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Equivariant bicycles on singular spaces

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Abstract – We give a geometric description of the equivariant Kasparov groups $KK_G(X, Y)$ for Thom-Mather stratified spaces X and Y. This generalizes a theorem of Connes and Skandalis.

Bicycles équivariants sur les espaces singuliers

Résumé – *Nous donnons une réalisation géométrique du foncteur bivariant équivariant de Kasparov* $KK_G(X, Y)$ *pour les espaces stratifiés de Thom-Mather. Cette réalisation généralise un théorème de Connes et Skandalis.*

Version française abrégée – Soient X et Y deux espaces stratifiés au sens de Thom-Mather [7]. Nous définissons une K-théorie bivariante KK(X, Y). Cette théorie est covariante en X et contravariante en Y. Pour construire le groupe KK(X, Y) on introduit la notion de cycle bivariant, que nous appelons bicycle, sur (X, Y). Il s'agit d'un quadruple (Z, E, f, g) tel que :

1. Z est un espace stratifié.
2. E est un fibré vectoriel complexe sur Z.
3. $f: Z \to X$ est une application continue et propre.
4. $g: Z \to Y$ est une application continue et non singulière normalement [6] dont le fibré normal possède une structure $Spin^c$.

Pour X et Y fixés on munit la classe de tous les bicycles d'une relation d'équivalence tout à fait analogue à celle de [1]. L'ensemble des classes d'équivalence est alors un groupe abélien et c'est le groupe KK(X, Y). Notre résultat principal donne, en utilisant la transversalité pour les espaces stratifiés, une construction simple et directe du produit d'intersection

$$KK(X, Y) \underset{Z}{\otimes} KK(Y, W) \to KK(X, W).$$

L'opérateur de Dirac détermine un isomorphisme de groupes abéliens

$$KK(X, Y) \to KK(C_0(X), C_0(Y))$$

qui transforme notre produit d'intersection en celui de Kasparov.

Soit maintenant G un groupe de Lie compact qui opère sur X et Y. Nous adaptons notre méthode à la situation équivariante, ce qui donne le groupe abélien $KK_G(X, Y)$. Dans ce cas notre définition du produit d'intersection

$$KK_G(X, Y) \underset{Z}{\otimes} KK_G(Y, W) \to KK_G(X, W)$$

utilise non seulement la transversalité, mais aussi, et surtout, la périodicité de Bott considérée géométriquement. Ainsi le système des groupes $KK_G(X, Y)$ apparaît comme le cadre naturel de la théorie d'intersection (topologique) équivariante.

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1. Introduction. – In [1] the first author and R. Douglas gave a geometric realization of K-homology. This approach to K-homology and K-theory was further developed in two papers of Connes and Skandalis ([2], [3]) in which they used bicycles (*i.e.* bivariant

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cycles) to construct elements of the Kasparov group $KK(C_0(X), C_0(Y))$. Connes and Skandalis referred to bicycles as correspondences. They considered the case when X was a locally compact Hausdorff topological space and Y was a manifold. The main point about bicycles is that the Kasparov product has a simple geometric description. In [2], they organized the collection of bicycles into a group $KK(X, Y)$, constructed a map

$$\mu : KK(X, Y) \to KK(C_0(X), C_0(Y))$$

and stated that it was an isomorphism. In this Note we outline the proof of this isomorphism together with two natural generalizations.

First, we allow Y to be a stratified space in the sense of Whitney or Mather [7]. For concreteness we have stated all theorems for a stratified space with Thom-Mather data since it is proved in [7] that a Whitney stratified space can be equipped with such data. To generalize to this case we rely heavily on Goresky's π-fiber condition and the notion of a normally non-singular map $f: Z \to Y$ between stratifed spaces. We generalize the definition of $f!$ to the case of a Spin^c normally non-singular map and prove the fundamental formula $f_1 ! \underset{Y}{\otimes} f_2 ! = (f_2 \circ f_1)!$ It is precisely the notion of normally non-singular map which makes this program possible. As a spin off we obtain geometric realizations of cobordism groups $\Omega^*(Y)$, compare [8], [6] and of bivariant cobordism groups $\Omega\Omega^*(X, Y)$. These realizations lead immediately to equivariant versions of the theories $KK^*_G(X, Y)$ and $\Omega\Omega^*_G(X, Y)$. The crucial difference between $KK^*_G(X, Y)$ and $\Omega\Omega^*_G(X, Y)$ is that Bott periodicity provides a replacement for transversality in $KK^*_G(X, Y)$ and no such construction is apparent for $\Omega\Omega^*_G(X, Y)$. Therefore $\Omega\Omega^*_G(X, Y)$ is considerably more difficult to compute. Another way to think of this is that $KK^*_G(X, Y)$ is the appropriate setting for (topological) equivariant intersection theory.

