

QUANTIZATION OF FOLIATIONS

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ABSTRACT

Using Gerstenhaber's deformation theory, we develop an analogue of the Poisson bracket for foliations associated to pre-symplectic manifolds. Our motivation for this is to understand better the algebra of Fourier integral operators associated to a coisotropic submanifold of the cotangent bundle by Guillemin, Sternberg and Uribe.

In this article, we will discuss a non-commutative analogue of the Poisson bracket for symplectic manifolds; we are motivated by the problem of finding a generalization of the symbol calculus for pseudodifferential operators associated to the algebras of Fourier integral operators discussed by Guillemin-Sternberg⁴ and Guillemin-Uribe⁵. Another motivation is to justify by some examples the following definition of Poisson structures in non-commutative geometry, which has been independently considered by Ping⁹. This definition will be explained in the first section, using Gerstenhaber's deformation theory².

Definition. *A Poisson structure on a (possibly non-commutative) algebra A is a two-cocycle $P \in Z^2(A, A)$ such that $P \circ P$ is a three-coboundary.*

Let M be a presymplectic manifold, that is, a manifold which carries a closed two-form of constant rank. This differential form defines a foliation F of the manifold, and we will study a Poisson structure on the convolution algebra of this foliation. This two-cocycle will depend on the presymplectic structure and on another piece of data which we call a Haar form. In this article, we only prove that $P \circ P$ is a coboundary when there is an invariant connection on the bundle $\tau = TM/F$; we will discuss the general case in another article.

Contact manifolds form an important class of presymplectic manifolds, where the kernel of the presymplectic form has dimension one. This case is simpler, because one may work with the crossed product algebra associated to the contact flow. Furthermore, there is a natural Haar form, namely the contact form. We study the

case of contact manifolds in the second section: it is already of interest, since it includes the case of geodesic flow on a Riemannian manifold.

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1. Poisson Structures on Non-commutative Algebras

Recall that a **Poisson algebra** is a commutative algebra A with Lie algebra structure $\{a_1, a_2\}$, such that

$$\{a_1, a_2 a_3\} = \{a_1, a_2\} a_3 + a_2 \{a_1, a_3\}.$$

The last term of this equation reverses the order of a_1 and a_2 , which means that it cannot be used as it stands to define a Poisson structure on a non-commutative algebra. In this section, we will explain how the correct notion of a Poisson structure on a non-commutative algebra is given by a two-cocycle P on A such that $P \circ P$ is a three-coboundary on A . The fact that P is a two-cocycle is the analogue of the Jacobi rule, while the fact that $P \circ P$ is a boundary is the analogue of Leibniz's rule. In this section, we will explain these formulas, which come from Gerstenhaber's deformation theory².

If A is an algebra, the space of **Hochschild k-cochains** on A is the complex $C^k(A, A) = \text{Hom}(A^{\otimes k}, A)$, with differential δ given by the formula

$$\begin{aligned} (\delta c)(a_1, \dots, a_{k+1}) &= a_1 c(a_2, \dots, a_{k+1}) \\ &+ \sum_{i=1}^k (-1)^i c(a_1, \dots, a_i a_{i+1}, \dots, a_{k+1}) + (-1)^{k+1} c(a_1, \dots, a_k) a_{k+1}. \end{aligned}$$

The cohomology of δ is called the Hochschild cohomology $H^\bullet(A, A)$ of A . For example, $H^0(A, A)$ is just the centre $Z(A)$ of A , while $H^1(A, A)$ is the space $\text{Out}(A) = \text{Der}(A)/\text{Inn}(A)$ of outer derivations of A .

The Hochschild cohomology of A carries a graded Lie bracket discovered by Gerstenhaber. Define an operation $C^k(A, A) \otimes C^l(A, A) \rightarrow C^{k+l-1}(A, A)$ by the formula

$$\begin{aligned} c_1 \circ c_2(a_1, \dots, a_{k+l-1}) \\ = \sum_{i=0}^{k-1} (-1)^{(k-i-1)(l-1)} c_1(a_1, \dots, a_i, c_2(a_{i+1}, \dots, a_{i+l}), a_{i+l+1}, \dots, a_{k+l-1}) \end{aligned}$$

for $c_1 \in C^k(A, A)$ and $c_2 \in C^l(A, A)$. Define the Gerstenhaber bracket

$$[c_1, c_2] = c_1 \circ c_2 - (-1)^{(k-1)(l-1)} c_2 \circ c_1.$$

This may be shown to define a graded Lie algebra structure on $C^{\bullet+1}(A, A)$ which descends to the Hochschild cohomology $H^{\bullet+1}(A, A)$.

Let $m \in C^2(A, A)$ be a two-cochain on A . The three-cochain $m \circ m$ is given by the formula

$$(m \circ m)(a_1, a_2, a_3) = m(a_1, m(a_2, a_3)) - m(m(a_1, a_2), a_3),$$

from which we see that that m defines an associative product on A if and only if $m \circ m = 0$. If m is the two-cochain corresponding to the product on A , then δc is given by the formula $\delta c = [m, c]$.

A **formal deformation** of an algebra A is an associative product on the vector space $A[[\nu]]$ over $\mathbb{C}[[\nu]]$ such that the induced product on $A = A[[\nu]]/\nu A[[\nu]]$ is the product of A . Such a deformation may be described by a cochain

$$m = \sum_{i=0}^{\infty} \nu^i m_i \in C^2(A, A)[[\nu]]$$

such that

- (1) $m \circ m = 0$, that is, $m(m(a_1, a_2), a_3) = m(a_1, m(a_2, a_3))$, and
- (2) $m_0(a_1, a_2) = a_1 a_2$.

