Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Mathematics 193 (2005) 18–39

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

André–Quillen cohomology and rational homotopy of function spaces

J. Block[1,*], A. Lazarev[2]

Department of Mathematics, University of Pennsylvania, 3E2A, David Rittenhouse Laboratory, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA

Received 19 August 2003; accepted 6 April 2004

Communicated by Mark Hovey
Available online 26 June 2004

Abstract

We develop a simple theory of André–Quillen cohomology for commutative differential graded algebras over a field of characteristic zero. We then relate it to the homotopy groups of function spaces and spaces of homotopy self-equivalences of rational nilpotent CW-complexes. This puts certain results of Sullivan in a more conceptual framework.

Keywords: Closed model category; Rational homotopy theory; Differential graded algebra; Derivation; Function space

1. Introduction

André–Quillen cohomology is a cohomology theory for commutative algebras originally introduced in [1,15]. It was subsequently generalized to cover simplicial algebras over operads [8], differential graded E_∞-algebras [14], and commutative S-algebras [2].

One of the purposes of the present paper is to give a simple and direct treatment of the André–Quillen cohomology in the category of commutative differential graded algebras (dga's) over a field of characteristic zero. This is done in Section 2. Our initial

* Corresponding author. Fax: +1-215-573-4063.
E-mail addresses: blockj@math.upenn.edu (J. Block), a.lazarev@bristol.ac.uk (A. Lazarev).
[1] Partially supported by the NSF Grant DMS-0204558.
[2] Partially supported by the EPSRC Grant GR/R84276/01.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2004.04.014

definition of André–Quillen cohomology of a dga A with coefficients in the differential graded (dg) module M over A is via an explicit cochain complex $C^*_{\mathrm{AQ}}(A, M)$ similar to the one introduced in [10]. We then produce various equivalent characterizations of André–Quillen cohomology, introduce the Gerstenhaber bracket on $C^*_{\mathrm{AQ}}(A, A)$ and show its homotopy invariance. In this connection we mention the recent paper [6] where analogues of some of our results were proved in the context of Hochschild cohomology.

In Section 3 we apply the developed techniques to computing the homotopy groups of function spaces. (We are dealing with *unpointed* spaces, however our machinery could be easily adapted to the pointed situation as well.) In particular, we are concerned with the group $\mathrm{hAut}(X)$ of homotopy classes of homotopy self-equivalences of a nilpotent CW-complex X. A well-known theorem of Sullivan [18] and Wilkerson [20] asserts that under suitable finiteness assumptions $\mathrm{hAut}(X)$ is an arithmetic group, that is, commensurable to the group of integer points of some algebraic group over $\mathbb{Q}$. An important step is to show that the group $\mathrm{hAut}(X_{\mathbb{Q}})$ is isomorphic to the group of $\mathbb{Q}$-points of an algebraic group. Here $X_{\mathbb{Q}}$ denotes the *rationalization* of the space X, i.e. its localization with respect to the homology theory $H_*(-, \mathbb{Q})$.

We reprove this result and identify the Lie algebra of this algebraic group. It turns out to be isomorphic to $H^0_{\mathrm{AQ}}(A^*(X), A^*(X))$, the zeroth André–Quillen cohomology of the Sullivan–de Rham algebra of X with coefficients in itself. Moreover, the Lie bracket corresponds to the Gerstenhaber bracket on $H^*_{\mathrm{AQ}}(A^*(X), A^*(X))$.

We also consider the question of computing the higher homotopy groups of a function space $F(X, Y)$ for two rational spaces X and Y. The answer is again formulated in terms of André–Quillen cohomology associated to the Sullivan–de Rham models of X and Y. This result was hinted at in [13].

There has been some previous work on the relation between spaces of automorphisms of rational spaces and derivations. It was described already in the original paper of Sullivan [18] and has been described in the paper of Schlessinger and Stasheff [17]. Tanre [19] describes models for $\mathrm{Aut}(X)$ and $\mathrm{BAut}(X)$.

The computation of the rational homotopy type of function spaces was realized for the first time by Haefliger [9].

2. André–Quillen cohomology of commutative differential graded algebras

Let $\mathcal{C}$ denote the category of commutative differential graded algebras over k, not necessarily connected. Here k is a field of characteristic 0. The differential is assumed to *raise* the degree by one. Then $\mathcal{C}$ admits a structure a closed model category as follows:

- weak equivalences are maps which induce an isomorphism on cohomology groups;
- fibrations are surjective maps;
- cofibrations are the maps which have the left lifting property with respect to acyclic fibrations.

That $\mathcal{C}$ is a closed model category is proved in the case of connected dga's in [3]. The general case is due to Hinich [11], who proved it in the still greater generality of algebras over an operad.

Let us describe the cofibrant objects in $\mathcal{C}$. First consider the operation of glueing cells to a dga (also called the Hirsch extension). Let A be a dga and V be a graded vector space. Let $f : V \to Z^*(A)$ be a linear map of degree 1 from V to the space of cocycles of A. Then define a new dga A_f whose underlying graded vector space is $A \otimes \Lambda V$. Here we denoted, following tradition, by ΛV the free graded commutative algebra on the vector space V. The differential on A is the old differential and the one on V is given by the map f. Then A_f is said to be obtained from A by glueing a (generalized) cell. Observe that A_f is a pushout of A by a free commutative algebra which justifies the name. A dga obtained from the trivial dga k is called a *cell* dga. Any cell dga is cofibrant and any cofibrant dga is a retract of a cell dga.

Now consider the category A-mod of dg modules over a dga A. This is also a closed model category where fibrations are surjective maps. Then a graded derivation of A with values in M of degree d is a map $\xi : A \to M$ of degree d which satisfies the Leibniz condition:

$$\xi(b_1 b_2) = \xi(b_1) b_2 + (-1)^{d|b_1|} b_1 \xi(b_2).$$

The set of all derivations form a complex, in fact a dg A-module $\mathrm{Der}^*(A, M)$.

Associated to a dga A is the dg module of its Kähler differentials Ω_A. It is defined in the usual manner as $\Omega_A := I/I^2$ where I is the kernel of the multiplication map $A \otimes A \to A$. It is a standard fact that there is an isomorphism of dg A-modules:

$$\mathrm{Hom}^*_A(\Omega_A, M) \cong \mathrm{Der}^*(A, M).$$

Let us now introduce the derived version of Ω_A also called the *André–Quillen homology* of A. First recall that the (homological) Hochschild complex of A with coefficients in itself is defined as the complex

$$C_*(A, A) = \{A \leftarrow A^{\otimes 2} \leftarrow \dots\}$$

with the standard bar differential. $C_*(A, A)$ is in fact itself a differential graded algebra with respect to the shuffle product, since A is graded commutative. Since A is a dga this is in fact a *bicomplex*. (The Hochschild differential lowers degree while the differential of A raises degree. Thus the total degree in the bicomplex is the difference of the two.) We will make use of the truncated version of $C_*(A, A)$ denoted by $\bar{C}_*(A, A)$. This is the same complex as $C_*(A, A)$ but starting with $A^{\otimes 2}$. Since A is commutative the complex $\bar{C}_*(A, A)$ splits off $C_*(A, A)$ as a direct summand.

Definition 2.1. The André–Quillen complex $C^{\mathrm{AQ}}_*(A, A)$ of a dga A is the quotient complex of $\bar{C}_*(A, A)$ by the subcomplex of decomposables, i.e., those elements which could be represented as shuffle products of two or more elements in $\bar{C}_*(A, A)$.

Remark 2.2. This complex (shifted) was defined by Harrison [10] in the case when A is a usual (ungraded) algebra. Its homology is also called the Harrison homology of A. It is well known that in characteristic zero case the shifted Harrison homology agrees with the André–Quillen homology defined by means of simplicial resolutions.

