

★-QUANTIZATIONS OF FOURIER–MUKAI TRANSFORMS

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Abstract. We study deformations of Fourier–Mukai transforms in general complex analytic settings. Suppose X and Y are complex manifolds, and let P be a coherent sheaf on $X \times Y$. Suppose that the Fourier–Mukai transform Φ given by the kernel P is an equivalence between the coherent derived categories of X and of Y . Suppose also that we are given a formal \star -quantization \mathbb{X} of X . Our main result is that \mathbb{X} gives rise to a unique formal \star -quantization \mathbb{Y} of Y . For the statement to hold, \star -quantizations must be understood in the framework of stacks of algebroids. The quantization \mathbb{Y} is uniquely determined by the condition that Φ deforms to an equivalence between the derived categories of \mathbb{X} and \mathbb{Y} . Equivalently, the condition is that P deforms to a coherent sheaf \tilde{P} on the formal \star -quantization $\mathbb{X} \times \mathbb{Y}^{op}$ of $X \times Y$; here \mathbb{Y}^{op} is the opposite of the quantization \mathbb{Y} .

Contents

1	Introduction	1404
1.1	Results	1404
1.2	Organization of the paper	1405
1.3	Relation to other works	1405
1.4	Extensions and generalizations	1407
1.5	Notation	1407
2	Formulation of Main Results	1407
2.1	\star -Quantizations	1407
2.2	Theorems A and B	1411
2.3	Plan of proof of Theorem A	1412
3	Generalities	1413
3.1	\mathcal{D} -modules on complex manifolds	1413
3.2	\mathcal{D} -modules on \star -quantizations	1419
3.3	Coalgebras in the category of \mathcal{D} -modules	1426
3.4	Neutralized \star -quantizations as \mathcal{D} -coalgebras	1428
4	Deformation Theory	1437
4.1	Proof of Theorem 2.18: neutralized case	1437
4.2	Proof of Theorem 2.18: general case	1442
4.3	Comments on tangible sheaves and \star -deformations	1447
4.4	Another proof of Theorem 2.18	1453
5	Computation of Cohomology	1458

5.1	An example: The case of complex tori	1458
5.2	Properties of the Fourier–Mukai transform	1460
5.3	Proof of Theorem 2.19	1464
5.4	Extension of Fourier–Mukai transforms to \star -quantizations	1465
6	Sheaves on \star -Quantizations Via Equivariant Stacks	1468
6.1	Equivariant stacks over the stack of quantizations	1469
6.2	Sections of equivariant stacks over quantizations	1473
6.3	Equivariance and R -linearity	1477
6.4	The 2-category of \star -quantizations	1478
	Acknowledgments	1480
	References	1480

1 Introduction

Derived equivalences of algebraic varieties are an important and essential topic in algebraic and complex geometry. The derived category of a scheme encodes much of its geometry and is also flexible enough to uncover non-trivial relations between non-isomorphic schemes.

It is natural to study how deformations of a scheme behave with respect to derived equivalences [Ari02, DP08, BBP07, Ben09]. An interesting phenomenon is that deformations of a scheme in a particular direction may correspond to deformations to a rather exotic object on the other side of a derived equivalence. These exotic objects are typically twisted or non-commutative spaces, and the deformed derived equivalences often give an interpretation of the deformations of the moduli spaces as moduli spaces in their own right. This leads to strong geometric statements such as the variational derived Torelli theorem [HMS09, HMS08].

1.1 Results. In this paper we study the deformations of Fourier–Mukai transforms in a general complex analytic setting; see Section 1.3 for a discussion of works related to the algebraic setting. We start with two complex manifolds X and Y together with a coherent Fourier–Mukai kernel P on $X \times Y$ implementing an equivalence between the coherent derived categories of X and Y . Then given an arbitrary formal \star -quantization of X we construct a unique \star -quantization of Y such that the Fourier–Mukai transform deforms to an equivalence of the derived categories of the quantizations. For this to hold we have to work with \star -quantizations in the framework of stacks of algebroids. To simplify the exposition we consider deformation quantizations over the ring $R = \mathbb{C}[[\hbar]]/(\hbar^{n+1})$. The proofs immediately extend to deformation quantizations over an arbitrary Artinian local \mathbb{C} -algebra R . In the case of deformations over a complete Noetherian local \mathbb{C} -algebra R (such as the familiar case $R = \mathbb{C}[[\hbar]]$), Theorem A still holds, because it can be reduced to Artinian quotients of R . On the other hand, Theorem B requires additional techniques (such as the results of [KS10]) to deal with the category of coherent sheaves on deformation quantizations.

Our main results are the following two theorems (see Theorems 2.15 and 2.16).

Theorem A. *Suppose that the support of $P \in \mathrm{Coh}(X \times Y)$ is proper over both X and Y , and that the integral transform $\Phi : D_{coh}^b(Y) \rightarrow D_{coh}^b(X)$ defined by P is fully faithful. Then:*

- *for any \star -quantization \mathbb{X} of X there exists a \star -quantization \mathbb{Y} of Y and a deformation of P to an \mathcal{O} -module \tilde{P} on $\mathbb{X} \times \mathbb{Y}^{op}$ (that is, \tilde{P} is R -flat, and the reduction $\tilde{P}/\hbar\tilde{P}$ is identified with P);*
- *the pair (\mathbb{Y}, \tilde{P}) is unique up to a 1-isomorphism, which is unique up to a unique 2-isomorphism.*

Theorem B. *Under the same assumptions we have:*

- *the deformation \tilde{P} gives a fully faithful integral transform $\tilde{\Phi} : D_{coh}^b(\mathbb{Y}) \rightarrow D_{coh}^b(\mathbb{X})$ between the coherent derived categories of \mathcal{O} -modules on \mathbb{Y} and \mathbb{X} ;*
- *if in addition Φ is an equivalence, then $\tilde{\Phi}$ is an equivalence as well.*

1.2 Organization of the paper. The paper is organized as follows. In Section 2 we review the setup of quantizations as stacks of algebroids and outline the main steps in the proof of Theorem A. (The proof of Theorem B is more straightforward.) There are two central results that feed into the proof of Theorem A.

One of them identifies the cohomology controlling the deformations of the pair (Y, P) to a pair (\mathbb{Y}, \tilde{P}) as in Theorem A. This is done in Section 4. The idea of the proof is to study deformations of an induced \mathcal{D} -module associated to P . This \mathcal{D} -module carries a comodule structure over a certain coalgebra, the necessary framework is explained in Section 4, and in full detail.

The second result is a vanishing theorem for the requisite cohomology assuming full faithfulness of Φ . We prove this result in Section 5, which also contains the proof of Theorem B. As a warm-up we start with the case of the usual Fourier–Mukai transform for complex tori. The general proof is independent of this case.