The above conditions for m is a formal deformation may be rewritten as

$$m_0 \circ m_k + m_1 \circ m_{k-1} + \cdots + m_{k-1} \circ m_1 + m_k \circ m_0 = 0.$$

For $k = 1$, this equation says that m_1 is a Hochschild two-cocycle, while for $k = 2$, we see that

$$\delta m_2 + m_1 \circ m_1 = 0,$$

so that the cocycle $m_1 \circ m_1$ is a coboundary. Motivated by this, we make the following definition.

Definition 1.1. A **Poisson structure** on an algebra A is a two-cocycle $P \in Z^2(A, A)$ such that $P \circ P \in B^3(A, A) \subset Z^3(A, A)$ is a three-coboundary.

If A has a Poisson structure P , its Hochschild cohomology $H^\bullet(A, A)$ carries a differential

$$\delta_P c = [P, c].$$

Indeed, $\delta_P \delta_P c = [P, [P, c]] = [P \circ P, c] = 0$. Furthermore, δ_P satisfies the equation

$$\delta_P [c_1, c_2] = [\delta_P c_1, c_2] + (-1)^{|c_1|-1} [c_1, \delta_P c_2].$$

The following observation is due to Drinfeld; we will not make use of it, but it shows a relationship between Poisson structures and commutative Poisson algebras.

Proposition 1.2. *If $P \in Z^2(A, A)$ is a Poisson structure on an algebra A , the centre $Z(A)$ of A is a Poisson algebra, with Poisson bracket*

$$\{a_1, a_2\} = P(a_1, a_2) - P(a_2, a_1).$$

Proof. We define a Poisson bracket on the centre $Z(A) = H^0(A, A)$ of A by the formula

$$\{a_1, a_2\} = [a_1, \delta_P a_2],$$

which once more lies in $H^0(A, A)$. Let us check that this equals $P(a_1, a_2) - P(a_2, a_1)$:

$$\begin{aligned} [a_1, \delta_P a_2] &= -(P \circ a_2) \circ a_1 = -P(a_2, \cdot) \circ a_1 + P(\cdot, a_2) \circ a_1 \\ &= P(a_1, a_2) - P(a_2, a_1). \end{aligned}$$

This shows that the Poisson bracket is antisymmetric. To check that it satisfies the Jacobi rule, we observe that $m(a_1, a_2) = a_1 a_2 + \nu p(a_1, a_2)$ is an associative product on $A[[\nu]]/(\nu^2)$. Then

$$\{a_1, a_2\} \cong m(a_1, a_2) - m(a_2, a_1) \pmod{\nu^2},$$

and the Jacobi rule is satisfied by a commutator.

Finally, the Poisson bracket satisfies Leibniz's rule because the Gerstenhaber bracket $[\cdot, \cdot]$ does,

$$\begin{aligned} \{a_1, a_2 a_3\} &= [a_1, \delta_P(a_2 a_3)] = [a_1, (\delta_P a_2) a_3] + [a_1, a_2 (\delta_P a_3)] \\ &= \{a_1, a_2\} a_3 + a_2 \{a_1, a_3\}. \quad \square \end{aligned}$$

Next, we consider the case where A is an algebra over \mathbb{C} with anti-involution $*$. This induces an involution on $C^\bullet(A, A)$, given by the formula

$$c^*(a_1, \dots, a_k) = (-1)^{k(k-1)/2} c(a_k^*, \dots, a_1^*)^*.$$

(The sign is just the parity of the permutation $\binom{1 \dots k}{k \dots 1}$.) The two-cochain $m \in C^2(A, A)$ associated to an associative $*$ -product $(a_1, a_2) \mapsto a_1 a_2$ satisfies the equation

$$m(a_1^*, a_2^*) = m(a_2, a_1)^*,$$

in other words, $m^* = -m$.

Lemma 1.3. *If $c_1 \in C^k(A, A)$ and $c_2 \in C^l(A, A)$, then $(c_1 \circ c_2)^* = c_1^* \circ c_2^*$.*

Proof.

$$\begin{aligned} (c_1^* \circ c_2^*)^*(a_1, \dots, a_{k+l-1}) &= (-1)^{(k+l-1)(k+l-2)/2} \sum_{i=0}^{k-1} (-1)^{(k-i-1)(l-1)} \\ &\quad c_1^*(a_{k+l-1}^*, \dots, a_{k+l-i}^*, c_2^*(a_{k+l-i-1}^*, \dots, a_{k-i}^*), a_{k-i-1}^*, \dots, a_1^*)^* \\ &= \sum_{i=0}^{k-1} (-1)^{i(l-1)} c_1(a_1, \dots, a_{k-i-1}, c_2(a_{k-i}, \dots, a_{k+l-i-1}), a_{k+l-i}, \dots, a_{k+l-1}). \end{aligned}$$

Replacing i by $k - i - 1$, we obtain the result. \square

This lemma shows that $[c_1, c_2]^* = [c_1^*, c_2^*]$, and hence that

$$\delta c^* = [m, c^*] = [m^*, c]^* = -(\delta c)^*.$$

In defining a deformation of an associative $*$ -product, we will suppose that it remains a $*$ -product with respect to the $*$ -operation on $A[[\nu]]$ given by the formula

$$\left(\sum_{i=0}^{\infty} \nu^i a_i \right)^* = \sum_{i=0}^{\infty} (-\nu)^i a_i^*.$$

This amounts to requiring that $m_i^* = (-1)^{i+1} m_i$, and motivates the following definition.