Theorem 2.3. *Let A be a cofibrant dga. Then there is a quasi-isomorphism of dg A-modules*:

$$\Omega_A \simeq C_*^{\mathrm{AQ}}(A, A).$$

Proof. Consider the map $f : A^{\otimes 3} \to A^{\otimes 2}$:

$$f : a \otimes b \otimes c \mapsto ab \otimes c - a \otimes bc + (-1)^{(|a|+|b|)|c|} ca \otimes b.$$

Clearly, $\operatorname{Im} f = \Omega_A$. There results a map of dg modules

$$C_*^{\mathrm{AQ}}(A, A) \to \Omega_A$$

and we want to prove that this map is a quasi-isomorphism for a cofibrant commutative dga A.

Without loss of generality, we assume that A is constructed from k by a series of Hirsch extensions. This gives A a filtration for which the associated graded algebra is simply the free commutative algebra on some set of generators with zero differential. This filtration lifts to Ω_A and $C_*^{\mathrm{AQ}}(A, A)$ so that the canonical map $C_*^{\mathrm{AQ}}(A, A) \to \Omega_A$ is a filtered map. Since it is clearly a quasi-isomorphism if A is free commutative with vanishing differential we conclude that the map is a quasi-isomorphism on the level of associated graded modules, therefore it was a quasi-isomorphism to begin with. $\square$

Now let us turn to the functor of derivations and its derived version. Let M be a dg module over a dga A and denote by $\widetilde{A}$ the cofibrant replacement of A. Then M is also a dg $\widetilde{A}$-module and we define the derived functor of derivations of A with values in M as $\mathrm{Der}^*(\widetilde{A}, M)$. (Strictly speaking, we have not set up things so that it is a derived functor; Der is not even a functor.) We have

$$\mathrm{Der}^*(\widetilde{A}, M) \cong \mathrm{Hom}_A^*(\Omega_{\widetilde{A}}, M) \simeq \mathrm{Hom}_A^*(C_*^{\mathrm{AQ}}(A, A), M).$$

The complex $\mathrm{Hom}_A^*(C_*^{\mathrm{AQ}}(A, A), M)$ embeds as a subcomplex into the truncated Hochschild complex

$$\bar{C}^*(A, M) := \mathrm{Hom}_A^*(\bar{C}_*(A, A), M)$$

consisting of those cochains which vanish on the shuffle products. (A shifted version of this complex is commonly called the Harrison cohomology complex of A with coefficients in M.) We denote this complex by $C^*_{\mathrm{AQ}}(A, M)$ and its cohomology by $H^*_{\mathrm{AQ}}(A, M)$. Therefore we proved the following

Theorem 2.4. *The cohomology of the complex* $C^*_{\mathrm{AQ}}(A, M)$ *is isomorphic to the cohomology of the differential graded module* $\mathrm{Der}^*(\widetilde{A}, M)$ *where* $\widetilde{A}$ *is a cofibrant replacement of* A.

Corollary 2.5. *Let* A *be a cofibrant dga and* M *is a dg* A*-module. Then there is a spectral sequence* $H^*_{\mathrm{AQ}}(H^*(A), H^*(M)) \Rightarrow H^*_{\mathrm{AQ}}(A, M)$.

Thus, we have two ways to compute the André–Quillen cohomology of a dga A with values in a dg A-module M. The first is to replace A with its cofibrant approximation and take its derivations in M. The second is via the functorial complex $C^*_{\mathrm{AQ}}(A, M)$. The method via the complex $C^*_{\mathrm{AQ}}(A, M)$ is better suited for theoretical purposes; in particular it gives rise to the spectral sequence as above. Another useful property of the complex $C^*_{\mathrm{AQ}}(A, M)$ is that it is a direct summand of the Hochschild complex. This is something that is not seen from the point of view of the derived functor of the derivations.

Derivations also admit the following useful interpretation in terms of square-zero extensions. Let A be a dga and M be a dg A-module. Denote by $A \ltimes M$ the dga which is isomorphic as a complex to $A \oplus M$ with multiplication defined as

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2).$$

We will call $A \ltimes M$ the *square-zero extension* of A by M. The dga $A \ltimes M$ is supplied with a dga map into A which is simply the projection onto the first component. We can thus view $A \ltimes M$ as an object in the *overcategory* of A, i.e., the category whose objects are dga's B supplied with a map $\varepsilon_B : B \to A$ and morphisms are obvious commutative triangles. We will denote this overcategory by $\mathcal{C}_A$. It inherits the structure of a closed model category from $\mathcal{C}$ so that a morphism in $\mathcal{C}_A$ is a cofibration if it is so considered as a map in $\mathcal{C}$. Note that A is also an object in $\mathcal{C}_A$ in an obvious fashion.

The association $M \rightsquigarrow A \ltimes M$ is a functor A-mod $\to \mathcal{C}_A$. It clearly preserves weak equivalences and therefore lifts to a functor between the corresponding homotopy categories.

Let B be an object in $\mathcal{C}_A$ which we could assume to be cofibrant without loss of generality. Then an A-module M has a structure of a B-module via the structure map $\varepsilon_B : B \to A$. An elementary calculation shows that a derivation $B \to M$ is nothing but a map $B \to A \ltimes M$ in $\mathcal{C}_A$. Furthermore, we have the following proposition:

Proposition 2.6. *There is a natural isomorphism*

$$H^0_{\mathrm{AQ}}(B, M) \cong [B, A \ltimes M]_{\mathcal{C}_A},$$

where $[-,-]_{\mathcal{C}_A}$ *denotes the set of homotopy classes of maps in* $\mathcal{C}_A$.

Proof. The map $b \mapsto b \otimes 1 - 1 \otimes b$ from $B \to B \otimes B$ can be considered as a map $B \to I$ where I is the kernel of the product map $B \otimes B \to B$. Composing this map with the projection $I \to I/I^2 = \Omega_B$ we get a map $d_B : B \to \Omega_B$, known as the *universal derivation* of the dga B. (It is standard to check that this is indeed a derivation). Denoting by $[-,-]_{B\text{-mod}}$ the set of morphisms in the homotopy category of B-mod we get a natural transformation

$$H^0_{\mathrm{AQ}}(B, M) = [\Omega_B, M]_{B\text{-mod}} \mapsto [B, A \ltimes M]_{\mathcal{C}_A} \tag{1}$$

which associates to a homotopy class of a map $f : \Omega_B \to M$ the composition

$$B \stackrel{(1,d_B)}{\to} B \ltimes \Omega_B \stackrel{(\varepsilon_B, f)}{\to} A \ltimes M.$$

To check that the map of (1) is an isomorphism consider the functor $\mathbb{S}$ which associates to a complex N the symmetric algebra $\mathbb{S}(N)$. The functor $\mathbb{S}$ is left adjoint to the forgetful functor from dga's to complexes and this adjunction passes to homotopy categories. Now it is easy to see that the map of 1 is an isomorphism if $B = \mathbb{S}(N)$. The general case follows by virtue of the following canonical split coequalizer which exists for any dga B:

$$\mathbb{S}^2 B \rightrightarrows \mathbb{S} B \to B. \qquad \square$$

Remark 2.7. On a technical note, observe that the projection $A \ltimes M \to A$ is a fibration from which it follows that $A \ltimes M$ is a fibrant object in $\mathcal{C}_A$. Therefore the set $[B, A \ltimes M]_{\mathcal{C}_A}$ does represent the set of morphisms in the homotopy category of $\mathcal{C}_A$.

The next thing we are going to describe is the *Gerstenhaber bracket* on the André–Quillen cohomology. It turns out that for a dga A the complex $C^*_{\mathrm{AQ}}(A, A)$ admits the structure of a dg Lie algebra. The most conceptual way to describe it is due to Schlessinger and Stasheff [16] which we will now recall.

Let *CA* be the cofree Lie coalgebra on *A*. It could be described as the quotient of the reduced tensor algebra $TA_+ = \oplus_{i=1}^{\infty} A^{\otimes i}$ by the image of the shuffle product. The usual bar differential descends from TA_+ to *CA* making it an acyclic Lie coalgebra. Then $C^*_{\mathrm{AQ}}(A, A)$ is naturally identified with the space $\mathrm{Coder}^*(CA, CA)$ of coderivations of the Lie coalgebra *CA*. See the appendix for background on Lie coalgebras and coderivations. Moreover, $\mathrm{Coder}^*(CA, CA)$ is naturally a dg Lie algebra with respect to the commutator bracket which we will call the Gerstenhaber bracket since it is a direct analogue of the bracket introduced in [7] on the Hochschild complex of an algebra.