The last section discusses an alternative approach to sheaves of \mathcal{O} -modules on \star -quantizations and the 2-category of \star -quantizations. This discussion is not used anywhere else in the paper but we include it as a unifying framework for dealing with geometric objects over \star -quantizations.

1.3 Relation to other works. It is instructive to compare our setup to the analogous problem in algebraic geometry. Quantizations of schemes can be approached in two ways. We can either consider quantizations geometrically in terms of stacks of algebroids as above, or we can look at deformations categorically as deformations of a dg enhancement of the category of coherent sheaves. Similarly a Fourier–Mukai transform for schemes can be studied either geometrically through its kernel or categorically as a functor between dg categories (the equivalence of the two approaches follows from derived Morita theory [Toë07] and the uniqueness of enhancements [LO10]).

This dualism fits with the generalized deformation proposal of Bondal [Bon93], Bondal–Orlov [BO02], and Kontsevich–Soibelman [KS09b], according to which the

primary object to deform should be a dg enhancement of the category of coherent sheaves. In concrete terms the dualism can be expressed as follows. Algebroid stack deformations of a scheme are controlled by its Hochschild cochain complex. A Fourier–Mukai equivalence between two schemes induces a L_∞ quasi-isomorphism between their Hochschild cochain complexes [Kel03]. This motivates Theorem A in the algebraic setting. This categorical interpretation of Theorem A is somewhat deficient since neither its statement nor its proof give an indication on how the deformed dg categories can be realized geometrically. A direct geometric approach to proving Theorem A for schemes was given in the recent work [Ari06].

Unfortunately there are serious difficulties with the categorical approach in the analytic context. The category of coherent sheaves on a complex manifold X is not a subtle enough invariant of X (see [Ver08]) and so its derived category does not carry enough information. Even if we choose to work with the more flexible coherent derived category we encounter the problem that the derived Morita theory is not available in analytic setting. It is possible that one can overcome this problem by replacing the coherent derived category with a larger category, or by simultaneously considering the derived coherent categories of $X \times Y$ for all test analytic spaces Y .

In this paper, we concentrate on the geometric approach. It needs to be suitably modified to fit the analytic setting; roughly speaking, the class of deformations must be restricted to match the local nature of function theory. This restriction (which is vacuous in the algebraic setting) is encoded in the notion of a \star -quantization described in Section 2.1. The class of \star -quantizations is very natural from the geometric point of view; it is equivalent to the conditions imposed in [NT01, PS04, KS09a, KS10].

Note. An interesting algebraic approach to Theorem A was suggested by an anonymous referee. The idea is to utilize the natural analog of Theorem A for deformations of dg algebras [Kel03]. Concretely let A and B be \mathbb{C} -dg algebras and let P be $A \otimes B^{\text{op}}$ -dg module such that the induced map

$$\phi_P : B \rightarrow \mathbb{R} \text{Hom}_A(P, P)$$

is a quasi-isomorphism of B -modules. In this situation [Kel03] associates with ϕ_P a canonical L_∞ morphism

$$\mathbb{R} \text{Hom}_{A \otimes A^{\text{op}}}(A, A) \rightarrow \mathbb{R} \text{Hom}_{B \otimes B^{\text{op}}}(B, B)$$

of the corresponding L_∞ algebras of Hochschild cochains. Since these L_∞ algebras control the deformation theories of A and B the dg algebra analogue of Theorem A follows. It is likely that one can combine this construction with the observation (see e.g. [BGNT05, Theorem 7.1.2 and Proposition 3.6.1]) that the quantizations of a complex space X are classified by the Maurer–Cartan elements in the dg Lie algebra of de Rham–Sullivan forms $C^\bullet(X, CC^\bullet(\mathcal{O}_X))$. This should lead to an order by order construction of the quantization \mathbb{Y} in Theorem A. Carrying out this argument rigorously will take a considerable effort, and we will not pursue the details here.

Instead we follow a different route that utilizes comodule structures on induced right \mathcal{D} -modules.

1.4 Extensions and generalizations. Theorems A and B are naturally formulated in greater generality. One variant involves replacing X and Y with \mathcal{O}^\times -gerbes over complex manifolds, so that \mathbb{X} and \mathbb{Y} are \star -stacks (see Section 2.5) rather than \star -quantizations. Although our proof remains valid in this setting, we leave this case to the reader in order to simplify the exposition.

It is also natural to attempt to relax the hypotheses of Theorems A and B by assuming that the Fourier–Mukai kernel P is an object of the derived category $D_{coh}^b(X \times Y)$ that is not necessarily concentrated in cohomological dimension 0. However, such generalization requires some technical results; we intend to return to this question in the future.

1.5 Notation

1.5.1. For a complex manifold M , $\text{Coh}(M)$ is the category of coherent \mathcal{O}_M -modules. Also, $D_{coh}^b(M)$ is the bounded derived category of coherent sheaves on M , and $D_{comp}^b(M) \subset D_{coh}^b(M)$ is the full subcategory of objects whose support is compact.

1.5.2. Let X and Y be complex manifolds and $P \in \text{Coh}(X \times Y)$ be a kernel object. Assume that $\text{supp}(P)$ is proper over Y . To simplify notation we set $Z := X \times Y$, and we write $p_X : Z \rightarrow X$ and $p_Y : Z \rightarrow Y$ for the two projections.

1.5.3. Let

$$\Phi = \Phi^P : D_{comp}^b(Y) \rightarrow D_{coh}^b(X)$$

be the integral transform with respect to P . Explicitly, Φ is given by

$$\Phi(F) = \mathbb{R}p_{X*}(p_Y^*(F) \otimes^{\mathbb{L}} P) \quad (F \in D_{comp}^b(Y)).$$

Since $\text{supp}(P)$ is proper over Y , the image of Φ is contained in $D_{comp}^b(X)$.

2 Formulation of Main Results

Let M be a complex manifold. We fix the base ring of deformation to be $R = \mathbb{C}[\hbar]/\hbar^{n+1}$. To check deformations over an arbitrary finite dimensional local \mathbb{C} -algebra, we need to consider local \mathbb{C} -algebras where the maximal ideal is not necessarily principal. All the arguments seem to extend to this case but we will not discuss such base algebras explicitly.

2.1 ★-Quantizations. Let us review the notions of \star -quantizations and coherent sheaves on them. Our exposition mostly follows [KS10] (see also [KS09a]).

Notation 2.1. Denote by $\mathcal{D}_M = \mathcal{D}iff_M(\mathcal{O}_M; \mathcal{O}_M)$ the sheaf of differential operators on M . More generally, let $\mathcal{D}iff_M(\mathcal{O}_M, \dots, \mathcal{O}_M; \mathcal{O}_M)$ be the sheaf of polydifferential operators

$$\mathcal{O}_M \times \dots \times \mathcal{O}_M \rightarrow \mathcal{O}_M.$$

We discuss (poly)differential operators in more detail in Section 3.1.