Definition 1.4. *A Poisson structure P on a $*$ -algebra is a Poisson structure P on the underlying algebra such that $P^* = P$.*

2. Contact Structures

To the contact flow on a contact manifold is naturally associated a crossed-product algebra $C_c^\infty(M) \rtimes \mathbb{R}$. In this section, we define a natural two-cocycle P on this algebra, and study the question of when this cocycle defines a Poisson structure, that is, when $P \circ P$ is a three-coboundary on $C_c^\infty(M) \rtimes \mathbb{R}$. This turns out to be a difficult problem, since it requires one to have a good understanding of $H^3(C_c^\infty(M) \rtimes \mathbb{R}, C_c^\infty(M) \rtimes \mathbb{R})$. By adapting an idea of Lichnerowicz, we give a sufficient condition for P to be a Poisson structure; the condition is that M has an invariant connection on its tangent bundle.

Let M be a $2n + 1$ -dimensional contact manifold M , with contact one-form θ ; thus, $\theta \wedge (d\theta)^n$ is a volume form. Denote the exact two-form $d\theta$ by ω . Any contact manifold has local coordinate charts $\{t, \xi_i, x^i\}$ which are canonical, in the sense that

$$\theta = dt + \sum_{i=1}^n \xi_i dx^i \quad \text{and} \quad \omega = \sum_{i=1}^n d\xi_i \wedge dx^i.$$

The kernel $F = \{X \in TM \mid \iota(X)\omega = 0\}$ is a one-dimensional sub-bundle of TM . Denote by $T \in \Gamma(M, F)$ the unique section of F such that $\theta(T) = 1$; T is called the contact (or Reeb) vector field of M . Let $\varphi_t : M \times \mathbb{R} \rightarrow M$ be the flow on M generated by T . In a canonical chart, $T = \partial/\partial t$, and $\varphi_t(s, \xi, x) = (s + t, \xi, x)$.

Let $\tau \subset TM$ be the **transverse bundle** of the contact manifold, defined by

$$\tau = \{X \in TM \mid \theta(X) = 0\}.$$

It is clear that $TM = F \oplus \tau$. In a canonical chart, τ is spanned by the vector fields

$$\Xi^i = \frac{\partial}{\partial \xi_i} \quad \text{and} \quad X_i = \frac{\partial}{\partial x^i} - \xi_i \frac{\partial}{\partial t}.$$

Examples of contact manifolds are provided by codimension one submanifolds of an exact symplectic manifold. Recall that a symplectic manifold (Φ, η) is called **exact** if its symplectic form may be written $\eta = d\alpha$ for some one-form α . There is a unique vector field Z on such a manifold such that $\alpha = \iota(Z)\eta$, and this vector field is conformally Hamiltonian, in the sense that $\mathcal{L}(Z)\eta = \eta$. If M is a codimension one submanifold of Φ which is transverse to the vector field Z , then the restriction of the differential form $\alpha \wedge (d\alpha)^n$ to M (where the dimension of Φ is $2n + 2$) will be a volume form on M , and hence the restriction of α to M defines a contact structure.

The most important example of such a contact manifold is given by the cosphere bundle of a Riemannian manifold

$$S^*M = \{\xi \in T^*M \mid |\xi|^2 = 1\}.$$

The flow generated by the contact vector field on S^*M may be identified with the geodesic flow, if we think of each covector $\xi \in S^*M$ as corresponding to a vector $X = \langle \xi, \cdot \rangle$.

The crossed-product algebra $C_c^\infty(M) \rtimes \mathbb{R}$ is the algebra of smooth functions $C_c^\infty(M \times \mathbb{R})$ with product

$$(fg)(t) = \int_{-\infty}^{\infty} f(s)(\varphi_s^*g)(t - s) ds.$$

Following Connes¹, we think of $C_c^\infty(M) \rtimes \mathbb{R}$ as an algebra of functions on the quotient of M by the flow φ_t . The algebra $C_c^\infty(M) \rtimes \mathbb{R}$ has an anti-involution, defined by the formula

$$f^*(t) = \bar{f}(-t).$$

Denote by $\Lambda \in \Gamma(M, \Lambda^2 TM)$ the two-vector on M such that the bundle map

$$\alpha \mapsto \iota(\alpha)\Lambda : T^*M \rightarrow TM$$

vanishes on F^* , the sub-bundle of T^*M spanned by θ , and is the inverse of the operator $X \mapsto \iota(X)\omega$ for $X \in \tau$. In a canonical chart, Λ is given by the formula

$$\Lambda = \sum_{i=1}^n X_i \wedge \Xi^i.$$

The bilinear map from $C_c^\infty(M)$ to itself given by the formula $[f, g] = \langle \Lambda, df \wedge dg \rangle$ is called the **Jacobi bracket** of M . From it, we may construct a bilinear map from $C_c^\infty(M) \rtimes \mathbb{R}$ to itself by the formula

$$\{f, g\}(t) = \int_{-\infty}^{\infty} [f(s), \varphi_s^* g(t-s)] ds,$$

We call $\{\cdot, \cdot\}$ the **Poisson bracket** on $C_c^\infty(M) \rtimes \mathbb{R}$; it is easily seen that it is compatible with the anti-involution on $C_c^\infty(M) \rtimes \mathbb{R}$, that is, $\{f^*, g^*\} = -\{g, f\}^*$.