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2. STRATIFIED SPACES AND NORMALLY NON-SINGULAR MAPS. — Throughout, by a stratified space Y we will mean a Thom-Mather stratified space ([7], [9]). Thus Y is partitioned into strata $Y = \bigcup_{S \in \mathscr{S}_Y} S$ and equipped with control data (T_S, π_S, ρ_S). Let $T_S(\varepsilon) = \{x \in T_S \mid \rho(x) < \varepsilon\}$. If a compact Lie group G acts continuously on Y, it should act smoothly on each strata, the stratification should be G-invariant and the control data compatible with the group action. A stratified G-subspace $X \subset Y$ is a locally closed G-subspace which is also stratified such that each stratum $S \in \mathscr{S}_X$ is contained in a unique stratum of Y. A stratified subspace satisfies the *π-fiber condition* [6] if for each stratum $S \in \mathscr{S}_Y$, there exists an $\varepsilon > 0$ such that

$$X \cap T_S(\varepsilon) = \pi_S^{-1}(X \cap S) \cap T_S(\varepsilon). \tag{1}$$

A G-map $f: X \to Y$ between stratified spaces is *stratified* if for each stratum $S \in \mathscr{S}_X$ there is a stratum $S' \in \mathscr{S}_Y$ so that $f: S \to S'$ is smooth. f is called *controlled* if for the strata S and S' above, there is an $\varepsilon > 0$ so that for $p \in T_S(\varepsilon)$ and $f(p) \in T_{S'}(\varepsilon)$ one has $\pi_{S'} f(p) = f \pi_S(p)$. f is called a *normally non-singular inclusion of condimension k* if f embeds X in Y as a π-fibered stratified G-subspace such that for each $x \in X$ contained in a stratum $S \in \mathscr{S}_X$ with $f(x) \in S' \in \mathscr{S}_Y$, one has $\dim(S') - \dim(S) = k$. And in general f is called a *normally non-singular map of codimension k* if there is a factorization of f

$$f: X \xrightarrow{i} Y \times \mathbf{R}^n \xrightarrow{p} Y \tag{2}$$

where i is a normally non-singular inclusion of codimension $n+k$ and p is the obvious projection, $\mathbf{R}^n$ is equipped with a linear action of G. The key property of normally non-singular maps is that they have normal bundles.

LEMMA 2.1. — 1. *If* $i: X \to Y$ *is a codimension* k *normally non-singular inclusion, then there exists a G-vector bundle* $\nu(i)$ *over* X *of rank* k *and an equivariant embedding* $\varphi: \nu(i) \to Y$ *mapping* $\nu(i)$ *onto an open neighborhood of* X.

2. *If* $f: X \to Y$ *is a normally non-singular map of codimension* k, *then the normal bundle* $\nu(i)$ *is unique up to stable isomorphism for any factorization* (2).

If $f: X \to Y$ is a normally non-singular G-map of codimension k, a Spin^c structure of f will be a G-equivariant Spin^c structure [3] on the normal bundle $\nu(i)$ for some factorization (2) of f. For now, the factorization (2) will be part of the data of the Spin^c structure. This will be cut down to size by the cobordism relation. Let f_i: $X_j \to Y$, $j=0, 1$, be two normally non-singular Spin^c maps of codimension k. Let $X_j \overset{i_j}{\to} Y \times \mathbf{R}^{n_j} \overset{p_j}{\to} Y$, $j=0, 1$ be the two factorizations in the definition of the Spin^c structure. f_0 and f_1 are called *cobordant* if there is a normally non-singular G-inclusion of codimension $N+k$ with Spin^c structure, $W \overset{i}{\to} Y \times \mathbf{R}^N \times [0, 1]$ so that $X_j = i^{-1}(Y \times \mathbf{R}^N \times \{j\})$ for $j=0, 1$ and the map $X_j = i^{-1}(Y \times \mathbf{R}^N \times \{j\}) \overset{i}{\to} Y \times \mathbf{R}^N$ should identify with the extension of i_j by embedding $\mathbf{R}^n$ into $\mathbf{R}^N$ as a subspace. Then $\nu(i)|_{X_j}$ is canonically isomorphic to $\nu(i_j) \oplus_{\mathbf{R}^{N-n_j}}$. We require that the Spin^c structure on $\nu(i)|_{X_j}$ be isomorphic to that on $\nu(i_j) \oplus_{\mathbf{R}^{N-n_j}}$.