DEFINITION 2.2. A \star -**product** on $\mathcal{O}_M \otimes_{\mathbb{C}} R$ is an associative R -linear product that is local and extends the product in \mathcal{O}_M . In other words, the product is given by a bidifferential operator:

$$f \star g = B(f, g), \quad B = B_0 + B_1 \hbar + \dots + B_n \hbar^n \in \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M) \otimes_{\mathbb{C}} R$$

$$(B_i \in \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M))$$

with $B_0(f, g) = fg$ being the product in \mathcal{O}_M .

A **neutralized \star -quantization** \widetilde{M} of M is a ringed space $(M, \mathcal{O}_{\widetilde{M}})$, where $\mathcal{O}_{\widetilde{M}}$ is a sheaf of R -algebras on M that is locally isomorphic to $\mathcal{O}_M \otimes_{\mathbb{C}} R$ equipped with a \star -product.

REMARK 2.3. Neutralized \star -quantizations are usually called simply \star -quantizations in the literature (see [NT01, Yek05]). For us, the primary objects are \star -quantizations in the sense of stacks of algebroids, as in [Kon01, PS04, Low08, KS10]; we therefore reserve the name ‘ \star -quantizations’ for these objects (Definition 2.9). One can then view neutralized \star -quantizations as \star -quantizations with an additional structure: neutralization, see Example 2.7. Roughly speaking, for \star -quantizations in the sense of stacks of algebroids, the sheaf of algebras $\mathcal{O}_{\widetilde{M}}$ is defined only locally; a neutralization involves a choice of a global sheaf of algebras (which may not exist).

In the terminology of [KS10], neutralized \star -quantizations are called DQ-algebras.

2.1.1. Let

$$\theta_i : \mathcal{O}_{U_i} \otimes_{\mathbb{C}} R \rightarrow \mathcal{O}_{\widetilde{M}}|_{U_i} \quad (M = \bigcup U_i) \tag{1}$$

be isomorphisms of Definition 2.2. The isomorphisms need not agree on double intersections, so in general $\theta_i|_{U_i \cap U_j} \neq \theta_j|_{U_i \cap U_j}$. However, the discrepancy is ‘as small as one may hope’: the composition

$$\theta_{ij} : \theta_i|_{U_i \cap U_j}^{-1} \circ \theta_j|_{U_i \cap U_j} : \mathcal{O}_{U_i \cap U_j} \otimes_{\mathbb{C}} R \rightarrow \mathcal{O}_{U_i \cap U_j} \otimes_{\mathbb{C}} R$$

is given by a differential operator:

$$\theta_{ij} = 1 + T_1 \hbar + \dots + T_n \hbar^n \in \mathcal{D}_{U_i \cap U_j} \otimes_{\mathbb{C}} R \quad (T_i \in \mathcal{D}_{U_i \cap U_j}),$$

see [KS10, Proposition 4.3]. (In the sense of Definition 4.18, $\mathcal{O}_{\widetilde{M}}$ acquires a natural \star -structure.) Note in particular that $\mathcal{O}_{\widetilde{M}}$ is R -flat and that the reduction $\mathcal{O}_{\widetilde{M}}/\hbar\mathcal{O}_{\widetilde{M}}$ is identified with \mathcal{O}_M .

Note that in the definition of \star -product, one often requires that $1 \in \mathcal{O}_M \otimes_{\mathbb{C}} R$ is the unit element. We prefer to avoid this restriction. The definition of neutralized \star -quantization is not affected by this choice: in (1), one can always choose isomorphisms θ_i and \star -products on $\mathcal{O}_{U_i} \otimes_{\mathbb{C}} R$ in such a way that 1 is the unit element.

Indeed, it is clear that any ★-product has a unit element (Lemma 2.4); changing θ_i , we can ensure that 1 is the unit.

LEMMA 2.4. *Any neutralized ★-quantization \widetilde{M} has a unit element $1 \in \mathcal{O}_{\widetilde{M}}$.*

Proof. Of course, this is a version of the well-known statement that deformations of unital algebras are unital (see for instance [GS83, Section 20]). Let us sketch the proof. Actually, we only need to assume that $\mathcal{O}_{\widetilde{M}}$ is a deformation of \mathcal{O}_M : that is, we assume that $\mathcal{O}_{\widetilde{M}}$ is a flat R -algebra with $\mathcal{O}_{\widetilde{M}}/\hbar\mathcal{O}_{\widetilde{M}} \simeq \mathcal{O}_M$.

It suffices to check that the unit exists locally on M since a unit element in an associative algebra is unique. Also, it suffices to prove existence of a right unit. Locally on M , $1 \in \mathcal{O}_M$ can be lifted to some section $\alpha \in \mathcal{O}_{\widetilde{M}}$. The multiplication maps $(\bullet) \cdot \alpha : \mathcal{O}_{\widetilde{M}} \rightarrow \mathcal{O}_{\widetilde{M}}$ and $\alpha \cdot (\bullet) : \mathcal{O}_{\widetilde{M}} \rightarrow \mathcal{O}_{\widetilde{M}}$ are bijective (since they are bijective on the associated graded). Let now $\beta \in \mathcal{O}_{\widetilde{M}}$ be such that $\alpha \cdot \beta = \alpha$. Then $(y \cdot \alpha) \cdot \beta = y \cdot \alpha$ for all local sections $y \in \mathcal{O}_{\widetilde{M}}$, and since every element in $\mathcal{O}_{\widetilde{M}}$ can be written as $y \cdot \alpha$ for some y , it follows that β is a right unit in $\mathcal{O}_{\widetilde{M}}$. \square

2.1.2. We now proceed to define non-neutralized (‘gerby’) quantizations. We first define the notion of a ★-stack. We view a ★-stack as a quantization of an \mathcal{O}^\times -gerbe on M . (In [KS10], ★-stacks are called DQ-algebroids, and \mathcal{O}^\times -gerbes are called invertible \mathcal{O} -algebroids). If the \mathcal{O}^\times -gerbe is neutral, we get a neutralized ★-quantization of M .

We use the following notation. Let \mathbb{M} be an R -linear stack of algebroids (introduced in [Kon01], see also [PS04, Low08, KS10]) over a space M . Given an open subset $U \subset M$ and two sections $\alpha, \beta \in \mathbb{M}(U)$, we write $\text{Hom}_{\mathbb{M}}(\alpha, \beta)$ for the R -module of homomorphisms in the category $\mathbb{M}(U)$, and $\mathcal{H}om_{\mathbb{M}}(\alpha, \beta)$ for the corresponding sheaf of homomorphisms on U . If $\alpha = \beta$, we write $\mathcal{E}nd_{\mathbb{M}}(\alpha)$ instead of $\mathcal{H}om_{\mathbb{M}}(\alpha, \alpha)$ for the corresponding sheaf of R -algebras.