Proposition 2.1. *The Poisson bracket satisfies the formula*

$$f_0\{f_1, f_2\} - \{f_0 f_1, f_2\} + \{f_0, f_1 f_2\} - \{f_0, f_1\} f_2 = 0$$

for all $f_i \in C_c^\infty(M) \rtimes \mathbb{R}$; that is, it is a two-cocycle.

Proof. Since $[T, \Lambda] = 0$, we see that $\varphi_t^*[f, g] = [\varphi_t^* f, \varphi_t^* g]$. Thus, we have

$$\begin{aligned} \{f_0 f_1, f_2\}(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_0(u) \varphi_u^* f_1(s-u), \varphi_s^* f_2(t-s)] du ds \\ &= (f_0\{f_1, f_2\})(t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_u^* f_1(s-u) [f_0(u), \varphi_s^* f_2(t-s)] du ds. \end{aligned}$$

Similarly,

$$\{f_0, f_1 f_2\}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_u^* f_1(s-u) [f_0(u), \varphi_s^* f_2(t-s)] du ds + (\{f_0, f_1\} f_2)(t).$$

From this, the desired formula is clear. \square

We will denote the two-cocycle corresponding to the Poisson bracket by P . In the rest of this section, we will give a sufficient condition for the three-cochain $P \circ P \in C^3(C_c^\infty(M) \rtimes \mathbb{R}, C_c^\infty(M) \rtimes \mathbb{R})$, given by the formula

$$\{\{f, g\}, h\} - \{f, \{g, h\}\}$$

to be a coboundary. To do this, we recall Lichnerowicz's notion of a contact connection⁸.

Definition 2.2. A **contact connection** on a contact manifold (M, θ) is a torsion-free connection on its tangent bundle such that $\nabla T = 0$ and $\nabla \omega = 0$. An *invariant contact connection* is a contact connection invariant under the contact flow.

The existence of an invariant contact connection seems to be rather rare: for example, restricting oneself to the case of a sphere bundle, there is such an invariant connection if the Riemannian manifold is flat, or locally symmetric of negative sectional curvature (for this second case, see Kanai⁷). However, Lichnerowicz has shown how an invariant contact connection may be constructed from any connection on the tangent bundle of M invariant under the contact flow.

Given an invariant contact connection on M , we may form the operator

$$\nabla^2 = \nabla(df) : C_c^\infty(M) \rightarrow \Gamma(M, S^2T^*M).$$

Since ∇ is invariant under the vector field T , it follows that $\varphi_s^* \nabla^2 = \nabla^2 \varphi_s^*$. Let P_2 be the two-cochain on $C_c^\infty(M) \times \mathbb{R}$ defined by the formula

$$P_2(f, g)(t) = \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, \nabla^2 f(s) \otimes \nabla^2(\varphi_s^* g)(t-s) \rangle ds,$$

where the tensor $\Lambda \otimes \Lambda$ is paired with $S^2T^*M \otimes S^2T^*M$ by the formula

$$\Lambda^{ij} \Lambda^{kl} \alpha_{ik} \beta_{jl}.$$

Note that $P_2(f^*, g^*) = P_2(g, f)^*$, that is, $P_2^* = -P_2$.

Proposition 2.3. *If ∇ is a contact connection on M invariant under the vector field T , then $\delta P_2 + P \circ P = 0$, in other words,*

$$\{\{f, g\}, h\} - \{f, \{g, h\}\} = fP_2(g, h) - P_2(fg, h) + P_2(f, gh) - P_2(f, g)h.$$

Proof. Since ∇ is a contact connection, it satisfies the formula $\nabla \Lambda = 0$: from this and its invariance, we see that

$$d \langle \Lambda, df(u) \wedge \varphi_u^* dg(s-u) \rangle = \langle \Lambda, \nabla^2 f(u) \otimes \varphi_u^* dg(s-u) + df(u) \otimes \varphi_u^* \nabla^2 g(s-u) \rangle,$$

from which it follows that

$$\begin{aligned} \{\{f, g\}, h\}(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, \nabla^2 f(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* dh(t-s) \\ &\quad + df(u) \otimes \varphi_u^* \nabla^2 g(s-u) \otimes \varphi_s^* dh(t-s) \rangle du ds \\ \{f, \{g, h\}\}(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, df(u) \otimes \varphi_u^* \nabla^2 g(s-u) \otimes \varphi_s^* dh(t-s) \\ &\quad + df(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* \nabla^2 h(t-s) \rangle du ds. \end{aligned}$$

Taking the difference, two terms cancel, and we obtain the formula

$$\begin{aligned} & \{\{f, g\}, h\}(t) - \{f, \{g, h\}\}(t) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, \nabla^2 f(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* dh(t-s) \\ & \quad - df(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* \nabla^2 h(t-s) \rangle du ds. \end{aligned}$$

On the other hand, using the formula $\nabla^2(fg) = \nabla^2 fg + 2df \otimes dg + f\nabla^2 g$, we see that

$$\begin{aligned} fP_2(g, h) - P_2(fg, h) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, \nabla^2 f(u) \varphi_u^* g(s-u) \otimes \varphi_s^* dh(t-s) \\ & \quad + df(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* dh(t-s) \rangle du ds. \end{aligned}$$

Similarly, we have

$$\begin{aligned} -P_2(f, gh) + P_2(f, g)h &= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \Lambda \otimes \Lambda, df(u) \otimes \varphi_u^* dg(s-u) \otimes \varphi_s^* dh(t-s) \\ & \quad + df(u) \varphi_u^* g(s-u) \otimes \varphi_s^* \nabla^2 h(t-s) \rangle du ds. \end{aligned}$$

The proposition follows. \square

3. Presymplectic Structures

We now turn to defining a Poisson bracket in the more general setting of presymplectic manifolds.