3. $KK_G(X, Y)$ AND $f!$. — Let $f: X \to Y$ be a Spin^c normally non-singular map. Factor f as (2) and let S be the bundle of spinors defining the Spin^c structure on $\nu(i)$. We define $f!$ by first defining $i!$ for embeddings i and $p!$ for projections p. Our definition of $i!$ and $p!$ is the same as in [2] and [3], pp. 1153-1154 where they call them $i_{\mathrm{im}}!$ and $p_{\mathrm{sub}}!$ Now set $f! = i! \underset{C_0(Y \times \mathbf{R}^n)}{\otimes} p! \in KK_G^k(C_0(X), C_0(Y))$.

Suppose $f_0: X_0 \to X_1$ and $f_1: X_1 \to Y$ are codimension k_0 and k_1 resp. normally non-singular Spin^c maps with factorizations

$$X_0 \overset{i_0}{\to} X_1 \times \mathbf{R}^{n_0} \overset{p_0}{\to} X_1$$

$$X_1 \overset{i_1}{\to} Y \times \mathbf{R}^{n_1} \overset{p_1}{\to} Y$$

and Spin^c structures S_0 and S_1 on the normal bundle $\nu(i_0)$ and $\nu(i_1)$. Then $f_1 \circ f_0$ has a factorization

$$X_0 \overset{i_0}{\to} X_1 \times \mathbf{R}^{n_0} \xrightarrow{i_1 \times \mathrm{Id}} Y \times \mathbf{R}^{n_0+n_1} \overset{p}{\to} Y$$

as $p \circ i$ where $i = (i_1 \times \mathrm{Id})\, i_0$. This induces a Spin^c structure on the normal bundle $\nu(i)$. Hence $f_1 \circ f_0$ becomes Spin^c.

THEOREM 3.1. — *For* f_0 *and* f_1 *as above we have*

$$(f_1 \circ f_0)! = f_0! \underset{C_0(X_1)}{\otimes} f_1!$$

One shows the theorem is true for compositions of normally non-singular inclusions and then for the composition of p_0 and p_1. Then the theorem follows from the

LEMMA 4.2 :

$$p_0! \underset{C_0(X_1)}{\otimes} i_1! = (i_1 \times \mathrm{Id})! \underset{C_0(Y \times \mathbf{R}^{n_0+n_1})}{\otimes} p!$$

as elements of $KK_G(C_0(X_1 \times \mathbf{R}^{n_0}), C_0(Y \times \mathbf{R}^{n_1}))$.

Let X and Y be stratified G-spaces. A G-*bicycle* from X to Y is a quadruple (Z, E, f, g) where Z is a stratified G-space, E is a complex G-vector bundle on Z, $f: Z \to X$ is a continuous and proper G-map and $g: Z \to Y$ is a normally non-singular Spin^c G-map. The codimension of g will be called the codimension of the bicycle. In complete analogy with [1] define an equivalence relation on bicycles generated by the following three moves.

1. *Cobordism.* — (Z_0, E_0, f_0, g_0) is cobordant to (Z_1, E_1, f_1, g_1) if there is a bicycle (Z, E, f, g) from X to $Y \times [0, 1]$, so that g is a cobordism between g_0 and g_1 and $f|_{g^{-1}(Y \times \{j\})} = f_j$, $j = 0, 1$ and $E|_{g^{-1}(Y \times \{j\})} = E_j$.

2. *Direct sum.* — $(Z, E_0 \oplus E_1, f, g) = (Z, E_0, f, g) \cup (Z, E_1, f, g)$.

3. *Vector bundle modification.* — This is carried out as in [1] Section 10 except that we must ensure that the space we end up with is stratified. Let F be a real G-vector bundle on Z. It is called *controlled* if there is control data on F, $(T_S, \pi_S, \rho_S)_{S \in \mathscr{S}_Z}$ (F has the same set of strata as Z) so that the vector bundle projection of F is a controlled map. This is no restriction on the vector bundle. If $\langle ., . \rangle$ is a metric on F, it is called controlled if $\langle ., . \rangle|_S$ is a smooth metric for each stratum S and for each stratum there is an $\varepsilon > 0$ so that $\langle e_x, e'_x \rangle = \langle \pi_S e_x, \pi_S e'_x \rangle$ for $e_x, e'_x \in T_S(\varepsilon)$. Given a controlled invariant metric on a controlled vector bundle, the sphere bundle $S(F)$ has a natural set of control data. Hence as in [1] we let F be an even dimensional Spin^c vector bundle over Z with a controlled metric on $F \oplus 1_Z$. Set $\hat{Z} = S(F \oplus 1_Z)$ and let $\hat{F}$ be the complex vector bundle on $\hat{Z}$ with $\rho_!(\hat{F}) = 1_Z$, $\rho: \hat{Z} \to Z$ is the natural projection. Then the third part of the equivalence relation is

$$(Z, E, f, g) = (\hat{Z}, \hat{F} \otimes \rho^* E, f \circ \rho, g \circ \rho).$$