On the other hand, André–Quillen cohomology of *A* could be described as $\mathrm{Der}^*(\widetilde{A}, \widetilde{A})$ where $\widetilde{A}$ is the cofibrant approximation of *A*. This gives another structure of a graded Lie algebra on $H^*_{\mathrm{AQ}}(A, A)$. The following result shows that these two structures coincide, and, moreover are invariants of the weak homotopy type of *A*.

Theorem 2.8. (1) *For two weakly equivalent cofibrant dga's A and B the dg Lie algebras* $\mathrm{Der}^*(A, A)$ *and* $\mathrm{Der}^*(B, B)$ *are quasi-isomorphic*;

(2) *for two weakly equivalent* (*not necessarily cofibrant*) *dga's A and B the dg Lie algebras* $\mathrm{Coder}^*(CA, CA)$ *and* $\mathrm{Coder}^*(CB, CB)$ *are quasi-isomorphic*;

(3) *for a cofibrant dga A the dg Lie algebras* $\mathrm{Der}^*(A, A)$ *and* $\mathrm{Coder}^*(CA, CA)$ *are quasi-isomorphic*.

Proof. The main problem is, of course, that $\mathrm{Der}^*(A, A)$ is not functorial with respect to A. To overcome this difficulty we use the properties of closed model categories. For part (1) let $f : A \to B$ be a weak equivalence between two cofibrant dga's. First assume that f is a fibration. Then there exists a right splitting $g : B \to A$ so that $f \circ g = \mathrm{id}_B$. Then define the map of dga's: $\mathrm{End}(B) \to \mathrm{End}(A)$ by assigning to a map $s : B \to B$ the map $g \circ s \circ f$. The condition $f \circ g = \mathrm{id}_B$ ensures that this map respects composition of endomorphisms. It follows that we have a map of dg Lie algebras $h : \mathrm{Der}^*(B, B) \to \mathrm{Der}^*(A, A)$ together with the commutative diagram of complexes:

$$\begin{tikzcd} \mathrm{Der}(A, A) \arrow[rr, "h"] \arrow[dr] & & \mathrm{Der}(B, B) \\ & \mathrm{Der}(A, B) \arrow[ur] & \end{tikzcd}$$

Here the southeast arrow and the northeast arrows are induced by f and g respectively. It follows that h is a quasi-isomorphism.

Similarly if $f : A \to B$ is an acyclic cofibration then it admits a left splitting $g : B \to A$ so that $g \circ f = \mathrm{id}_A$ and we have a map of dg Lie algebras $\mathrm{Der}^*(B, B) \to \mathrm{Der}^*(A, A)$. In the general case we use the presentation of f as a composition of a cofibration and a fibration.

The argument for part (2) is more difficult, although the idea is the same. The most conceptual way is to introduce a closed model category structure on dg Lie coalgebras and notice that for a dga A the dg Lie coalgebra CA is a fibrant–cofibrant object. The corresponding result for dg coalgebras is due to Hinich [12].

While there is little doubt that this could be done a detailed proof would require rewriting much of Hinich's paper (with appropriate modifications). Since we do not need the full strength of the model category structure we give an alternative proof here which is of some independent interest.

Let $f : A \to B$ be a quasi-isomorphism between two dga's and denote by $\widetilde{f}$ the induced map of dg Lie coalgebras $CA \to CB$. Note that $\widetilde{f}$ is a quasi-isomorphism. Moreover, note that CA and CB have filtrations inherited from the tensor coalgebras and $\widetilde{f}$ is a *filtered* quasi-isomorphism, that is it induces a quasi-isomorphism on each graded component.

We claim that $\widetilde{f}$ could be factored as

$$CA \xrightarrow{\widetilde{g}} D \xrightarrow{\widetilde{h}} CB,$$

where

- both maps $\widetilde{g}$ and $\widetilde{h}$ are filtered quasi-isomorphisms of dg Lie coalgebras;
- D is a dg Lie coalgebra that is free as a Lie coalgebra;
- the map $\widetilde{g} : CA \to D$ admits a filtered left inverse;
- the map $\widetilde{h} : D \to CB$ admits a filtered right inverse.

This claim clearly implies the result we need by standard spectral sequence arguments. We will now begin to prove the required factorization. It is convenient for us to work with *pro-free* Lie algebras instead of cofree Lie coalgebras. See the appendix for definitions and results about pro-free and pronilpotent Lie algebras. Note that pro-free Lie algebras are special cases of pronilpotent Lie algebras. Let *LA* and *LB* denote the (graded) *k*-linear dual to *CA* and *CB*. Then *LA* and *LB* are pro-free Lie algebras on the *k*-vector space A^* and B^*, and are thus pronilpotent. Moreover, *LA* and *LB* have differentials making them dg Lie algebras and there is a continuous map $\widetilde{f}^* : LB \to LA$. Further, *LA* and *LB* possess natural filtrations by bracket length and $\widetilde{f}^*$ is a filtered quasi-isomorphism.

Under these conditions we will construct a dg Lie algebra D' so that $\widetilde{f}^*$ factors as

$$LB \xrightarrow{\widetilde{h}^*} D' \xrightarrow{\widetilde{g}^*} LA$$

so that:

- $\widetilde{h}^*$ and $\widetilde{g}^*$ are continuous filtered quasi-isomorphisms of dg Lie algebras;
- D' is a dg Lie algebra which is pro-free as a Lie algebra;
- the map $\widetilde{h}^* : LB \to D$ admits a continuous filtered left inverse;
- the map $\widetilde{g}^* : D' \to LA$ admits a continuous filtered right inverse.

Taking continuous duals this is easily seen to be equivalent to the statement of the claim.

We use an analogue of the mapping cylinder construction in topology. Let $L\langle t\rangle$ denote the pro-free Lie algebra with two generators t, dt such that $|dt| = |t| + 1$. We introduce a differential in $L\langle t\rangle$ by setting $d(t) = dt$ and $d(dt) = 0$. Then $L\langle t\rangle$ is a filtered dg Lie algebra which is filtered contractible (which means that the graded components corresponding to its filtration are contractible). For a filtered dg Lie algebra l the free Lie product of l and $L\langle t\rangle$ has an induced filtration and we denote by $l\langle t\rangle$ its completion with respect to this filtration. For a finite indexing set I we introduce the notation $l\langle t_\alpha\rangle_{\alpha\in I}$ to denote the completion of the free Lie product of l and the collection of Lie algebras $L\langle t_\alpha\rangle$ with $\alpha \in I$. If I is infinite we denote by $l\langle t_\alpha\rangle_{\alpha\in I}$ the inverse limit of $l\langle t_\alpha\rangle_{\alpha\in J}$ where J ranges through finite subsets of I. Note that the natural inclusion $L \hookrightarrow L\langle t_\alpha\rangle$ is a filtered quasi-isomorphism having a filtered right inverse (which sends all t_α to zero).

Now let

$$D' := LB\langle t_\alpha\rangle,$$

where α runs through the set of homogeneous elements of LA. We then have a factorization of $\widetilde{f}^*$ as

$$LB \to D' \to LA,$$

where the first arrow is the obvious inclusion and the second arrow is the map which coincides with $\widetilde{f}^*$ on LB and takes each element t_α in $LB\langle t_\alpha\rangle$ to $\alpha \in LA$. Note that the latter map is a *surjection* and a filtered quasi-isomorphism. The following lemma ensures that it has a (filtered) right inverse.