Definition 3.1. *A presymplectic manifold M is a manifold endowed with a closed two-form ω of constant rank.*

Let M be a presymplectic manifold, and let $F \subset TM$ be the subbundle on which ω vanishes. The following lemma shows that F is integrable; it is a special case of the Frobenius integrability theorem.

Lemma 3.2. *The bundle F is integrable and thus defines a foliation of M .*

Proof. For all X and $Y \in \Gamma(M, F)$ and $Z \in \Gamma(M, TM)$, we see that

$$\begin{aligned} 0 &= d\omega(X, Y, Z) = X \cdot \omega(Y, Z) - Y \cdot \omega(X, Z) + Z \cdot \omega(X, Y) \\ & \quad - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \\ &= -\omega([X, Y], Z), \end{aligned}$$

and hence that $[X, Y]$ is a section of F . \square

We have the following Darboux theorem for presymplectic manifolds.

Proposition 3.3. *If (M, ω) is a presymplectic manifold, there is for each $x \in M$ a coordinate neighborhood U with coordinates $\{x_1, \dots, x_q, y_1, \dots, y_p\}$ such that*

$$\omega = \sum_{i=1}^{q/2} dx_i \wedge dx_{i+1}.$$

Proof. Let $U = U_1 \times U_2 \subset \mathbb{R}^q \times \mathbb{R}^p$ be a foliation chart, where the leaves of U have the form $\{x\} \times U_2$. The form ω is basic with respect to the subbundle F , that is, for any $X \in \Gamma(M, F)$ one has $\iota(X)\omega = 0$ and $\iota(X)d\omega = 0$. Hence for any such foliation chart, ω is pulled back from a closed two-form on U_1 , which is in fact a symplectic form. The theorem follows from the Darboux theorem for symplectic manifolds applied to U_1 . \square

Given a foliated manifold (M, F) , we let $\tau = TM/F$ denote the transverse bundle to F .

Recall that a submanifold M of a symplectic manifold (Φ, η) is called **coisotropic** if the tangent space $T_x M \subset T_x \Phi$ contains a Lagrangian subspace of $T_x \Phi$ for all $x \in M$. In the following proposition, we show that in fact all presymplectic manifolds have this form (see Gotay³).

Proposition 3.4. *Let (M, ω) be a presymplectic manifold. Then there is a symplectic manifold (Φ, η) and an embedding $i : M \rightarrow \Phi$ such that $\omega = i^*\eta$ and M is a coisotropic submanifold of Φ .*

Proof. Let F^* be the dual of the integrable bundle F , let $\pi : F^* \rightarrow M$ be the projection to the base, and let $i : M \rightarrow F^*$ be the embedding by the zero section. If we choose a splitting of the short exact sequence

$$0 \rightarrow F \rightarrow TM \rightarrow \tau \rightarrow 0,$$

we obtain an embedding j of F^* in T^*M . Let $j^*\omega_{T^*M}$ be the pull-back of the canonical symplectic form on T^*M to F^* ; it is easy to see that $i^*j^*\omega_{T^*M} = 0$, and hence that the restriction of $\eta = \pi^*\omega + j^*\omega_{T^*M}$ to M equals ω . Furthermore, there is a tubular neighborhood $\Phi \subset F^*$ of M so that η restricted to Φ is non-degenerate. \square

In the rest of this section, we will show that presymplectic structures are essentially the same thing as transverse symplectic structures, as defined by Haefliger⁶; this result is not needed to read the rest of the paper. Recall that a transversal N to a foliation (M, F) is an immersion $N \rightarrow M$ which is transverse to F , such that the dimension of N is equal to the codimension of the foliation. We say that N is complete if it intersects each leaf of M ; every foliation has a complete transversal.

Definition 3.5. A *transverse symplectic structure* on a foliated manifold (M, F) is a symplectic structure ω on a complete transversal N such that for all diffeomorphisms h in the holonomy pseudogroup we have $h^*\omega = \omega$.

It follows from this definition that any transversal N of a foliation (M, F) with a transversal symplectic structure inherits a symplectic structure. Recall that a foliation is given by a covering U_i and submersions $f_i : U_i \rightarrow N_i$. The N_i have the same dimension as the codimension of the foliation and there are transition functions $h_{ij} : f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ which satisfy $f_i = h_{ij} \circ f_j$. Since we may identify the manifolds N_i with transversals, we obtain symplectic forms ω_i on N_i . The diffeomorphisms h_{ij} are the generators of the holonomy pseudogroup and thus $\omega_j = h_{ij}^*\omega_i$ on $f_j(U_i \cap U_j)$. The pullback $f_i^*\omega_i$ is closed and

$$f_i^*\omega_i = (g_{ij} \circ f_j)^*\omega_i = f_j^* \circ h_{ij}^*\omega_i = f_j^*\omega_j$$

so that the forms $f_i^*\omega_i$ agree on overlaps and so define a global closed two-form $\omega \in \Omega^2(M)$. Of course, ω is not non-degenerate, but its kernel is precisely the bundle F of vectors tangent to the foliation. Hence we have proved the following result.

Proposition 3.6. A *transverse symplectic structure* on (M, F) determines in a unique manner a *presymplectic structure* ω on M such that F is the kernel of the form ω .

