} \times T$ is again a model for $\underline{EG}$ according to the recognition principle for $\underline{EG}$ [2, proposition 1.8]. Consider the closed subspace $\underline{EG} \times T_0 \subseteq \underline{EG} \times T$. Then it is easy to see that

$$K_j^{G,c}(\underline{EG} \times T_0) \cong \bigoplus_{v\in\Sigma_0} K_j^{G_v,c}(\underline{EG}_v)$$

using the fact that $\underline{EG}$ is also a model for $\underline{EH}$ for any subgroup $H \subseteq G$. Similarly,

$$K_j^{G,c}(\underline{EG} \times T, \underline{EG} \times T_o) \cong \bigoplus_{e\in\Sigma_1} K_{j-1}^{G_e,c}(\underline{EG}_e)\,.$$

One checks that the maps μ are compatible and that one has the following commutative diagram with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & K_j^{G,c}(\underline{EG} \times T_0) \arrow[r] \arrow[d, "\cong"] & K_j^{G,c}(\underline{EG} \times T) \arrow[r] \arrow[d, "\cong"] & K_j^{G,c}(\underline{EG} \times T, \underline{EG} \times T_o) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{v\in\Sigma_0} K_j^{G_v,c}(\underline{EG}_v) \arrow[r] \arrow[d, "\mu"] & K^{j\,G,c}(\underline{EG}) \arrow[r] \arrow[d, "\mu"] & \bigoplus_{e\in\Sigma_1} K_{j-1}^{G_e,c}(\underline{EG}_e) \arrow[r] \arrow[d, "\mu"] & \cdots \\
\cdots \arrow[r] & \bigoplus_{v\in\Sigma_0} K_j(C_r^*(G_v)) \arrow[r] & K_j(C_r^*(G)) \arrow[r] & \bigoplus_{e\in\Sigma_1} K_{j-1}(C_r^*(G_e)) \arrow[r] & \cdots
\end{tikzcd}$$

and so by the 5-lemma we are done. ∎

Also, the lemma below follows from the commutativity of K-theory with direct limits (of directed systems of injections of groups).

LEMMA 1.4 *Let $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \cdots$ be an increasing sequence of groups for which* BC *is true. Then* BC *is true for $\bigcup_i \Gamma_i$.*

2 A Class of Groups for Which BC Holds

We now construct a class of countable discrete groups for which BC is true. Let $\mathcal{C}_0$ be the class of groups acting on trees with trivial isotropy groups for all edges and all vertices. Thus $\mathcal{C}_0$ is the class of (not necessarily finitely generated) free groups. Let $\mathcal{C}_1$ be the class of groups that act on trees so that all isotropy groups are in $\mathcal{C}_0$. Inductively, let $\mathcal{C}_n$ be the class of groups acting on trees with isotropy groups from the class $\mathcal{C}_{n-1}$, and finally let $\mathcal{C}$ be the union of all the $\mathcal{C}_n$'s, which we call the Cappell class; this is essentially the class of groups for which Cappell calculates the L-groups in [5]. Equivalently $\mathcal{C}_n$ consists of the class of fundamental groups of graphs of groups where the edge and vertex groups lie in $\mathcal{C}_{n-1}$. Recall that a group Γ is called locally free if the group generated by any finite set of elements is free, or equivalently, Γ is the union of directed system of free groups. Note that $\mathcal{C}_1$ already contains the class of locally free groups since if $\Gamma = \bigcup_i \Gamma_i$ where the Γ_i are free, then Γ is the fundamental group of the graph of groups

$$\begin{array}{ccccccccc} \circ & \text{————} & \circ & \text{————} & \circ & \text{————} & \circ & \cdots \\ \Gamma_1 & \Gamma_1 & \Gamma_2 & \Gamma_2 & \Gamma_3 & \Gamma_3 & \Gamma_4 & \end{array}$$

PROPOSITION 2.1 *If* $\Gamma \in \mathcal{C}$, *then* BC *is true for* Γ.

PROOF: This follows from Corollary 1.3 and Lemma 1.4. ∎

LEMMA 2.2 1. *If* $\Gamma_1 \subseteq \Gamma_2 \in C$, *then* $\Gamma_1 \in \mathcal{C}$.

2. *If* Γ_1 *and* $\Gamma_2 \in \mathcal{C}$, *then* $\Gamma_1 \times \Gamma_2 \in \mathcal{C}$.

3. *Let* $\Gamma \in \mathcal{C}$ *and suppose* $1 \to \mathbb{Z} \to Q \to \Gamma \to 1$ *is a central extension by* $\mathbb{Z}$. *Then* $Q \in \mathcal{C}$.