Lemma 2.9. *Let $s : l \to g$ be a continuous surjective map of dg Lie algebras which are pro-free Lie algebras. Assume that s is a filtered quasi-isomorphism. Then s admits a continuous filtered right inverse.*

Proof. Choose a filtered Lie algebra map $i : g \to l$ for which $i \circ s = \mathrm{id}_l$ (we do not claim that i is a map of dg Lie algebras), A.11. Then we have an isomorphism of vector spaces $l \cong I \oplus i(g)$ where I is the kernel of s. For $a \in g$ we have

$$d(i(a)) = i(d(a)) + \xi(a).$$

It is straightforward to check that the map $\xi : g \to I$ is a cocycle in the complex $\mathrm{Der}^*_f(g, I)$ of filtration-preserving derivations of g with values in I (the g-module structure on I is provided by the map i). Furthermore, it is possible to find a right inverse to s compatible with differentials if and only if ξ is a coboundary in $\mathrm{Der}^*_f(g, I)$. Since s is a filtered quasi-isomorphism, I is filtered contractible. Denote by $F_n(I)$ the nth filtration component of I i.e., $F_n(I)$ consists of those elements in I which have bracket length $\geqslant n$. Then $F_n(I)/F_{n+1}(I)$ has zero homology. We have

$$\mathrm{Der}^*_f(g, I) \cong \varprojlim \mathrm{Der}^*(g, F_n(I)).$$

However since $F_n(I)/F_{n+1}(I)$ is contractible the obvious induction shows that $\mathrm{Der}^*(g, F_n(I))$ is likewise contractible so $\mathrm{Der}^*_f(g, I)$ is contractible and our lemma is proved. $\square$

Finally for (3) we only need to note that the canonical projection

$$\mathrm{Coder}^*(CA, CA) \to \mathrm{Der}^*(A, A)$$

is a map of dg Lie algebras. The required result then follows from Theorem 2.4 and part (2) of the present theorem which has just been proved. □

3. Rational homotopy groups of function spaces

Before discussing the rational homotopy of function spaces we need to establish the following facts about the relation of spaces of maps and that of their Postnikov stages. These are presumably well known but we have been unable to find a proper reference. Let $\mathrm{hAut}(X)$ denote the group of homotopy classes of homotopy equivalences of a space X. The set of homotopy classes of (unpointed) maps $X \to Y$ is denoted by $[X, Y]$, as usual.

Proposition 3.1. *Let X and Y be connected CW-complexes with $\dim X = n$ and Y is nilpotent. Let X_n and Y_n denote their nth Postnikov stages. Then*

(1) $[X, Y] \to [X_n, Y_n]$ *is a bijection.*
(2) *If X is nilpotent then* $\mathrm{hAut}(X) \to \mathrm{hAut}(X_n)$ *is an isomorphism of groups.*

Proof. Note first that part (2) of the Theorem is an immediate consequence of part (1).

For (1) consider the following maps of sets:

$$[X, Y] \to [X, Y_n] \leftarrow [X_n, Y_n].$$

Let us prove first that $[X, Y] \to [X, Y_n]$ is a bijection. For simplicity, we assume that the Postnikov tower of Y consists of principal fibrations. In the general case we could argue similarly, replacing the Postnikov tower of Y by its principal refinement (which exists since Y is nilpotent). A map $X \to Y_n$ lifts to a map $X \to Y_{n+1}$ if and only if an obstruction class lying in $H^{n+2}(X, \pi_{n+1}(Y))$ is zero. This is ensured by our assumption that the dimension of X is less than or equal to n. Furthermore, the group $H^{n+1}(X, \pi_{n+1}(Y))$ acts on the set $[X, Y_{n+1}]$ so that the set of orbits is precisely $[X, Y_n]$. Since this group is zero we conclude that there is a bijection between sets $[X, Y_n]$ and $[X, Y_{n+1}]$. Arguing by induction up the Postnikov tower of Y we see that $[X, Y] \to [X, Y_n]$ is a bijection.

We will now show that the map $[X, Y_n] \leftarrow [X_n, Y_n]$ is bijective. Since associating to a space its nth Postnikov stage is a functor in the homotopy category, it follows that any map $X \to Y_n$ extends to $X_n \to Y_n$. Furthermore, assuming again that the Postnikov tower of Y consists of principal fibrations we see that the ambiguities in choosing extensions lie in the relative cohomology groups $H^k(X, X^n, \pi_k(Y_n))$. For $k \leqslant n$ these groups vanish since X is an n-dimensional complex whereas for $k > n$ they vanish since Y_n has no homotopy above dimension n. Therefore any map $X \to Y_n$ extends to $X_n \to Y_n$ uniquely up to homotopy. □

We will also need the following linearized version of the preceding result.

Proposition 3.2. *Let A and B be two connected dga's so that the cohomology of A vanishes above dimension n and $A \to B$ be a fixed dga map making B into an A dg-module. Let A_n and B_n denote the nth Postnikov stages of A and B respectively. Then*

(1) $H^0_{\mathrm{AQ}}(A, B) \to H^0_{\mathrm{AQ}}(A_n, B_n)$ *is a bijection.*
(2) $H^0_{\mathrm{AQ}}(A, B) \to H^0_{\mathrm{AQ}}(A_n, B_n)$ *is an isomorphism of Lie algebras.*

Proof. Let us first explain the construction of the map $H^0_{\mathrm{AQ}}(A, B) \to H^0_{\mathrm{AQ}}(A_n, B_n)$ figuring in the formulation of the theorem. We assume from the very beginning that A and B are minimal which results in no loss of generality. Then $H^0_{\mathrm{AQ}}(A, B)$ is simply the zeroth cohomology of the complex $\mathrm{Der}^*(A, B)$ (this, of course, is not a dg Lie algebra unless $A = B$). Next, A_n and B_n are subalgebras of A and B generated by the polynomial generators in degrees $\leqslant n$. Because of the minimality A_n and B_n are closed under the differential. Clearly a derivation of degree 0 $A \to B$ descends to a derivation $A_n \to B_n$. Moreover, cycles in $\mathrm{Der}^0(A, B)$ map to cycles in $\mathrm{Der}^0(A_n, B_n)$ and boundaries to boundaries. Thus, the map $H^0_{\mathrm{AQ}}(A, B) \to H^0_{\mathrm{AQ}}(A_n, B_n)$ is well-defined and clearly is a Lie algebra map in the case $A = B$. It is further obvious that part (1) of the proposition is a consequence of part (2).

As in the proof of Proposition 3.1 we assume, purely for notational simplicity, that the Postnikov tower of A consists of principal fibrations. That means that for a polynomial generator x of A of degree k the element dx belongs to A_{k-1}. If this is not the case, then one could argue similarly, using the principal refinement of the Postnikov tower of A which exists since A is minimal.

Let us denote by d_A and d_B the differentials in A and B and, by abuse of notation, also the differentials in A_n and B_n.

Surjectivity: Let $\xi : A_n \to B_n$ be a derivation of zero degree such that $\xi \circ d_B = d_A \circ \xi$. In other words, ξ is a cycle in $\mathrm{Der}^0(A_n, B_n)$. Then ξ could be extended to A_{n+1} if and only if for any generator $x \in A$ in degree $n+1$ the element $\xi(d_A x)$ is a coboundary in B. Note that $d_A x \in A_n$ so $\xi(d_A x)$ is defined. We have

$$d_B \xi(d_A x) = \xi(d_A d_A x) = 0$$

which means that $\xi(d_A x)$ is an $n+2$-cocycle in B. Since all $n+1$-cocycles are coboundaries we see that ξ can indeed be extended to A_{n+1}. Using induction up the Postnikov tower of A we see that ξ could be extended to a derivation $A \to B$.

Injectivity: Suppose that $\xi \in \mathrm{Der}^0(A, B)$ determines a boundary in $\mathrm{Der}^0(A_n, B_n)$; we will then show that ξ is a boundary in $\mathrm{Der}^0(A, B)$. Indeed, considering ξ as an element in $\mathrm{Der}^0(A_n, B_n)$ we have

$$\xi = d_B \circ \eta + \eta \circ d_A,$$

where η is a derivation $A_n \to B_n$ of degree -1. Take a generator $x \in A$ in dimension $n+1$; we want to define $\eta(x)$ so that the following equality were true:

$$\xi(x) = d_B \circ \eta(x) + \eta \circ d_A(x).$$

For this, it is necessary and sufficient that $(\xi - d_B \circ \eta)(x)$ be a coboundary in B. We have

$$\begin{aligned} d_A[(\xi - d_B \circ \eta)(x)] &= \xi(d_A x) - d_A(\eta(d_B x)) \\ &= \eta(d_B \circ d_B x) \\ &= 0. \end{aligned}$$

In other words, $(\xi - d_B \circ \eta)(x)$ is an $n+1$-cocycle in B. Since all $n+1$-cocycles are coboundaries we conclude that ξ restricted to A_{n+1} is a coboundary in $\mathrm{Der}^0(A_{n+1}, B_{n+1})$. Induction up the Postnikov tower of A finishes the proof. $\square$

Let X be a nilpotent space of finite type. Denote by A its minimal model. Then A is an augmented commutative differential graded algebra over $\mathbb{Q}$ which is a polynomial algebra on $\pi_*(X)$. The differential on A, $d_A : A \to A$ is a derivation of degree 1 for which $d_A(A) = A_+ \cdot A_+$. Here we denoted by A_+ the set of elements in A having positive degree.