Denote by $KK_G(X, Y)$ the group defined by imposing this equivalence relation on all G-bicycles from X to Y and defining addition by disjoint union. $KK_G(X, Y)$ inherits a $\mathbf{Z}/2\mathbf{Z}$ grading from the parity of the codimension of the bicycle.

Remark. — To define groups $\Omega\Omega_G^*(X, Y)$ one forgets about the bundle E and parts 2 and 3 of the equivalence relation.

4. THE PRODUCT IN THE TRANSVERSE CASE. — In this section we show that the geometric form of the Kasparov product described in [3], carries over to the stratified case. Define a map $\mu: KK_G^*(X, Y) \to KK_G^*(C_0(X), C_0(Y))$ by

$$\mu(Z, E, f, g) = f_*((E) \underset{C_0(Z)}{\otimes} g!) = [f] \underset{Z}{\otimes} (E) \underset{Z}{\otimes} g!$$

where $(E) \in KK_G(C_0(Z), C_0(Z))$ corresponds to the Kasparov bimodule $(\mathscr{E}, 0)$ where $\mathscr{E}$ is the Hilbert module completion of $C_0(Z, E)$ under the obvious $C_0(Z)$-valued inner product, and $[f]$ is the obvious Kasparov module coming from a proper G-map.

Let $\Xi_0 = (Z_0, E_0, f_0, g_0)$ be a bicycle from X to Y and $\Xi_1 = (Z_1, E_1, f_1, g_1)$ a bicycle from Y to V. Assume f_1 is a controlled map. g_0 and f_1 are said to be *transverse* if given the factorization $Z_0 \overset{i_0}{\to} Y \times \mathbf{R}^n$ of g_0 where i_0 is a Spin^c normally non-singular inclusion, one has that $\bar{f}_1 = f_1 \times \mathrm{Id}: Z_1 \times \mathbf{R}^n \to Y \times \mathbf{R}^n$ is transverse to $i_0(Z_0)$ in the sense of stratified spaces. (This can be stated without appealing to a factorization.) Now form $\bar{f}_1^{-1}(i_0(Z_0))$. This is homeomorphic to $Z_0 \times_Y Z_1$ and by [6], Proposition 5.2, the inclusion $\bar{f}_1^{-1}(i_0(Z_0)) \xrightarrow{\tilde{q}_1} Z_1 \times \mathbf{R}^n$ is a normally-non-singular inclusion. Consider the diagram

$$(3) \qquad \begin{array}{ccccc} & & Z_0 \times_Y Z_1 = Z_0 \times_{Y \times \mathbf{R}^n} Z_1 \times \mathbf{R}^n & & \\ & \swarrow q_0 & & \tilde{q}_1 \searrow & \\ Z_0 & & & & Z_1 \times \mathbf{R}^n \\ & \searrow i_0 & & \swarrow \bar{f}_1 & \\ & & Y \times \mathbf{R}^n & & \end{array}$$

Then $q_0^* \nu(i_0) \cong \nu(\tilde{q}_1)$. So $\nu(\tilde{q}_1)$ is naturally Spin^c. Thus we may form $q_1 = p \circ \tilde{q}_1$. Let $\Xi = (Z, E, f, g)$ where $Z = Z_0 \times_{Y \times \mathbf{R}^n} Z_1 \times \mathbf{R}^n$, $E = q_0^* E_0 \otimes q_1^* E_1$, $f = f_0 \circ q_0$ and $g = g_1 \circ q_1$. Ξ is called the composition of the bicycles Ξ_0 and Ξ_1 and is denoted $\Xi_0 \circ \Xi_1$.