PROOF: We will prove (3); the proofs of (1) and (2) are similar. If $\Gamma \in \mathcal{C}_0$, that is, Γ is free, then $Q \cong \Gamma \times Z$, which is in $\mathcal{C}_1$. Now assume $\Gamma \in \mathcal{C}_n$. Then $\Gamma = \pi_1(\mathfrak{G})$ where $\mathfrak{G}$ is a graph of groups such that all vertex and edge groups lie in $\mathcal{C}_{n-1}$, and $Q = \pi_1(\mathfrak{G}')$ where $\mathfrak{G}'$ is a graph of groups where the edge and vertex groups are central extensions by $\mathbb{Z}$ by elements of $\mathcal{C}_{n-1}$. By the induction hypothesis, these vertex and edge groups lie in $\mathcal{C}_n$. Hence $Q \in \mathcal{C}_{n+1}$. ∎

Now we show that the Kan-Thurston construction can be carried out in category $\mathcal{C}$. First, start with $\mathbb{F}_2 = \langle a, b \rangle$. Map $\varphi : \mathbb{F}_2 \to \mathbb{F}_2$ by

$$\begin{aligned} a &\longmapsto [a, b] \\ b &\longmapsto [b^{-1}, a] \, . \end{aligned}$$

Clearly, $H_i(\mathbb{F}_2, \mathbb{Z}) \to H_i(\mathbb{F}_2, \mathbb{Z})$ is the zero map. Also, φ is injective. Let $A_0 \equiv \lim(\mathbb{F}_2, \varphi)$. Now we use the following lemma from [3]:

LEMMA 2.3 *Let $F \subseteq G$, G an acyclic group. Then $G *_F G$ embeds in $(A \times F) *_F G$. If A is acyclic, then $(A \times F) *_F G$ is also.*

THEOREM 2.4 *For any 0-connected simplicial complex X, there is an aspherical simplicial complex TX and a map $t : TX \to X$ that is a homology equivalence and $TX = K(\Gamma, 1)$ where $\Gamma \in \mathcal{C}$ and $t : \Gamma \to \pi_1(X)$ is a surjection.*

PROOF: The idea of the proof is clear enough. One builds TX inductively a simplex at a time. Suppose X is constructed from Y by attaching a simplex σ to $\partial\sigma \subseteq Y$. Then by the induction hypothesis $T(\partial\sigma)$ exists, satisfying the theorem. Now embed $\pi_1(T(\partial\sigma))$ into an acyclic group C and form $T(\sigma)$ by attaching $K(C, 1)$ to $K(\pi_1(\partial\sigma), 1)$ and form TX by attaching $T(\sigma)$ to $T(\partial\sigma)$. Thus $\pi_1(TX) = \pi_1(TY) *_{\pi_1(T(\partial\sigma))} \pi_1(T(\sigma))$, and so $\pi_1(TX) \in \mathcal{C}$ if $C \in \mathcal{C}$. To handle this, we follow Maunder [7] and construct C along with TX in the induction. We include the details for completeness and to make sure that the group in the end is actually in $\mathcal{C}$.

We prove the theorem first for finite simplicial complexes X by induction, and the theorem is completed by simply writing any simplicial complex as the direct limit of its finite subcomplexes. We will construct for every finite X a simplicial pair and a map of pairs $t : (UX, TX) \to (CX, X)$ where CX is the cone on X satisfying the following inductive hypothesis:

Hypothesis. For each subcomplex $Z \subseteq X$ we have

1. $t : t^{-1}(Z) \to Z$ is a homology equivalence and $t : \pi_1(t^{-1}(Z)) \to \pi_1(Z)$ is a surjection, and
2. $t : t^{-1}(CZ) \to CZ$ is a homology equivalence and $t : \pi_1(t^{-1}(CZ)) \to \pi_1(CZ)$ is a surjection, and
3. $t : \pi_1(t^{-1}(CZ)) \to \pi_1(UX)$, $t : \pi_1(t^{-1}Z) \to \pi_1(TX)$, $t : \pi_1(TX) \to \pi_1(UX)$ are all injective.

We start the induction by letting $t : (UX, TX) \to (CX, X)$ be the identity if X is 0-dimensional. Now we induct on the number of simplices of dimension ≥ 1. Now suppose that the hypothesis is true for any simplicial complex X with $\leq N - 1$ simplices of dimension ≥ 1. Suppose to X we glue a simplex σ to $\partial\sigma \subseteq X$ to form Y. Now let TY be TX with a copy of $t^{-1}(C\partial\sigma) \subseteq UX$

attached along $t^{-1}(\partial\sigma)$ and extend t to TY by sending $t^{-1}(C\partial\sigma)$ to $C\partial\sigma$, via t, identified with σ (by sending the cone point to the barycenter of σ).