Let $L := \mathrm{Der}^* A$, the set of all graded derivations of A. The differential on L is given by the formula $d(\eta) = [\eta, d_A]$ where $\eta \in L$. The condition $d_A \circ d_A = 0$ ensures that the operator $[-, d_A]$ in L has square 0.

Since A is a cofibrant object in the closed model category of differential graded algebras the cohomology of L represents the derived functor of derivations. In other words, $H^*(L) \cong H^*_{\mathrm{AQ}}(A, A)$ as we saw in Section 2. The set of derivations of degree 0, that is $H^0_{\mathrm{AQ}}(A, A)$ is then a conventional (ungraded) Lie algebra.

Remark 3.3. Consider the graded Lie algebra $B^*(L)$. The condition that A is minimal nilpotent ensures that every element in $B^*(L)$ is nilpotent. In particular $B^0(L)$ is a nilpotent (ungraded) Lie algebra over $\mathbb{Q}$.

We will now briefly recall the notion of homotopy in the category of dga's restricting ourselves to self-maps. The details may be found in [3].

Let $A[t, dt]$ denote the differential graded algebra obtained from A by adjoining polynomial variables t, dt subject to the relation $(dt)^2 = 0$. The differential $d_{A[t,dt]}$ on $A[t, dt]$ is induced from the one on A. More precisely, denote by ∂_t the partial derivative with respect to t. Then for $h \in A[t, dt]$ we have

$$d_{A[t,dt]}(h) = d_A(h) + (\partial_t h)dt.$$

There are two dga maps $A[t, dt] \to A$ given by

$$e_0 : h \to h|_{t=0} \quad \text{and} \quad e_1 : h \to h|_{t=1}.$$

Let $F : A \to A$ be a dga self-map of A. Then F is said to be homotopic to the identity if there exists a dga map $A \to A[t, dt]$ such that its composition with e_0

is the identity map on A whereas its composition with e_1 is F. Note that any map $A \to A[t, dt]$ could be written as $F + Gdt$ where F and G are maps $A \to A[t]$. The set of dga self-maps of A homotopic to the identity forms a normal subgroup in the set of all automorphisms of A. The corresponding quotient group is the group of *homotopy self-equivalences* of A and will be denoted by $\mathrm{hAut}(A)$. It is isomorphic to the group of homotopy classes of homotopy self-equivalences of $X_{\mathbb{Q}}$, the rationalization of the space X.

Theorem 3.4. *Let X denote a nilpotent CW complex which is either finite or has a finite Postnikov tower, and let $A^*(X)$ denote its Sullivan–de Rham model. The group* $\mathrm{hAut}(A^*(X)) \cong \mathrm{hAut}(X_{\mathbb{Q}})$ *is the group of $\mathbb{Q}$-points of an affine algebraic group scheme over $\mathbb{Q}$ whose Lie algebra is $H^0_{\mathrm{AQ}}(A^*(X), A^*(X))$.*

Remark 3.5. Sullivan [18] sketched the proof of the fact that $\mathrm{hAut}(X_{\mathbb{Q}})$ is an algebraic group. The explicit identification of its Lie algebra in terms of André–Quillen cohomology is new.

Proof. First of all we replace $A^*(X)$ by its minimal model denoted by A. Then $\mathrm{hAut}(A) \cong \mathrm{hAut}(A^*(X))$. By 3.1 and 3.2, the case of finite CW complex reduces to that of the finite Postnikov tower. Thus we may assume that A is a nilpotent minimal algebra with generators of bounded degree. Now because of the finiteness assumptions the group of algebra automorphisms of A (i.e., not taking into account the differential) is algebraic. Further the condition that an algebra map $A \to A$ commutes with d_A is algebraic from which it follows that the group $\mathcal{Z}$ of dga automorphisms of A is also an algebraic group over $\mathbb{Q}$. It is obtained from its *reductive* part (coming from the quadratic part of the differential d_A) by iterated extensions by the *additive group scheme*.

The Lie algebra of the group $\mathcal{Z}$ is just the set of degree 0 derivations of A which commute with the differential in A. Therefore this is the Lie algebra $Z_0(L)$ of zero degree cocycles of L. The normal Lie subalgebra $B^0(L)$ of $Z^0(L)$ is nilpotent and there is a corresponding normal subgroup $\mathcal{B}$ in the algebra self-maps of A obtained by exponentiating $B^0(L)$. We will show that this subgroup consists precisely of those self-maps which are homotopic to the identity. Thus, $\mathrm{hAut}(A)$ is the quotient of an affine algebraic group by a normal affine algebraic subgroup and is thus affine algebraic.

Granting this for a moment, we see that the Lie algebra of $\mathrm{hAut}(A) = \mathcal{Z}/\mathcal{B}$ is the quotient Lie algebra $Z_0(L)/B_0(L)$ which is precisely $H^0_{\mathrm{AQ}}(A, A) \cong H^0_{\mathrm{AQ}}(A^*(X), A^*(X))$ by the results of Section 2.

Let $F + Gdt : A \to A[t, dt]$ be a homotopy for which $F|_{t=0} = \mathrm{id}_A$. First examine the condition that it is a dga map. For $h_1, h_2 \in A$ we have:

$$\begin{aligned}(F + Gdt)(h_1h_2) &= F(h_1h_2) + G(h_1h_2)dt \\ &= F(h_1)F(h_2) + [F(h_1)G(h_2) + G(h_1)F(h_2)]dt \\ &= (F + Gdt)(h_1)(F + Gdt)(h_2).\end{aligned}$$

So we get two conditions:

(1) $F(h_1h_2) = F(h_1)F(h_2)$ which means simply that F is an algebra map and
(2) $G(h_1h_2) = F(h_1)G(h_2) + G(h_1)F(h_2)$.

The second condition means that G is an F-derivation. Setting $t = 0$ in the second equation we get

$$G|_{t=0}(h_1h_2) = h_1G(g_2) + G(h_1)h_2.$$

In other words $G|_{t=0}$ is a usual derivation of A.

Next, the condition that $F + Gdt$ is a map of *differential* algebras means that

$$(F + Gdt) \circ d_A = d_{A[t,dt]} \circ (F + Gdt).$$

Applying both sides of this equation to $h \in A$ and equating coefficients at t and dt we get two identities:

(1) $F \circ d_A = d_A \circ F$,
(2) $\partial_t F = G \circ d_A + d_A \circ G = [G, d_A]$.

The first condition above simply means that $F = F(t)$ commutes with the differential; i.e., determines a family of maps of complexes $A \to A$. Setting $t = 0$ in the second equation we get $\partial_t F|_{t=0} = [G|_{t=0}, d_A]$ which means that $\partial_t F|_{t=0}$ is a coboundary in L.

Now let F_1 be an element in $\mathcal{B}$. Then F can be expressed as $F_1 = \exp([G_0, d_A])$ where G_0 is a derivation of A. Let $F = F(t) = \exp(t[G_0, d_A])$ and $G = G(t) = G_0 \exp(t[G_0, d_A])$. Then $(F(t), G(t))$ is the homotopy from F_1 to id. Indeed, $F(t)$ is an algebra map for each t and $G(t)$ is an $F(t)$-derivation.