DEFINITION 2.5. *Let \mathbb{M} be an R -linear stack of algebroids on M . Given an open set $U \subset M$ and $\alpha \in \mathbb{M}(U)$, we denote the ringed space $(U, \mathcal{E}nd_{\mathbb{M}}(\alpha))$ by \widetilde{U}_α . We say that \mathbb{M} is a ★-stack if \widetilde{U}_α is a neutralized ★-quantization of U for all U and α .*

REMARK 2.6. Note that for any $\alpha \in \mathbb{M}(U)$, $\beta \in \mathbb{M}(V)$ (where $U, V \subset M$ are open sets), the sheaves $\mathcal{E}nd_{\mathbb{M}}(\alpha)|_{U \cap V}$ and $\mathcal{E}nd_{\mathbb{M}}(\beta)|_{U \cap V}$ are locally isomorphic. Therefore, in Definition 2.5, it suffices to check that $(\widetilde{U}_i)_{\alpha_i}$ is a neutralized ★-quantization for some open cover $M = \bigcup U_i$ and for some $\alpha_i \in \mathbb{M}(U_i)$.

EXAMPLE 2.7: NEUTRALIZED QUANTIZATIONS. By definition, a **neutralization** of a ★-stack M is a global section $\alpha \in \mathbb{M}(M)$. Then \widetilde{M}_α is a neutralized ★-quantization of M .

Conversely, given a neutralized ★-quantization \widetilde{M} , define the stack of algebroids \mathbb{M} by letting \mathbb{M}_U be the category of locally free right $\mathcal{O}_{\widetilde{M}}|_U$ -modules of rank 1. One can easily see that \mathbb{M} has a natural structure of a ★-stack; it carries a natural neutralization $\alpha = \mathcal{O}_{\widetilde{M}} \in \mathbb{M}_M$ (the free module) such that $\widetilde{M} = \widetilde{M}_\alpha$. In this way, we can view neutralized ★-quantizations as ★-stacks equipped with an additional structure: neutralization.

EXAMPLE 2.8: COMMUTATIVE \star -STACKS. We say that a \star -stack \mathbb{M} is *commutative* if for every local section α , the sheaf of endomorphisms $\mathcal{E}nd_{\mathbb{M}}(\alpha)$ is a sheaf of commutative algebras. It is easy to see that for commutative \mathbb{M} , the sheaf $\mathcal{E}nd_{\mathbb{M}}(\alpha)$ on an open set $U \subset \mathbb{M}$ does not depend on the choice of $\alpha \in \mathbb{M}(U)$. As U varies, these sheaves glue to a sheaf $\mathcal{O}_{\underline{M}}$ on M . The ringed space $\underline{M} = (M, \mathcal{O}_{\underline{M}})$ is an R -deformation of M as a complex analytic space, and \mathbb{M} is simply an \mathcal{O}^\times -gerbe on \underline{M} .

2.1.3. Let \mathbb{M} be a \star -stack over $R = \mathbb{C}[\hbar]/\hbar^{n+1}$. For any $n' < n$, $R' = \mathbb{C}[\hbar]/\hbar^{n'+1}$ is a quotient of R ; it is easy to see that \mathbb{M} induces a \star -stack \mathbb{M}' over R' , which we call the *reduction* of \mathbb{M} to R' . Explicitly, for every open U , we let $(\mathbb{M}_U)'$ be the category whose objects are the same as \mathbb{M}_U , while the space of morphisms between $\alpha, \beta \in (\mathbb{M}_U)'$ equals

$$\Gamma(U, \mathcal{H}om_{\mathbb{M}}(\alpha, \beta) \otimes_R R').$$

We then let \mathbb{M}' be the stack associated with the pre-stack $U \mapsto (\mathbb{M}_U)'$. It is not hard to check that \mathbb{M}' is a \star -stack over R' .

DEFINITION 2.9. A \star -**quantization of M** is a \star -stack \mathbb{M} on M together with a neutralization of the reduction of \mathbb{M} to \mathbb{C} .

REMARK 2.10. Equivalently, a \star -quantization can be defined as follows (cf. [KS10, Example 2.1.2 and Proposition 2.1.3]). Let \mathbb{M} be a \star -stack. The structure of a \star -quantization on \mathbb{M} is given by an \mathcal{O} -module ℓ on \mathbb{M} such that for every open set U and every $\alpha \in \mathbb{M}(U)$, the action of $\mathcal{O}_{\tilde{U}_\alpha} = \mathcal{E}nd_{\mathbb{M}}(\alpha)$ on ℓ_α factors through $\mathcal{O}_U = \mathcal{E}nd_{\mathbb{M}}(\alpha)/\hbar \mathcal{E}nd_{\mathbb{M}}(\alpha)$, and that ℓ_α is invertible as an \mathcal{O}_U -module.

Explicitly, such ℓ is given as follows. For every open set U and every $\alpha \in \mathbb{M}(U)$, we specify an invertible \mathcal{O}_U -module ℓ_α , which is functorial in α and respects restriction to open subsets of U . Then the sheaf $\mathcal{E}nd_{\mathbb{M}}(\alpha)$ acts on ℓ_α . We require that the action of $f \in \mathcal{E}nd_{\mathbb{M}}(\alpha)$ on ℓ_α factors through \mathcal{O}_U .

Note that for any $\alpha, \beta \in \mathbb{M}_U$, we obtain an isomorphism

$$\mathcal{H}om_{\mathbb{M}}(\alpha, \beta)/\hbar \mathcal{H}om_{\mathbb{M}}(\alpha, \beta) \simeq \ell_\alpha^{-1} \otimes \ell_\beta$$

that agrees with composition and restriction. In a sense, ℓ_α is an anti-derivative of the reduction modulo \hbar of the cocycle $\mathcal{H}om_{\mathbb{M}}(\alpha, \beta)$.

REMARK 2.11. (1) Let \mathbb{M} be a \star -quantization of M . One can easily define the *opposite* \star -quantization \mathbb{M}^{op} ; for any open U , the category \mathbb{M}_U^{op} is the opposite of \mathbb{M}_U .

(2) Let \mathbb{M}, \mathbb{N} be \star -quantizations of complex manifolds M and N , respectively. It is not hard to define the *product* $\mathbb{M} \times \mathbb{N}$, which is a \star -quantization of $M \times N$.

2.1.4. We now turn to \mathcal{O} -modules on \star -quantizations. We define them as representations of stacks of algebroids (as in for instance [KS10]); equivalent approaches are discussed in Section 6.

Let \mathbf{Sh}_R be the stack of sheaves of R -modules on M : to an open set $U \subset M$, it assigns the category $\mathbf{Sh}_R(U)$ of sheaves of R -modules on U .

DEFINITION 2.12. Let \mathbb{M} be a \star -stack. An $\mathcal{O}_{\mathbb{M}}$ -**module** is a 1-morphism $\tilde{F} : \mathbb{M} \rightarrow \mathbf{Sh}_R$ of stacks of R -linear categories over M .