Let (M, ω) be a presymplectic manifold. If N is a transversal to the foliation F defined by ω , the restriction of the form ω to N defines a symplectic structure on N , since $TN \cong \tau|_N$ and the restriction of ω to N coincides with the form ω_τ on τ .

Proposition 3.7. If (M, ω) is a presymplectic manifold, there is a naturally induced *transverse symplectic structure* on (M, F) which corresponds to ω by the above proposition.

Proof. Let N denote any complete transversal. Then as above, ω restricts to define a symplectic structure ω_N on N . We only have to show that ω is invariant under the holonomy pseudogroup. Use the notation above for $f_i : U_i \rightarrow N_i$ and ω_i for the restrictions of ω to N_i . The holonomy pseudogroup is generated by the h_{ij} and we will show that $\omega_j = h_{ij}^*\omega_i$. But this is a direct consequence of the fact that $\pi : TN_i \rightarrow \tau|_{N_i}$ preserves the symplectic forms ω . \square

4. Haar Forms

Let (M, F) be a foliation, with $\dim(F) = p$ and $\dim(M) = p + q$. In this section, we will study the geometry of a foliation enriched with three extra pieces

of geometric structure:

- (1) a **connection**, that is, a choice of a splitting of the short exact sequence

$$0 \rightarrow F \rightarrow TM \rightarrow \tau \rightarrow 0;$$

- (2) a **Haar measure**, that is, a nowhere-vanishing section of the longitudinal density bundle $|F|$ on M ;
 (3) an orientation of the bundle F .

The two pieces of data (2) and (3) are equivalent to the giving of a nowhere-vanishing section of the bundle $\Lambda^p F^*$ over M . Let $\alpha \in \Omega^p(M)$ be a differential form which restricts to such a nowhere-vanishing section of $\Lambda^p F^*$; we will call such a differential form a **Haar form**. Sullivan¹¹ has observed that such a form determines the above three pieces of data: the orientation of F and the nowhere-vanishing density are defined by the restriction of α to F , while the splitting of $TM \cong \tau \oplus F$ is obtained by identifying τ with the kernel of the surjective map

$$\Gamma(M, TM) \ni X \mapsto \iota(X)\alpha \in \Gamma(M, \Lambda^{p-1} F^*).$$

Conversely, to the above data, we may associate a p -form α for which $\iota(Y)\alpha = 0$ if Y is a section of the supplementary bundle τ determined by the connection, and $\alpha(X_1, \dots, X_p) = 1$ if (X_1, \dots, X_p) is an oriented frame of F of volume one. This Haar form was first studied by Rummeler¹⁰. A Haar form of this type is called a **pure** Haar form. Informally, we have a fibration

$$\{\text{Haar forms}\} \rightarrow \{\text{connections}\} \times \{\text{Haar measures}\} \times \{\text{orientations}\}$$

Let E be a vector bundle on M . The connection defines a decomposition of the space of differential forms on M , according to the number of longitudinal and transverse indices:

$$\Omega^{i,j}(M, E) = \Gamma(M, \Lambda^i F^* \otimes \Lambda^j \tau^* \otimes E).$$

For example, a Haar form α lies in

$$\Omega^{p,0}(M) \oplus \sum_{n \geq 2} \Omega^{p-n,n}(M),$$

and α is pure if and only if it lies in $\Omega^{p,0}(M)$. A covariant derivative ∇ on E may be decomposed into three homogeneous components

$$\nabla = \nabla_1 + \nabla_0 + \nabla_{-1},$$

where $\nabla_s : \Omega^{i,j}(M, E) \rightarrow \Omega^{i+s, j-s+1}(M, E)$. To see this, it suffices to check the result for sections of E and elements of $\Omega^{1,0}(M) = \Gamma(M, F^*)$, for which it is obvious, and elements of $\Omega^{0,1}(M) = \Gamma(M, \tau^*)$, for which it follows from the fact that F is integrable.

If α is a Haar form, there is a unique section $k \in \Gamma(M, \tau^*)$ such that

$$d_0\alpha - k \wedge \alpha \in \sum_{s \geq 1} \Omega^{p-s, s+1}(M);$$

we call k the **mean curvature** of the Haar form α . If α is pure, we see that we actually have the formula $d_0\alpha = k \wedge \alpha$. Calling k the mean curvature is justified by the following result of Rummmler¹⁰.

Proposition 4.1. *Let (M, F) be a foliation with Haar form α . Let g be any Riemannian metric on M for which the sub-bundles F and τ determined by the connection are orthogonal, and which induces the Haar measure on the bundle F . Then k equals the mean curvature of the leaves of the foliation determined by F in M .*

We say that a Haar form α is **taut** if its mean curvature vanishes. An example of a taut Haar form is given by a contact manifold (M, α) . Since the two-form $d\alpha$ has constant rank q , its kernel F is a one-dimensional bundle, which defines a foliation of M . The differential form α is a taut Haar form for this foliation, and in fact it is pure as well.

5. The Holonomy Groupoid

The **holonomy groupoid** \mathcal{G} of the foliation (M, F) is a smooth manifold (although not necessarily Hausdorff) of dimension $2p+q$, with smooth maps $s : \mathcal{G} \rightarrow M$ and $t : \mathcal{G} \rightarrow M$ called the source and target. We can pull back the foliation on M by either of these maps: these foliations coincide, and define a foliation on \mathcal{G} of codimension q . If $\mathcal{G}^{(2)}$ is the fibre-product

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G}^2 \mid t(\gamma_1) = s(\gamma_2)\},$$

then the multiplication map $m(\gamma_1, \gamma_2) = \gamma_1 \circ \gamma_2$ is a smooth map from $\mathcal{G}^{(2)}$ to \mathcal{G} , and $\partial(\gamma_1, \gamma_2) = t(\gamma_1) = s(\gamma_2)$ is a smooth map from $\mathcal{G}^{(2)}$ to M .