Conversely, suppose that $(F(t), G(t))$ is a homotopy from id to $F(1)$, we need to show that $F(1)$ belongs to $\mathcal{B}$. Here we will use the fact that A is nilpotent and minimal but our proof extends under the condition that F_1 is homotopic to the identity through automorphisms, that is $F(t)$ is an automorphism for all t. This condition follows automatically for minimal algebras since a weak equivalence between minimal nilpotent algebras is necessarily an isomorphism.

We can write

$$F(t) = \mathrm{Id}_A + \sum_{j=1}^{\infty} F_j t^j.$$

Since $F : A \to A[t, dt]$ we must have that the sum is locally finite in the sense that for $a \in A$, $F_j(a) = 0$ for $j \gg 0$. F^{-1} exists formally:

$$F^{-1}(t) = \mathrm{Id}_A + \sum_{i=1}^{\infty} (-1)^i \left(\sum_{j=1}^{\infty} F_j t^j \right)^i$$

and this infinite sum is locally finite by the condition above.

Since $\partial_t F = [G, d_A]$ we have, taking into account that F commutes with d_A:

$$\begin{aligned}(\partial_t F)F^{-1} &= [G, d_A]F^{-1} \\ &= G \circ d_A \circ F^{-1} - d_A \circ G \circ F^{-1} \\ &= [GF^{-1}, d_A].\end{aligned}$$

Since $G(t)$ is an $F(t)$-derivation $GF^{-1}(t)$ will be a usual derivation. Noting that

$$(\partial_t F(t))F^{-1}(t) = \partial_t \log F(t)$$

we have the following equation:

$$\partial_t \log F(t) = [GF^{-1}, d_A].$$

Therefore

$$\begin{aligned}F(1) &= \exp\left(\int_0^1 [GF^{-1}, d_A]dt\right) \\ &= \exp\left[\int_0^1 GF^{-1}dt, d_A\right].\end{aligned}$$

Here the integral is carried out formally. Moreover, $\int_0^1 GF^{-1}dt$ is a locally finite expression and still a derivation. Therefore $F(1)$ is contained in $\mathcal{B}$. $\square$

Remark 3.6. One naturally wonders whether there is a relationship between the whole complex $C^*_{\mathrm{AQ}}(A^*(X), A^*(X))$ together with its Gerstenhaber bracket and the space $\mathrm{Aut}(X)$ of homotopy self-equivalences of X. Schlessinger and Stasheff [17] provide an affirmative answer to this question in the case when X is simply connected. Namely, they show that the complex consisting of derivations of A lowering the degree by $k > 1$ serves as a Lie model for the universal covering of the space $\mathrm{BAut}(X)$. That implies that the Whitehead product in $\pi_{>1}\,\mathrm{BAut}(X)$ and the Gerstenhaber bracket in $H^{<0}_{\mathrm{AQ}}(A^*(X), A^*(X))$ agree whereas our theorem compares $\pi_1\,\mathrm{BAut}(X)$ and $H^0_{\mathrm{AQ}}(A, A)$. It seems likely that the theorem of Schlessinger–Stasheff could be extended to the nilpotent case as well.

Remark 3.7. One can also express the tangent Lie algebra to $\mathrm{hAut}(X_{\mathbb{Q}})$ in terms of the Quillen or dg Lie algebra model of X, at least when X is simply connected (cf. [5] concerning Quillen models). Namely, if $L(X)$ is a Quillen model for X consider the Chevalley–Eilenberg complex $C^*_{\mathrm{CE}}(L(X), \mathbb{Q})$ computing the Lie algebra cohomology of L with trivial coefficients. Then $C^*_{\mathrm{CE}}(L(X), \mathbb{Q})$ is a cofibrant dga and serves as a Sullivan model for X. Furthermore, the dg Lie algebra of derivations of $C^*_{\mathrm{CE}}(L(X), \mathbb{Q})$

coincides up to a shift of dimensions with the complex $C^*_{\mathrm{CE}}(L(X), L(X))$ computing the Lie algebra cohomology of $L(X)$ with coefficients in itself. The complex $C^{*-1}_{\mathrm{CE}}(L(X), L(X))$ carries a dg Lie algebra structure so that $H^1_{\mathrm{CE}}(L(X), L(X))$, its cohomology in the total degree 1 is an ungraded Lie algebra. We conclude that the Lie algebra of $\mathrm{hAut}(X_Q)$ is isomorphic to $H^1_{\mathrm{CE}}(L(X), L(X))$.

Our next result is concerned with the more general problem of computing $\pi_i F(X, Y)$ for $i \geqslant 1$. Here X, Y are rational nilpotent CW-complexes of finite type and $F(X, Y)$ is the space of continuous maps from X into Y. Recall that $A^*(X)$ and $A^*(Y)$ denote the Sullivan–de Rham algebras of X and Y, respectively. In view of the previous remark it is natural to expect that the homotopy groups of $F(X, Y)$ are expressed in terms of the André–Quillen cohomology of $A^*(Y)$ with values in $A^*(X)$. When $X = Y$ and X is simply connected the corresponding statement was proved by Schlessinger and Stasheff by combining their deformation theory with the fact that $\mathrm{BAut}(X)$ is the classifying space for X-fibrations. This machinery is not available for $X \neq Y$ and we use Proposition 2.6 instead. Note also that the Gerstenhaber bracket no longer exists on the complex $C^*_{\mathrm{AQ}}(A, B)$ for $A \neq B$.

Assume that there is given a map $f : X \to Y$ which determines a basepoint in $F(X, Y)$. The map f induces a map $A^*(Y) \to A^*(X)$ making $A^*(X)$ into a differential graded module over $A^*(Y)$.

Theorem 3.8. *There is an isomorphism of sets*

$$\pi_n(F(X, Y), f) \cong H^{-n}_{\mathrm{AQ}}(A^*(Y), A^*(X)).$$

If $n \geqslant 2$ then this is an isomorphism of Abelian groups.

Proof. Consider the de Rham algebra $A^*(X \times S^n)$ where $n \geqslant 1$. Clearly, $A^*(X \times S^n)$ is weakly equivalent as a commutative dga to the dga $A^*(X) \ltimes A^*(X)[n]$ where $A^*(X)[n]$ is the square-zero ideal which is isomorphic as a $\mathbb{Q}$-vector space to $A^*(X)$ and whose grading is shifted by n.

Denoting by $F_*(?, ?)$ the function space between *pointed* topological spaces we have an obvious homotopy fibre sequence:

$$F_*(S^n, F(X, Y)) \to F(S^n, F(X, Y)) \to F(*, F(X, Y)) = F(X, Y),$$

where the second arrow is induce by the inclusion of the basepoint in S^n. (The basepoint of $F(X, Y)$ is f and we take $F_*(S^n, F(X, Y))$ to be the fibre *over* f.) Standard adjunction gives the homotopy fibre sequence

$$F_*(S^n, F(X, Y)) \to F(X \times S^n, Y) \to F(X, Y). \tag{2}$$

Consider now the category $\mathcal{T}_X$ of spaces *under* X, i.e., spaces supplied with a map from X with morphisms being the obvious commutative triangles. This is a topological closed model category and $X \times S^n, Y \in \mathcal{T}_X$. Note that $X \times S^n$ is a cogroup object in the homotopy category of $\mathcal{T}_X$, abelian when $n > 1$. Likewise $A^*(X \times S^n)$ is (equivalent to) an abelian group object in $\mathcal{C}_{A^*(X)}$. For $n > 1$ the cogroup structure on $X \times S^n$ corresponds to the group structure on $A^*(X \times S^n)$ under the Sullivan–de Rham functor $A^* : \mathcal{T}_X \to \mathcal{C}_{A^*(X)}$.