2.1.5. Let $\tilde{F} : \mathbb{M} \rightarrow \mathbf{Sh}_R$ be an $\mathcal{O}_{\mathbb{M}}$ -module. For any open $U \subset M$ and any $\alpha \in \mathbb{M}(U)$, $\tilde{F}(\alpha)$ is a sheaf of R -modules on U . Moreover, the sheaf of R -algebras $\mathit{End}_{\mathbb{M}}(\alpha)$ acts on $\tilde{F}(\alpha)$. We denote the resulting $\mathcal{O}_{\tilde{U}_\alpha}$ -module by \tilde{F}_α . In particular, if $\alpha \in \mathbb{M}(M)$ is a neutralization, the resulting functor

$$\begin{array}{ccc} (\mathcal{O}_{\mathbb{M}}\text{-modules}) & \longrightarrow & (\mathit{End}_{\mathbb{M}}(\alpha)\text{-modules}) \\ \tilde{F} & \longrightarrow & \tilde{F}_\alpha \end{array}$$

is an equivalence.

DEFINITION 2.13. An $\mathcal{O}_{\mathbb{M}}$ -module \tilde{F} is **coherent** if \tilde{F}_α is coherent for all open $U \subset M$ and all $\alpha \in \mathbb{M}(U)$. The category of coherent $\mathcal{O}_{\mathbb{M}}$ -modules is denoted by $\mathbf{Coh}(\mathbb{M})$. The **bounded coherent derived category** $D_{coh}^b(\mathbb{M})$ is the derived category of complexes of $\mathcal{O}_{\mathbb{M}}$ -modules that have finitely many non-zero cohomology objects each of which is coherent.

DEFINITION 2.14. The **support** of an $\mathcal{O}_{\mathbb{M}}$ -module \tilde{F} is the subset in M which is the union of the supports of \tilde{F}_α taken over all open $U \subset M$ and all $\alpha \in \mathbb{M}(U)$. Denote by $D_{comp}^b(\mathbb{M}) \subset D_{coh}^b(\mathbb{M})$ is the full subcategory of complexes whose cohomology objects are compactly supported.

2.2 Theorems A and B. We want to study the \star -quantizations of Fourier–Mukai equivalences between complex manifolds. Guided by the case of complex tori [BBP07], the general algebraic case [Ari06], and by the derived Morita theory [Toë07] it is natural to conjecture that Fourier–Mukai equivalences propagate along any \star -quantization of one of the manifolds involved.

In fact in the hypotheses of Theorem A it is not necessary to assume properness of the support of P over X . One should then consider the functor Φ only on the subcategory $D_{comp}^b(Y) \subset D_{coh}^b(Y)$ (see Theorem 2.15 below). Theorem 2.15 clearly implies Theorem A but in fact according to Lemma 5.7 the two theorems are equivalent if the support of P is proper over X .

Theorem 2.15. Suppose that the support of $P \in \mathbf{Coh}(X \times Y)$ is proper over Y , and that the integral transform $\Phi : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$ defined by P is fully faithful. Then:

- for any \star -quantization \mathbb{X} of X there exists a \star -quantization \mathbb{Y} of Y and a deformation of P to an \mathcal{O} -module \tilde{P} on $\mathbb{X} \times \mathbb{Y}^{op}$ (that is, \tilde{P} is R -flat, and the reduction $\tilde{P}/\hbar\tilde{P}$ is identified with P);

- the pair (\mathbb{Y}, \tilde{P}) is unique up to a 1-isomorphism, which is unique up to a unique 2-isomorphism.

We also have slightly more general version of Theorem B:

Theorem 2.16. *Under the assumptions of Theorem 2.15 we have:*

- the deformation \tilde{P} gives a fully faithful integral transform $\tilde{\Phi} : D_{comp}^b(\mathbb{Y}) \rightarrow D_{comp}^b(\mathbb{X})$;
- if in addition the support of P is proper over X , then \tilde{P} gives a fully faithful integral transform $\tilde{\Phi} : D_{coh}^b(\mathbb{Y}) \rightarrow D_{coh}^b(\mathbb{X})$;
- if the support of P is proper over X and $\tilde{\Phi}$ is an equivalence, then $\tilde{\Phi} : D_{coh}^b(\mathbb{Y}) \rightarrow D_{coh}^b(\mathbb{X})$ is an equivalence as well.

2.3 Plan of proof of Theorem A

2.3.1. Let P be a coherent sheaf on $Z = X \times Y$ whose support is proper over Y . Denote by $\mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)$ the sheaf of differential operators $\mathcal{O}_Z \rightarrow P$ that are \mathcal{O}_X -linear. We consider $\mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)$ as a (non-coherent) \mathcal{O}_Z -module under right multiplications by functions. See Section 3.1 for details.

Note that a section $s \in P$ defines an operator $f \mapsto fs$; this yields a homomorphism

$$P \rightarrow \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P). \tag{2}$$

2.3.2. Consider now in the derived category of \mathcal{O}_Z -modules the derived sheaf of homomorphisms

$$\mathbb{R} \mathcal{H}om(P, \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)).$$

Its derived push-forward to Y is a complex of \mathcal{O}_Y -modules (or, more properly, an object of the derived category)

$$\mathcal{E}(P) := \mathbb{R}p_{Y,*} \mathbb{R} \mathcal{H}om(P, \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)).$$

The homomorphism (2) induces a map $\mathcal{O}_Z \rightarrow \mathbb{R} \mathcal{H}om(P, \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P))$, which yields a map $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$.

REMARK 2.17. The derived sheaf of homomorphisms $\mathbb{R} \mathcal{H}om(P, \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P))$ can be viewed as the derived sheaf of relative differential operators from P to itself. In fact, if P is locally free, we have a quasi-isomorphism

$$\mathcal{D}iff_{Z/X}(P; P) \simeq \mathbb{R} \mathcal{H}om(P, \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)),$$

where $\mathcal{D}iff_{Z/X}(P; P)$ is the sheaf of \mathcal{O}_X -linear differential operators from P to itself. (This is a relative version of Lemma 4.17.) Moreover, the statement remains true for P of the form $P = i_{\Gamma*}V$, where $\Gamma \subset Z$ is a closed submanifold such that the projection of Γ to X is submersive, $i_{\Gamma} : \Gamma \hookrightarrow Z$ is the embedding, and V is a locally free coherent sheaf on Γ . These hypotheses hold for the Fourier–Mukai transform for complex tori.

We separate Theorem 2.15 into two parts: one concerns deformation theory, while the other is about integral transforms.