Definition 5.1. *A \mathcal{G} -equivariant vector bundle E on (M, F) is a vector bundle E over M along with an isomorphism between the pull-backs s^*E and t^*E on \mathcal{G} .*

*An **invariant** connection on a \mathcal{G} -equivariant bundle E is a connection ∇ on E such that the two connections $s^*\nabla$ and $t^*\nabla$ correspond to each other under the identification of s^*E with t^*E .*

An important example of an equivariant bundle is the transverse bundle τ of the foliation. On this bundle, it is natural to restrict attention to Bott connections.

Definition 5.2. A **Bott connection** on the transverse bundle τ is a connection such that

$$\nabla_Z X_\tau = [Z, X]_\tau$$

for $Z \in \Gamma(M, F)$ and $X \in \Gamma(M, TM)$. Here, $X \mapsto X_\tau$ denotes the image of a vector field under projection to $\Gamma(M, \tau)$.

If $X_\tau = Y_\tau$, then $[Z, X - Y] \in \Gamma(M, F)$, so that

$$\nabla_Z X_\tau - \nabla_Z Y_\tau = [Z, X - Y]_\tau = 0.$$

Thus, the above definition is meaningful.

Definition 5.3. The **torsion** of a Bott connection on the transverse bundle τ is the section $T \in \Omega^{0,1}(M, \text{End}(\tau))$ given by the formula

$$T(X_\tau)Y_\tau = \nabla_X Y_\tau - \nabla_Y X_\tau - [X, Y]_\tau$$

for $X, Y \in \Gamma(M, TM)$. (It is easily shown that this is well-defined.)

If E is an equivariant vector bundle on (M, F) , we may define $\Gamma_c(\mathcal{G}, E)$ in the same way as in the case $E = \mathbb{R}$: its elements are finite sums of smooth sections of s^*E of compact support defined in distinguished charts of \mathcal{G} (see Connes¹). Given a Haar form α on (M, F) , the convolution $e * f$ of two sections $e \in C_c^\infty(\mathcal{G}, E)$ and $f \in C_c^\infty(\mathcal{G}, F)$ is the section of the equivariant bundle $E \otimes F$ defined by restricting $e \boxtimes f \in C_c^\infty(\mathcal{G} \times \mathcal{G}, E \boxtimes F)$ to $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ and applying the integral over the fibres m_* to the differential form $e \boxtimes f|_{\mathcal{G}^{(2)}} \partial^* \alpha$:

$$e * f = m_*(e \boxtimes f|_{\mathcal{G}^{(2)}} \partial^* \alpha).$$

The following proposition is an easy generalization of the result Connes¹, that convolution defines an associative product on $C_c^\infty(\mathcal{G})$.

Proposition 5.4. If E_i are \mathcal{G} -equivariant bundles on (M, F) , $i = 1, 2, 3$, and $e_i \in \Gamma_c(\mathcal{G}, E_i)$, then $e_1 * (e_2 * e_3) - (e_1 * e_2) * e_3 \in \Gamma_c(\mathcal{G}, E_1 \otimes E_2 \otimes E_3)$ vanishes.

It is on the non-commutative algebra $C_c^\infty(\mathcal{G})$ that we will define a Poisson structure, where \mathcal{G} is the holonomy groupoid attached to a pre-symplectic manifold M . If the foliation of M is given by a nowhere-zero vector field X , the convolution algebra $C_c^\infty(\mathcal{G})$ is the crossed-product algebra $C_c^\infty(M) \rtimes \mathbb{R}$ associated to the flow of X , as long as the flow has no periodic orbits with holonomy of finite order. This is why we have employed the algebra $C_c^\infty(M) \rtimes \mathbb{R}$ when M is a contact manifold.

6. The Poisson Bracket

If (M, F) is a foliation with Haar form α , and if E is an equivariant bundle on E with invariant connection ∇ , let

$$\nabla_0 : \Gamma_c(\mathcal{G}, E) \rightarrow \Omega_c^{0,1}(\mathcal{G}, E)$$

be the zero-component of the connection $s^*\nabla = t^*\nabla : \Gamma_c(M, E) \rightarrow \Omega_c^1(\mathcal{G}, E)$. The following result shows that ∇_0 violates Leibniz's rule by a term involving the mean curvature of the Haar form α .

Proposition 6.1. *If E and F are equivariant bundles with invariant connections, and $e \in \Gamma_c(\mathcal{G}, E)$ and $f \in \Gamma_c(\mathcal{G}, F)$, we have the formula*

$$\nabla_0(e * f) = (\nabla_0 e) * f + e * (\nabla_0 f) + e * (t^*k)f.$$

Proof. If we apply Leibniz's rule to the formula $e * f = m_*(e \boxtimes f|_{\mathcal{G}^{(2)}} \partial^* \alpha)$, we see that

$$\begin{aligned} \nabla_0(e * f) &= m_* \left((\nabla_0 e \boxtimes f|_{\mathcal{G}^{(2)}} + e \boxtimes \nabla_0 f|_{\mathcal{G}^{(2)}}) \partial^* \alpha \right) + m_*(e \boxtimes f|_{\mathcal{G}^{(2)}} \partial^* d_0 \alpha) \\ &= (\nabla_0 e) * f + e * (\nabla_0 f) + m_*(e \boxtimes f|_{\mathcal{G}^{(2)}} \partial^* k \wedge \partial^* \alpha). \quad \square \end{aligned}$$