THEOREM 4.1. — *For the* G-*bicycles* Ξ_0 *and* Ξ_1 *as above, one has*

$$\mu(\Xi_0) \underset{C_0(Y)}{\otimes} \mu(\Xi_1) = \mu(\Xi_0 \circ \Xi_1).$$

THEOREM 4.2. — *Suppose* $\Xi_j = (Z_j, E_j, f_j, g_j)$, $j = 0, 1$ *are two* G-*bicycles from* X *to* Y *which are cobordant. Then* $\mu(\Xi_0) = \mu(\Xi_1)$ *in* $KK_G(C_0(X), C_0(Y))$.

Remark. — The usual equivalence relation put on orientations is subsumed in our cobordism relation (*see* Quillen [8]). One could have imposed this relation on orientation from the beginning.

PROPOSITION 4.3. — 1. *If* $\Xi_i = (Z_i, E_i, f_i, g_i)$, $i = 0, 1$ *are two bicycles from* X *to* Y *which are cobordant then* $\mu(\Xi_0) = \mu(\Xi_1)$.

2. *For any* F *as in the definition of vector bundle modification,*

$$\mu(\hat{Z}, \hat{F} \otimes \rho^*(E), f \circ \rho, g \circ \rho) = \mu(Z, E, f, g).$$

3. $\mu(\Xi)$ *is well defined. That is, it only depends on the equivalence class of* Ξ *in* $KK_G(X, Y)$.

THEOREM 4.4. — $\mu : KK_G^*(X, Y) \to KK_G^*(C_0(X), C_0(Y))$ *is an isomorphism.*

The proof proceeds in several steps. The first is to prove the isomorphism in the case where X is a point. This is done by explicitly constructing an inverse to μ by using a clutching construction. The rest of the theorem is proved by establishing appropriate functorial properties of $KK_G(., Y)$ as a functor in the first variable.

5. THE PRODUCT IN THE NON-TRANSVERSE CASE. — Here we show how to use Bott-periodicity instead of transversality to obtain the Kasparov product. Let $\Xi_0 = (Z_0, E_0, f_0, g_0)$ be a bicycle from X to Y and $\Xi_1 = (Z_1, E_1, f_1, g_1)$ a bicycle from Y to V. Consider a factorization of g_0 as $Z_0 \xrightarrow{i} Y \times \mathbf{R}^n$ where i is a normally non-singular Spin^c inclusion of codimension k and assume k is even. Let $\nu(i)$ be the normal bundle and $\varphi : \nu(i) \to Y \times \mathbf{R}^n$ an equivariant embedding as a tubular neighborhood. Equip $\nu(i)$ with a controlled invariant metric. $\nu(i)$ has a Spin^c-structure so that we may perform vector-bundle modification with respect to $\nu(i)$. Let $(\hat{Z}_0, \hat{F} \otimes \rho^*(E_0), f \circ \rho, g \circ \rho)$ be the modified bicycle. We will now alter the map $g \circ \rho$ to obtain an equivalent bicycle but which has good transversal properties. Consider the one-parameter family of maps j_t: $S(\nu(i) \otimes 1_{Z_0}) \to Y \times \mathbf{R}^n$ given by $j_t(\zeta, x) = \varphi(t . \zeta)$. For some value of t, t_0, j_{t_0} is

transverse to $\bar{f}=f\times \mathrm{Id}:Z_1\times \mathbf{R}^n\to Y\times \mathbf{R}^n$, while $j_0=g\circ\rho$. Let $\tilde{\Xi}_0$ be the G-bicycle $(\hat{Z}_0, \hat{F}\otimes\rho^*(E_0), f\circ\rho, j_{t_0})$. We may now carry out the Kasparov product as before.

This allows one to define equivariant intersection numbers. For example, let V be a complex manifold with a G-action. Let X and Y denote invariant smooth subvarieties. Then we may form the bicycles $\Xi=(X, 1_X, p, i)$ and $\Theta=(Y, 1_Y, i, p)$ where p is the map to a point and i is the inclusion into V. By forming the product $\Xi\circ\Theta$ we arrive at an element of $KK_G(pt., pt.)=R(G)$ which is the intersection number.

1. If X and Y intersect transversally and in complementary dimension, then $\Xi\circ\Theta=m.1$ where m is the ordinary intersection number of X and Y and 1 is the trivial one dimensional representation of G.

2. Let $X=Y=p$ be a fixed point of the action. Then $\Xi\circ\Theta=\sum_i(-1)^i\Lambda^i(T_p^{\mathbf{C}}(M))$ where $T_p^{\mathbf{C}}(M)$ is the complex cotangent space. In this case we see how $\Xi\circ\Theta$ is an obstruction to equivariantly pulling X and Y apart.

There is an explicit excess intersection formula in the spirit of [5] and [4].

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