Theorem 2.18. *Suppose that $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$ is a quasi-isomorphism. Then the conclusion of Theorem 2.15 holds: For any \star -quantization \mathbb{X} of X there exists a \star -quantization \mathbb{Y} of Y and a \mathcal{O} -module \tilde{P} on $\mathbb{X} \times \mathbb{Y}^{op}$ deforming P . The pair (\mathbb{Y}, \tilde{P}) is unique up to a 1-isomorphism, which is unique up to a unique 2-isomorphism.*

Theorem 2.19. *Let P satisfy the hypotheses of Theorem 2.15. Then $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$ is a quasi-isomorphism.*

Clearly, Theorems 2.18 and 2.19 imply Theorem 2.15. Their proofs occupy Sections 4 and 5.

3 Generalities

We prove Theorem 2.18 by replacing the \mathcal{O} -module P with the induced (right) \mathcal{D} -module $P_{\mathcal{D}}$. In addition to being a \mathcal{D} -module, $P_{\mathcal{D}}$ carries a coaction of a certain coalgebra. The idea of the proof is that deformations of $P_{\mathcal{D}}$ as a comodule in the category of \mathcal{D} -modules are easier to study than deformations of P . If P satisfies certain smoothness conditions, deformations of $P_{\mathcal{D}}$ can be interpreted in terms of \star -structures, as we explain in Section 4.3. (However, \star -structures are not used in the proof.) The approach to \star -structures via \mathcal{D} -modules is based on \star -pseudotensor structure of [BD04]. We are very grateful to V. Drinfeld for drawing our attention to this approach.

In this section, we study differential operators and \mathcal{D} -modules on \star -quantizations. We also explain the correspondence between \mathcal{O} -modules and comodules in the category of \mathcal{D} -modules.

3.1 \mathcal{D} -modules on complex manifolds. Let us start by reviewing differential operators (see [Kas03, Bjö93]) on a (non-quantized) complex manifold M .

DEFINITION 3.1. *Let Q and P be \mathcal{O}_M -modules, and let $A : Q \rightarrow P$ be a \mathbb{C} -linear map. Given a sequence of functions $f_0, f_1, \dots, f_N \in \mathcal{O}_M$ for some $N \geq 0$, define a sequence of \mathbb{C} -linear maps $A_k : Q \rightarrow P$ by $A_{-1} := A$, $A_k := f_k A_{k-1} - A_{k-1} f_k$. We say that A is a **differential operator** if for every point $x \in M$ and every section $s \in Q$ defined at x , there exists a neighborhood U of x and $N \geq 0$ such that for any open subset $V \subset U$ and any choice of functions f_0, \dots, f_N on V , $A_N(s|_V)$ vanishes.*

Clearly, differential operators $Q \rightarrow P$ form a sheaf that we denote by $\mathcal{D}iff_M(Q; P)$.

REMARK 3.2. Generally speaking, the bound N (which could be thought of as the order of the differential operator A) depends on s and on U . If we assume that $Q \in \text{Coh}(M)$, one can choose a uniform bound N that does not depend on s , however, it may still be local (that is, depend on U).

For example, consider $A : \mathcal{O}_M \rightarrow P$. Then A is a differential operator if and only if it can be locally written as

$$f \mapsto \sum_{\alpha=(\alpha_1, \dots, \alpha_n)} \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} a_\alpha,$$

for sections $a_\alpha \in P$ such that locally only finitely many of a_α 's are non-zero. Here (x_1, \dots, x_n) is a local coordinate system on M .

3.1.1. It is helpful to restate Definition 3.1 by splitting it into steps. Let P and Q be \mathcal{O}_M -modules.

- A \mathbb{C} -linear map $A : \mathcal{O}_M \rightarrow P$ is a **differential operator of order at most N** if for any local functions $f_0, f_1, \dots, f_N \in \mathcal{O}_M$, we have $A_N = 0$.
- A \mathbb{C} -linear map $A : \mathcal{O}_M \rightarrow P$ is a **differential operator** if locally there exists N such that it is a differential operator of order at most N .
- A \mathbb{C} -linear map $A : Q \rightarrow P$ is a **differential operator** if for every $s \in Q$, the map

$$\mathcal{O}_M \rightarrow P : f \mapsto A(fs)$$

is a differential operator.

It is now easy to see that a composition of differential operators is a differential operator.

3.1.2. Consider the sheaf of differential operators

$$\mathcal{D}_M = \mathcal{D}iff_M(\mathcal{O}_M; \mathcal{O}_M).$$

It is a sheaf of algebras on M . Note that \mathcal{D}_M has two structures of an \mathcal{O}_M -module given by pre- and post-composition with the multiplication on \mathcal{O}_M . With respect to either structure, \mathcal{D}_M is a locally free \mathcal{O}_M -module of infinite rank.

Let P be an \mathcal{O}_M -module. Consider the **induced** right \mathcal{D}_M -module

$$P_{\mathcal{D}} := P \otimes_{\mathcal{O}_M} \mathcal{D}_M.$$

Here the tensor product is taken with respect to the left action of \mathcal{O}_M on \mathcal{D}_M . We identify $P_{\mathcal{D}}$ with the sheaf $\mathcal{D}iff_M(\mathcal{O}_M; P)$ via the isomorphism

$$P \otimes_{\mathcal{O}_M} \mathcal{D}_M \rightarrow \mathcal{D}iff_M(\mathcal{O}_M; P) : p \otimes A \mapsto A(\bullet)p.$$

Under this identification, the action of \mathcal{D}_M on $\mathcal{D}iff_M(\mathcal{O}_M; P)$ comes from its action on \mathcal{O}_M .

Note that P can be reconstructed from $P_{\mathcal{D}}$ as $P = P_{\mathcal{D}} \otimes_{\mathcal{D}_M} \mathcal{O}_M$ (here \mathcal{O}_M is considered as a left \mathcal{D}_M -module). The induction functor is adjoint to the forgetful functor from \mathcal{D}_M -modules to \mathcal{O}_M -modules. Both of these statements hold in the derived sense:

LEMMA 3.3. *Let P be an \mathcal{O}_M -module.*

- (a) *For any right \mathcal{D}_M -module F , there are natural isomorphisms*

$$\begin{aligned} \text{Ext}_{\mathcal{D}_M}^i(P_{\mathcal{D}}, F) &= \text{Ext}_{\mathcal{O}_M}^i(P, F) & (i \geq 0) \\ \mathcal{E}xt_{\mathcal{D}_M}^i(P_{\mathcal{D}}, F) &= \mathcal{E}xt_{\mathcal{O}_M}^i(P, F) & (i \geq 0) \end{aligned}$$

of vector spaces and sheaves of vector spaces, respectively.