It follows that the space $F_*(S^n, F(X, Y))$ is weakly equivalent to the function space in $\mathcal{T}_X$ from $X \times S^n$ to Y. Therefore, denoting by $[-,-]_{\mathcal{T}_X}$ the set of homotopy classes of maps in $\mathcal{T}_X$ we have

$$\pi_n F_*(X, Y) \cong [X \times S^n, Y]_{\mathcal{T}_X}.$$

Since the homotopy category of finite type rational spaces in $\mathcal{T}_X$ is anti-equivalent to the homotopy category of finite type dga's *over* $A^*(X)$ we have

$$\begin{aligned}
\pi_n F_*(X, Y) &\cong [A^*(Y), A^*(X \times S^n)]_{\mathcal{C}_{A^*(X)}} \\
&\cong [A^*(Y), A^*(X) \ltimes A^*(X)[n]]_{\mathcal{C}_{A^*(X)}} \\
&\cong H^0_{\mathrm{AQ}}(A^*(Y,)A^*(X)[n]) \quad \text{(by Proposition 2.6)} \\
&\cong H^{-n}_{\mathrm{AQ}}(A^*(Y), A^*(X)). \qquad \square
\end{aligned}$$

Remark 3.9. If X is a point then the space of maps $X \to Y$ is simply Y. The previous theorem then gives an identification of $\pi_*(Y)$ with $H^*_{\mathrm{AQ}}(A^*(Y), \mathbb{Q})$. This is not surprising since by Theorem 2.4 $H^*_{\mathrm{AQ}}(A^*(Y), \mathbb{Q})$ is identified with the homology of the complex $\mathrm{Der}^*(M(A^*(Y)))$. Here $M(A^*(Y))$ denotes the minimal model of Y. Clearly this homology is simply the dual space of indecomposables in $M(A^*(Y))$ so we recover a standard result in rational homotopy theory.

More generally, suppose that the map $f : X \to Y$ is homotopic to the trivial map. Thus, via the augmentation map $A^*(Y) \to \mathbb{Q}$, $A^*(X)$ is an $A^*(Y)$-module. Then it is easy to see that there is an isomorphism of complexes

$$C^*_{\mathrm{AQ}}(A^*(Y), A^*(X)) \cong C^*_{\mathrm{AQ}}(A^*(Y), \mathbb{Q}) \hat{\otimes} A^*(X).$$

Here $\hat{\otimes}$ denotes completed tensor product. Therefore

$$\begin{aligned}
H^*_{\mathrm{AQ}}(A^*(Y), A^*(X)) &\cong H^*_{\mathrm{AQ}}(A^*(Y), \mathbb{Q}) \hat{\otimes} H^*(X) \\
&\cong \pi_*(Y) \hat{\otimes} H^*(X).
\end{aligned}$$

So we recover an isomorphism obtained in [4]:

$$\pi_n(F(X, Y), f) \cong \prod_{k=1}^{\infty} \pi_k(Y) \otimes H^{k-n}(X).$$

Remark 3.10. In the case X and Y are simply connected the considerations similar to those in Remark 3.7 yield a bijection of sets for $n=1$ and an isomorphism of Abelian groups for $n>1$:

$$\pi_n F(X,Y) \cong H^{1-n}(L(X), L(Y)),$$

where $L(X)$ and $L(Y)$ are Quillen models of X and Y, respectively. It seems likely that the last isomorphism continues to hold under no finiteness assumptions on the rational spaces X and Y.

Acknowledgments

J.B. and A.L. thank Yves Felix for comments relating to earlier work on this subject.

Appendix A

In this appendix we collect certain facts about profinite Lie algebras which are used in the main body of the paper. These facts appear to be standard but we are unaware of any published references. Below k denotes a fixed field of characteristic zero.

Definition A.1. A profinite set is an inverse limit of finite sets. Profinite sets form a category $\mathcal{PS}$ in which morphisms are continuous maps between profinite sets with respect to the inverse limit topology.

Remark A.2. It is well known that $\mathcal{PS}$ is equivalent to the category of totally disconnected compact Hausdorff topological spaces. We will not use this fact, however.

Definition A.3. A profinite k-vector space is an inverse limit of finite dimensional k-vector spaces. Profinite spaces form a category $\mathcal{P}\mathrm{Vect}_k$ in which morphisms are continuous linear maps between profinite spaces with respect to the inverse limit topology.

Remark A.4. Note that the category of profinite vector spaces (as well as that of profinite sets) is closed under taking arbitrary inverse limits.

Definition A.5. Let $U = \lim_{\leftarrow} U_\alpha$, $V = \lim_{\leftarrow} V_\beta$ be profinite spaces. Then their completed tensor product $U\hat{\otimes}V$ is the following profinite space.

$$U\hat{\otimes}V = \lim_{\substack{\leftarrow \\ \alpha,\beta}} U_\alpha \otimes V_\beta.$$

Proposition A.6. *The category* $\mathcal{P}\mathrm{Vect}_k$ *is anti-equivalent to the category* Vect_k *of discrete k-vector spaces and all linear maps. Under this anti-equivalence the tensor product in* Vect_k *corresponds to the completed tensor product in* $\mathcal{P}\mathrm{Vect}_k$.

Proof. Note that any vector space is a direct limit of its finite-dimensional subspaces. The functor $\mathrm{Vect}_k \mapsto \mathcal{P}\mathrm{Vect}_k$ is specified by taking the dual vector space $V \mapsto V^* := \mathrm{Hom}_k(V, k)$. The inverse functor associates to any profinite vector space U its continuous dual $U^* := \mathrm{Hom}_{\mathrm{cont}}(U, k)$. The second statement of the proposition is obvious. $\square$

Corollary A.7. *Any continuous surjective map $U \to V$ in $\mathcal{P}\mathrm{Vect}_k$ admits a continuous splitting.*

Proof. Indeed, using the anti-equivalence of $\mathcal{P}\mathrm{Vect}_k$ and Vect_k we see that this is equivalent to the statement that any monomorphism of discrete k-vector spaces is split which is clearly true. $\square$

For a finite set X denote by $k\langle X\rangle$ the vector space spanned by the elements in X. More generally, let $X = \lim_{\leftarrow} X_\alpha$ be a profinite set and denote by $k\langle X\rangle$ the profinite vector space

$$k\langle X\rangle = \lim_{\leftarrow} k\langle X_\alpha\rangle.$$

Clearly this defines a functor $\mathcal{PS} \mapsto \mathcal{P}\mathrm{Vect}_k$.

Proposition A.8. *The functor $X \mapsto k\langle X\rangle$ is left adjoint to the forgetful functor $\mathcal{P}\mathrm{Vect}_k \mapsto \mathcal{PS}$.*

Proof. Let $X = \lim_{\leftarrow} X_\alpha$ be a profinite set and $V = \lim_{\leftarrow} V_\beta$ be a profinite vector space. Then we have isomorphisms of sets

$$\begin{aligned} \mathcal{P}\mathrm{Vect}_k(k\langle X\rangle, V) &\cong \lim_{\leftarrow \beta} \lim_{\rightarrow \alpha} \mathrm{Vect}_k(k\langle X_\alpha\rangle, V_\beta) \\ &\cong \mathcal{PS}(X, V). \end{aligned}$$

These isomorphisms are obviously natural in X and V. $\square$

Definition A.9. A pronilpotent Lie algebra is an inverse limit of finite-dimensional *nilpotent* Lie algebras. Pronilpotent Lie algebras form a category $\mathcal{P}\mathrm{Lie}_k$ in which morphisms are continuous Lie algebra maps. For clarity, we state everything for the ungraded case, but the definitions and proofs go through in the graded case as well.

For a finite-dimensional vector space V denote by $\mathcal{L}V$ the free Lie algebra on V; it is well known that $\mathcal{L}V$ can be identified with the space of primitive elements in TV, the tensor algebra on V. The Lie algebra $\mathcal{L}V$ has a filtration with respect to the bracket length. Denote by LV the completion of $\mathcal{L}V$ with respect to this filtration. Alternatively, LV is the space of primitive elements in the completed tensor algebra $TV\hat{} := \prod_{i=0}^{\infty} V^{\hat{\otimes} i}$. Note that LV is a pronilpotent Lie algebra and if g is a finite-dimensional nilpotent Lie algebra then there is a natural bijection of sets:

$$\mathcal{P}\mathrm{Lie}_k(LV, g) \cong \mathrm{Vect}_k(V, g). \tag{A.1}$$

More generally, let $V = \lim_{\leftarrow} V_\alpha$ be a pronilpotent space. Then $LV := \lim_{\leftarrow} LV_\alpha$. We will call LV the pro-free Lie algebra on V; it is still a pronilpotent Lie algebra. The correspondence $V \mapsto LV$ is clearly a functor $\mathcal{P}\mathrm{Vect}_k \mapsto \mathcal{P}\mathrm{Lie}_k$.