Corollary 6.2. *The operator*

$$De = \nabla_0 e + \frac{1}{2}((t^*k)e + e(s^*k))$$

*satisfies Leibniz's rule: $D(e * f) = (De) * f + e * (Df)$.*

We can now define a two-cocycle on the convolution algebra $C_c^\infty(\mathcal{G})$ associated to a presymplectic manifold (M, ω) with Haar form α ; this will be our candidate Poisson bracket. Using the invariant two-form ω , we obtain an isomorphism between the bundles τ and τ^* on \mathcal{G} and hence a skew symmetric form on τ^* , which we denote by Λ . Then the Poisson bracket of two functions f_1 and $f_2 \in C_c^\infty(\mathcal{G})$ is defined by the formula

$$\{f_1, f_2\} = \langle \Lambda, Df_1 * Df_2 \rangle.$$

Note that $\{f_1^*, f_2^*\} = -\{f_2, f_1\}^*$. The following result generalizes Proposition 2.1, and its proof is essentially the same.

Proposition 6.3. *The Poisson bracket is a two-cocycle, that is, for all $f_i \in C_c^\infty(\mathcal{G})$, it satisfies the formula*

$$f_0 * \{f_1, f_2\} - \{f_0 * f_1, f_2\} + \{f_0, f_1 * f_2\} - \{f_0, f_1\} * f_2 = 0.$$

We can show that the Poisson bracket introduced above is a Poisson structure on $C_c^\infty(\mathcal{G})$, as long as there is an invariant Bott connection on the transverse bundle τ . The following construction is once more inspired by similar results of Lichnerowicz.

Definition 6.4. Let (M, ω) be a presymplectic manifold. A **presymplectic connection** on M is an invariant Bott connection on τ such that $\nabla\Lambda = 0$.

Proposition 6.5. Let (M, ω) be a presymplectic manifold. There is a polynomial map from the space of invariant Bott connections on τ to the space of presymplectic connections.

Proof. We will give an explicit formula which, given an invariant Bott connection, produces another connection which is an invariant Bott connection, is torsion-free, and satisfies the equation $\nabla\omega = 0$. Since this equation implies that $\nabla\Lambda = 0$, this will give a proof of the proposition.

We start by replacing ∇ by the torsion-free connection $\nabla' = \nabla - \frac{1}{2}T$, where $T \in \Omega^{0,1}(M, \text{End}(\tau))$ is the torsion of the connection ∇ . Note that ∇' is an invariant Bott connection.

The longitudinal covariant derivative $\nabla'\omega$ vanishes, because ω is invariant under the holonomy groupoid and ∇' is a Bott connection. It follows that S , defined by the equation

$$\omega(S(Z)X, Y) = \frac{1}{3} \{(\nabla'_Z\omega)(X, Y) + (\nabla'_X\omega)(Z, Y)\},$$

is an element of $S \in \Omega^{0,1}(M, \text{End}(\tau))$.

We define a new torsion-free, invariant Bott connection by the formula $\nabla'' = \nabla' + S$. It follows that

$$\begin{aligned} (\nabla''_Z\omega)(X, Y) &= (\nabla'_Z\omega)(X, Y) - \omega(S(Z)X, Y) - \omega(X, S(Z)Y) \\ &= \frac{1}{3} \{(\nabla'_X\omega)(Y, Z) + (\nabla'_Y\omega)(Z, X) + (\nabla'_Z\omega)(X, Y)\} \\ &= d\omega(X, Y, Z) = 0, \end{aligned}$$

since ∇' is torsion-free and $d\omega = 0$. This shows that $\nabla''\omega = 0$. \square

Thus, suppose that we have a presymplectic connection ∇ on M . Denote by

$$D^2 : C_c^\infty(\mathcal{G}) \rightarrow \Gamma_c(\mathcal{G}, \tau^* \otimes \tau^*)$$

the composition of $D : C_c^\infty(\mathcal{G}) \rightarrow \Gamma_c(\mathcal{G}, \tau^*)$ with $D^{\tau^*} : \Gamma_c(\mathcal{G}, \tau^*) \rightarrow \Gamma_c(\mathcal{G}, \tau^* \otimes \tau^*)$; since ∇ is torsion-free, D^2 actually takes values in $\Gamma_c(\mathcal{G}, S^2\tau^*)$. Let P_2 be the two-cochain on $C_c^\infty(\mathcal{G})$ defined by the formula

$$P_2(f, g) = \langle \Lambda \otimes \Lambda, D^2 f * D^2 g \rangle,$$

where the tensor $\Lambda \otimes \Lambda$ is paired with $S^2\tau^* \otimes S^2\tau^*$ by $\Lambda^{ij}\Lambda^{kl}\alpha_{ik}\beta_{jl}$. The following result is a generalization of Proposition 2.3.

Proposition 6.6. *Let ∇ be a presymplectic connection on (M, ω) . Then*

$$\{\{f, g\}, h\} - \{f, \{g, h\}\} = f * P_2(g, h) - P_2(f * g, h) + P_2(f, g * h) - P_2(f, g) * h,$$

so that the two-cochain $P(f, g) = \{f, g\}$ defines a Poisson structure on the algebra $C_c^\infty(\mathcal{G})$.

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