(b) For any left \mathcal{D}_M -module F , there is a natural isomorphism

$$\mathcal{T}or_i^{\mathcal{D}_M}(P_{\mathcal{D}}, F) = \mathcal{T}or_i^{\mathcal{O}_M}(P, F)$$

of sheaves of \mathcal{O}_M -modules. In particular, for $F = \mathcal{O}_M$, there is an isomorphism

$$P_{\mathcal{D}} \otimes_{\mathcal{D}_M}^{\mathbb{L}} \mathcal{O}_M = P.$$

Proof. The isomorphisms are clear for $i = 0$; for $i > 0$, it is enough to resolve F by injective \mathcal{D}_M -modules in (a) and flat \mathcal{D}_M -modules in (b). Note that injective \mathcal{D}_M -modules remain injective over \mathcal{O}_M ; the same holds for flat \mathcal{D}_M -modules. Alternatively the statement follows from the fact that \mathcal{D}_M is a locally free and hence flat as an \mathcal{O}_M -module. \square

Let us interpret differential operators between \mathcal{O}_M -modules in term of the induced \mathcal{D}_M -modules. Directly from the definition, we derive the following claim.

LEMMA 3.4. Let Q, P be \mathcal{O}_M -modules.

(a) Any differential operator $A \in \mathcal{D}iff_M(Q; P)$ induces a morphism of \mathcal{O}_M -modules

$$Q \rightarrow \mathcal{D}iff_M(\mathcal{O}_M; P) = P_{\mathcal{D}} : s \mapsto A(\bullet s) \quad (s \in Q).$$

This provides an identification $\mathcal{D}iff_M(Q; P) = \mathcal{H}om_{\mathcal{O}_M}(Q, P_{\mathcal{D}})$.

(b) Any differential operator $A \in \mathcal{D}iff_M(Q; P)$ induces a map

$$\mathcal{D}iff_M(\mathcal{O}_M; Q) \rightarrow \mathcal{D}iff_M(\mathcal{O}_M; P) : B \mapsto A \circ B.$$

This provides an identification

$$\begin{aligned} \mathcal{D}iff_M(Q; P) &= \mathcal{H}om_{\mathcal{D}_M}(\mathcal{D}iff_M(\mathcal{O}_M; Q), \mathcal{D}iff_M(\mathcal{O}_M; P)) \\ &= \mathcal{H}om_{\mathcal{D}_M}(Q_{\mathcal{D}}, P_{\mathcal{D}}). \end{aligned}$$

REMARK 3.5. (1) Note that $P_{\mathcal{D}}$ has two structures of an \mathcal{O}_M -module. Indeed, besides the right action of \mathcal{D}_M (coming from its action on \mathcal{O}_M), it has a left action of \mathcal{O}_M coming from its action on P . Unless stated otherwise, we ignore the latter action; in particular, the structure of \mathcal{O}_M -module on $P_{\mathcal{D}}$ comes from the action of \mathcal{O}_M on itself. Equivalently, after we form the tensor product $P \otimes_{\mathcal{O}_M} \mathcal{D}_M$ using the left action of \mathcal{O}_M on \mathcal{D}_M , we consider it as an \mathcal{O}_M -module using the right action of \mathcal{O}_M on \mathcal{D}_M rather than the left action. This is the structure used in Lemma 3.4(a).

(2) The two statements of Lemma 3.4 are related by the isomorphism of Lemma 3.3(b).

(3) In homological calculations it is often useful to derive the notion of a differential operator. In view of Lemma 3.4, it is natural to consider

$$\mathbb{R} \mathcal{H}om_{\mathcal{O}_M}(Q, P_{\mathcal{D}}) = \mathbb{R} \mathcal{H}om_{\mathcal{D}_M}(Q_{\mathcal{D}}, P_{\mathcal{D}})$$

as the higher derived version of the sheaf of differential operators.

We have a similar formalism for polydifferential operators.

DEFINITION 3.6. Let P be an \mathcal{O}_M -module.

- A \mathbb{C} -polylinear map $A : \mathcal{O}_M \times \cdots \times \mathcal{O}_M \rightarrow P$ (of n arguments) is a **polydifferential operator of (poly)order at most** (N_1, \dots, N_n) if it is a differential operator of order at most N_j in the j -th argument whenever the remaining $n - 1$ arguments are fixed.
- A \mathbb{C} -polylinear map $A : \mathcal{O}_M \times \cdots \times \mathcal{O}_M \rightarrow P$ is a **polydifferential operator** if locally there exist (N_1, \dots, N_n) such that A is a polydifferential operator of order at most (N_1, \dots, N_n) .
- Let P_1, \dots, P_n be \mathcal{O}_M -modules. A \mathbb{C} -polylinear map

$$A : P_1 \times \cdots \times P_n \rightarrow P$$

is a **polydifferential operator** if for any local sections $s_1 \in P_1, \dots, s_n \in P_n$, the map

$$\mathcal{O}_M \times \cdots \times \mathcal{O}_M \rightarrow P : (f_1, \dots, f_n) \mapsto A(f_1 s_1, \dots, f_n s_n)$$

is a polydifferential operator.

We denote the sheaf of polydifferential operators by $\mathcal{D}iff_M(P_1, \dots, P_n; P)$.

LEMMA 3.7. Let P and P_1 be \mathcal{O}_M -modules and suppose that P_1 is coherent.

- (a) Consider on the sheaf $\mathcal{D}iff_M(P_1; P)$ the two structures of an \mathcal{O}_M -module coming from the action of \mathcal{O}_M on P and on P_1 . Denote the resulting \mathcal{O}_M -modules Q and Q_1 . Then the tautological map $Q \rightarrow Q_1$ is a differential operator, and the same holds for its inverse.
- (b) Let P_2 be another \mathcal{O}_M -module. Then a map $A : P_2 \times P_1 \rightarrow P$ is a bidifferential operator if and only if the map

$$P_2 \rightarrow \mathcal{D}iff_M(P_1; P) : s \mapsto A(s, \bullet)$$

is a differential operator. The statement holds for both \mathcal{O}_M -module structures on $\mathcal{D}iff_M(P_1; P)$.

Proof. (a) Let (x_1, \dots, x_n) be a local coordinate system on M . To show that $Q \rightarrow Q_1$ is a differential operator, we need to prove that for any $A \in \mathcal{D}iff_M(P_1; P)$, there is a formula

$$fA = \sum_{\alpha=(\alpha_1, \dots, \alpha_n)} a_\alpha \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f \right) \quad (f \in \mathcal{O}_M)$$

for some $a_\alpha \in \mathcal{D}iff_M(P_1; P)$ such that locally, only finitely many of a_α are non-zero. Such a formula exists because locally on M , the order of A is finite, since P_1 is coherent. The proof that $Q_1 \rightarrow Q$ is a differential operator is similar.

- (b) Note that by part (a), we have an identification $\mathcal{D}iff_M(P_2; Q) = \mathcal{D}iff_M(P_2; Q_1)$. Therefore, it suffices to prove the claim for one of the two \mathcal{O}_M -module structures. However the statement is obvious for the \mathcal{O}_M -module Q . \square

EXAMPLE 3.8. Let P be an \mathcal{O}_M -module. By Lemma 3.7, we get identifications

$$\mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P) = \mathcal{D}iff_M(\mathcal{O}_M; P_{\mathcal{D}}) = P_{\mathcal{D}} \otimes_{\mathcal{O}_M} \mathcal{D}_M = P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M.$$

In fact, we obtain two such identifications corresponding to two actions of \mathcal{O}_M on $P_{\mathcal{D}}$. Let us write them explicitly.