Proposition A.10. *The functor $V \mapsto LV$ is left adjoint to the forgetful functor $\mathcal{P}\mathrm{Lie}_k \mapsto \mathcal{P}\mathrm{Vect}_k$.*

Proof. Let $LV = \lim_{\leftarrow} LV_\alpha$ where V_α are finite-dimensional vector spaces and $g = \lim_{\leftarrow} g_\beta$ where g_β are finite-dimensional nilpotent Lie algebras. We have

$$\begin{aligned}
\mathcal{P}\mathrm{Lie}_k(LV, g) &\cong \lim_{\leftarrow\,\beta} \mathcal{P}\mathrm{Lie}_k(LV, g_\beta) \\
&\cong \lim_{\leftarrow\,\beta} \lim_{\rightarrow\,\alpha} \mathcal{P}\mathrm{Lie}_k(LV_\alpha, g_\beta) \\
&\cong \lim_{\leftarrow\,\beta} \lim_{\rightarrow\,\alpha} \mathrm{Vect}_k(V_\alpha, g_\beta) \quad \text{by formula (A.1)} \\
&\cong \mathcal{P}\mathrm{Vect}_k(V, g). \qquad \square
\end{aligned}$$

Proposition A.11. *Let $g \in \mathcal{P}\mathrm{Lie}_k$ and LV be a pro-free Lie algebra. Then any continuous surjective map of Lie algebras $g \to LV$ admits a continuous splitting $LV \to g$.*

Proof. By Proposition A.10 a continuous splitting of $g \to LV$ in $\mathcal{P}\mathrm{Lie}_k$ is equivalent to the continuous splitting of $g \to V$ in $\mathcal{P}\mathrm{Vect}_k$. The latter exists by Corollary A.7. $\square$

Definition A.12. A continuous derivation of a pronilpotent Lie algebra g is a continuous map $f : g \to g$ which is a derivation of g in the usual sense. The space of all derivations of g will be denoted by $\mathrm{Der}(g)$. Clearly $\mathrm{Der}(g)$ is itself a Lie algebra under the commutator bracket.

Proposition A.13. *Any continuous derivation of a pro-free Lie algebra LV is determined by its restriction on V so there is an isomorphism of sets*

$$\mathrm{Der}(LV) \cong \mathcal{P}\mathrm{Vect}_k(V, LV).$$

Proof. The proof is similar to that of Proposition A.10. Instead of formula (A.1) we use the easily checked bijection of sets

$$\mathrm{Der}(LV) \cong \mathrm{Vect}_k(V, LV)$$

for a finite-dimensional space V. $\square$

Next, we discuss the relation of pronilpotent Lie algebras and Lie coalgebras.

Definition A.14. A Lie coalgebra over k is a (discrete) k-vector space V together with a map $\delta : V \to V \otimes V$ such that the k-dual map $V^* \otimes V^* \hookrightarrow (V \otimes V)^* \to V^*$ is skew-symmetric and satisfies the Jacobi identity. If V is finite dimensional we say that it is *conilpotent* if the corresponding Lie algebra on V^* is nilpotent. In general, we say that a Lie coalgebra is conilpotent if it is a union of its finite-dimensional conilpotent Lie subcoalgebras. We denote the category of conilpotent Lie coalgebras over k by $\mathcal{C}\mathrm{Lie}_k$.

Proposition A.15. *The category $\mathcal{C}\mathrm{Lie}_k$ is anti-equivalent to the category $\mathcal{P}\mathrm{Lie}_k$.*

Proof. The proof is the same as that of Proposition A.6. □

We define the a coderivation of a conilpotent Lie coalgebra V as the linear map $V \to V$ such that the dual map $V^* \to V^*$ is a continuous derivation of the associated pro-free Lie algebra V^*. Thus $\mathrm{Coder}(V) \cong \mathrm{Der}(V^*)$ is itself a Lie algebra.

Example A.16. Let H be a Hopf algebra over k and denote by I its augmentation ideal. Let $\Delta : H \to H \otimes H$ be the diagonal in H and $\tau : H \otimes H \to H \otimes H$ be the switch map. Denote by $\Delta' : H \to H \otimes H$ the composition $\tau \circ \Delta$. Then the map $\Delta - \Delta' : H \to H \otimes H$ induces a Lie cobracket on the space of indecomposables I/I^2. The corresponding Lie algebra will be isomorphic to the space of primitive elements in H^*. Now let $H = TV$, the tensor coalgebra on a space V endowed with the so-called cut coproduct:

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_k [v_1 \otimes \ldots \otimes v_k] \otimes [v_{k+1} \otimes \ldots \otimes v_n].$$

Together with the shuffle product the diagonal Δ makes TV into a Hopf algebra. Its space of indecomposables is the cofree Lie coalgebra whose dual is the pro-free Lie algebra LV^*.

We see, that the space of coderivations of a cofree Lie coalgebra on a space V is identified with the space of linear maps $TV \to V$.

References

[1] M. André, Méthode simpliciale en algébre homologique et algébre commutative, Lecture Notes in Mathematics, Vol. 32, Springer, Berlin, New York, 1967 (in French).

[2] M. Basterra, André–Quillen cohomology of commutative S-algebras, J. Pure Appl. Algebra 144 (2) (1999) 111–143.

[3] A.K. Bousfield, V.K.A.M. Gugenheim, On PL de Rham theory and rational homotopy type, Mem. Amer. Math. Soc. 8 (179) (1976).

[4] E.H. Brown, R.H. Szczarba, On the rational homotopy type of function spaces, Trans. Amer. Math. Soc. 349 (12) (1997) 4931–4951.

[5] Y. Félix, S. Halperin, J.-K. Thomas, Rational Homotopy Theory, Graduate Texts in Mathematics, Vol. 205, Springer, New York, 2001.
[6] Y. Félix, L. Menichi, J.-K. Thomas. Duality in Gerstenhaber algebras, arXiv: math.AT/0211229.
[7] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. 78 (2) (1963) 267–288.
[8] P. Goerss, M. Hopkins, André–Quillen (co)-Homology for Simplicial Algebras over Simplicial Operads, Contemporary Math., Vol. 265, Amer. Math. Soc., Providence, RI, 2000, pp. 41–85.
[9] A. Haefliger, Rational homotopy of the space of sections of a nilpotent bundle, Trans. Amer. Math. Soc. 273 (2) (1982) 609–620.
[10] D.K. Harrison, Commutative algebras and cohomology, Trans. Amer. Math. Soc. 104 (1962) 191–204.
[11] V. Hinich, Homological algebra of homotopy algebras, Comm. Algebra 25 (10) (1997) 3291–3323.
[12] V. Hinich, DG coalgebras as formal stacks, J. Pure Appl. Algebra 162 (2–3) (2001) 209–250.
[13] A. Lazarev, Cohomology theories for highly structured ring spectra, arXiv: math.AT/0211275.
[14] M. Mandell, Topological André–Quillen cohomology and E-infinity André–Quillen cohomology, Adv. Math. 177 (2) (2003) 227–279.
[15] D. Quillen, On the (co-)homology of commutative rings, Applications of categorical algebra, Proceedings of the Symposium on Pure Mathematics, Vol. XVII, New York, 1968, Amer. Math. Soc., Providence, RI, 1970, pp. 65–87.
[16] M. Schlessinger, J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, J. Pure Appl. Algebra 3 (2–3) (1985) 313–322.
[17] M. Schlessinger, J. Stasheff, Deformation theory and Rational Homotopy type, preprint.
[18] D. Sullivan, Infinitesimal computations in topology, Inst. Hautes Etudes Sci. Publ. Math. 47 (1977) 269–331 (1978).
[19] D. Tanré, Modéles de Chen, Quillen, Sullivan et applications aux fibrations de Serre, Lecture Notes in Mathematics, Vol. 1025, Springer, Berlin, 1983.
[20] C. Wilkerson, Applications of minimal simplicial groups, Topology 15 (2) (1976) 111–130.