To construct UY, consider S to be the union of the original copy of $t^{-1}(C\partial\sigma) \subseteq UX$ to the new copy just glued. Then S is a $K(\pi, 1)$ where $\pi = G *_F G$, $G = \pi_1(t^{-1}(C\partial\sigma))$, and $F = \pi_1(t^{-1}\partial\sigma)$. Then $\pi \subseteq C = (A_0 \times F) *_F G$. Since G and A_0 are acyclic, so is C by the lemma above. Let $f : X \to K(C, 1)$ be the map realizing the injection of groups. Then form $UY = (UX \cup_{t^{-1}(C\partial\sigma)} S) \cup_S \mathrm{Cyl}(f)$ where $\mathrm{Cyl}(f)$ denotes the mapping cylinder of f. Extend $t : UY \to Y$ by $K(C, 1)$ to the barycenter of $C\sigma$. Then

1. $t : TY \to Y$ and $t : UY \to CY$ are homology equivalences. (This follows from Mayer-Vietoris).
2. TY and UY are aspherical. This follows from the fact that all the maps of fundamental groups from subcomplexes are injective, allowing one to lift the picture to the universal cover and then using Mayer-Vietrois (see [7]).
3. The rest of the inductive hypotheses are easy to check.

In the end, we've constructed TX purely from amalgamated free product starting from the group $A_0 \in \mathcal{C}$ and direct limits. ∎

We now recall an example from Burghelea [4]. Let $X = \mathbb{C}P^\infty$ and E the universal circle bundle over X. Then let $\widehat{\Gamma} = TX$. So in $H^2(TX, \mathbb{Z}) = \mathbb{Z}$ there is a generator corresponding to E. Let $1 \to \mathbb{Z} \to \Gamma \to \widehat{\Gamma} \to 1$ be the central extension corresponding to this generator. The $K(\Gamma, 1)$ is homology equivalent to E. According to Burghelea,

$$HP_i(\mathbb{C}\Gamma) \cong \bigoplus_{\gamma\in\langle\Gamma\rangle_f} \widehat{K}_i(B\Gamma_\gamma) \oplus \bigoplus_{\gamma\in\langle\Gamma\rangle_i} \widehat{\widehat{K}}_i(B\Gamma_\gamma/\gamma)\,,$$

and in this case, $\gamma \in \mathbb{Z} \subseteq \Gamma$, the generator of $\mathbb{Z}$, will contribute the term $\widehat{\widehat{K}}_i(B\Gamma\gamma/\gamma = TX) = \widehat{\widehat{K}}_i(X) \neq 0$. On the other hand, $\widehat{\Gamma} \in \mathcal{C}$ so $\Gamma \in \mathcal{C}$ and therefore BC is true for Γ. Lest one think this might be occurring because Γ is not finitely presented, we have the following:

Amplification. There is a group $\Gamma \in \mathcal{C}$, which is finitely presented, for which the second terms above are nonzero.

This is accomplished by using the version of the Higman embedding theorem that is in [6]. This theorem says that if Γ is a recursively generated and presented group, then it can be embedded in a finitely presented one. The construction only uses free products with amalgamation and HNN extensions and thus preserves category $\mathcal{C}$. So let Γ be as constructed above. It is clear that it is recursively generated and presented (if the triangulation of $\mathbb{C}P^{\infty}$ is), and so embeds in P, a finitely presented group. So $\mathbb{Z} \subseteq \Gamma \subseteq P$ and the centralizer of $\mathbb{Z}$ in P is exactly Γ again since P is constructed from Γ out of HNN and free products with amalgamation. (One can check that the HNN extensions involved in the proof do not enlarge the centralizer of $\mathbb{Z}$.). So the same term occurs in the periodic cyclic homology of $\mathbb{C}\Gamma$.

Acknowledgment. The author would like to thank Paul Baum and Shmuel Weinberger for many helpful conversations.

Bibliography

[1] Baum, P. , and Connes, A., Chern character for discrete groups, pp. 163–232 in: *A Fete of Topology*, Academic Press, Boston, 1988.

[2] Baum, P., Connes, A., and Higson, N., Classifying space for proper actions and K-theory of group C*-Algebras, pp. 240–291 in: *C*-Algebras 1943–1993, A Fifty-Year Celebration*, Contemp. Math. 167, Amer. Math. Soc., Providence, 1994.

[3] Baumslag, G., Dyer, E., and Heller, A., The topology of discrete groups, *J. Pure Appl. Algebra* 16(1), 1980, pp. 1–47.

[4] Burghelea, D. The cyclic homology of the group rings, *Comm. Math. Helv.* 60, 1983, pp. 354–365.

[5] Cappell, S., On the homotopy invariance of higher signatures, *Inven. Math.* 33, 1976, pp. 171–179.

[6] Manin, Y., *A Course in Mathematical Logic*, Springer, Berlin, 1977.

[7] Maunder, C. R. F., A short proof of a theorem of Kan and Thurston, *Bull. London Math. Soc.* 13, 1981, pp. 325–327.

[8] Pimsner, M., K-groups of crossed products with groups acting on trees, *Inven. Math.* 86, 1986, pp. 603–634.

JONATHAN BLOCK
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
E-mail: `blockj@math.upennn.edu`

Received January 1996.