First, consider $P_{\mathcal{D}}$ as an \mathcal{O}_M -module using the action of \mathcal{O}_M on P (this \mathcal{O}_M -module is denoted by Q in Lemma 3.7). This corresponds to forming the tensor product

$$P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M$$

using the left action of \mathcal{O}_M on both copies of \mathcal{D}_M . We then obtain an isomorphism

$$\begin{aligned} P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M &\longrightarrow \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P) \\ p \otimes A_1 \otimes A_2 &\longmapsto A_2(\bullet)A_1(\bullet)p \quad (p \in P; A_1, A_2 \in \mathcal{D}_M). \end{aligned}$$

Now consider $P_{\mathcal{D}}$ as an \mathcal{O}_M -module using the action of \mathcal{O}_M on itself (this is denoted by Q_1 in Lemma 3.7). This corresponds to forming the tensor product

$$P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M$$

using the structure of \mathcal{O}_M -bimodule on the first copy of \mathcal{D}_M . This provides an isomorphism

$$\begin{aligned} P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M &\longrightarrow \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P) \\ p \otimes A_1 \otimes A_2 &\longmapsto A_1(\bullet A_2(\bullet))p \quad (p \in P; A_1, A_2 \in \mathcal{D}_M). \end{aligned}$$

Of course, similar formulas apply to the identification

$$P \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes \cdots \otimes_{\mathcal{O}_M} \mathcal{D}_M = \mathcal{D}iff_M(\mathcal{O}_M, \dots, \mathcal{O}_M; P)$$

for differential operators in more than two variables.

REMARK 3.9. Lemma 3.7 provides an interpretation of polydifferential operators using \mathcal{D} -modules. However, a more canonical interpretation can be given using \star -operations on \mathcal{D} -modules defined in [BD04], see Remark 3.38 for a short summary.

3.1.3. Notice that in our study of $\mathcal{D}iff_M(Q; P)$, we never imposed any restrictions, such as quasi-coherence, on the \mathcal{O}_M -module P (on the other hand, coherence of Q is necessary for Lemma 3.7). For this reason, the above results extend immediately to the following situation: Let $p_X : Z \rightarrow X$ be a morphism of complex manifolds, let Q be an \mathcal{O}_X -module (preferably coherent), and let P be an \mathcal{O}_Z -module (or, more

generally, a $p_X^{-1}\mathcal{O}_X$ -module). Then we can consider differential operators $p_X^{-1}Q \rightarrow P$ defined using the action of $p_X^{-1}\mathcal{O}_X$. Such differential operators form a sheaf on Z that we denote by $\mathcal{D}iff_Z(p_X^{-1}Q; P)$. In particular, if $Q = \mathcal{O}_X$, we obtain the sheaf

$$\mathcal{D}iff_Z(p_X^{-1}\mathcal{O}_X; P) = P \otimes_{p_X^{-1}\mathcal{O}_X} p_X^{-1}\mathcal{D}_X;$$

it is the right $p_X^{-1}\mathcal{D}_X$ -module induced by P .

Similarly, suppose we are given two morphisms $p_X : Z \rightarrow X$ and $p_Y : Z \rightarrow Y$, and suppose P_1 (resp. P_2, P) is an \mathcal{O}_X -module (resp. an \mathcal{O}_Y -module, an \mathcal{O}_Z -module). We can then consider bidifferential operators $p_X^{-1}P_1 \times p_Y^{-1}P_2 \rightarrow P$; they form a sheaf on Z that we denote by $\mathcal{D}iff_Z(p_X^{-1}P_1, p_Y^{-1}P_2; P)$. One can also consider polydifferential operators in more than two variables in the same setting. The details are left to the reader.

3.1.4. All of the above remains valid in the relative setting. Namely, let $p_X : Z \rightarrow X$ be a submersive morphism of complex manifolds. Given \mathcal{O}_Z -modules P, Q , we define the sheaf of **relative differential operators** from P to Q to be

$$\mathcal{D}iff_{Z/X}(P; Q) := \{A \in \mathcal{D}iff_Z(P; Q) : A(p_X^{-1}f) = (p_X^{-1}f)A \text{ for all } f \in \mathcal{O}_X\}.$$

In particular, set $\mathcal{D}_{Z/X} := \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; \mathcal{O}_Z)$. For any \mathcal{O}_Z -module P , the **induced** right $\mathcal{D}_{Z/X}$ -module

$$P_{\mathcal{D}/X} := P \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z/X}$$

is identified with $\mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)$. Note that P can be reconstructed from $P_{\mathcal{D}/X}$ as

$$P = P_{\mathcal{D}/X} \otimes_{\mathcal{D}_{Z/X}} \mathcal{O}_Z.$$

The induction functor is adjoint to the forgetful functor from $\mathcal{D}_{Z/X}$ -modules to \mathcal{O}_Z -modules, and the relative versions of Lemmas 3.3 and 3.4 hold. The proofs are parallel to the absolute case.

LEMMA 3.10. *Let P be an \mathcal{O}_Z -module.*

(a) *For any right $\mathcal{D}_{Z/X}$ -module F , there are natural isomorphisms*

$$\begin{aligned} \text{Ext}_{\mathcal{D}_{Z/X}}^i(P_{\mathcal{D}/X}, F) &= \text{Ext}_{\mathcal{O}_Z}^i(P, F) & (i \geq 0) \\ \mathcal{E}xt_{\mathcal{D}_{Z/X}}^i(P_{\mathcal{D}/X}, F) &= \mathcal{E}xt_{\mathcal{O}_Z}^i(P, F) & (i \geq 0) \end{aligned}$$

of vector spaces and sheaves of vector spaces, respectively.

(b) *For any left $\mathcal{D}_{Z/X}$ -module F , there is a natural isomorphism*

$$\text{Tor}_i^{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}, F) = \text{Tor}_i^{\mathcal{O}_Z}(P, F)$$

of sheaves of \mathcal{O}_Z -modules. In particular, for $F = \mathcal{O}_Z$, there is an isomorphism

$$P_{\mathcal{D}/X} \otimes_{\mathcal{D}_{Z/X}}^{\mathbb{L}} \mathcal{O}_Z = P.$$

LEMMA 3.11. *Let Q, P be \mathcal{O}_Z -modules.*

(a) *Any differential operator $A \in \mathcal{D}iff_{Z/X}(Q; P)$ induces a map*

$$Q \rightarrow \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P) = P_{\mathcal{D}/X} : s \mapsto A(\bullet s) \quad (s \in Q).$$

This provides an identification $\mathcal{D}iff_{Z/X}(Q; P) = \mathcal{H}om_{\mathcal{O}_Z}(Q, P_{\mathcal{D}/X})$.

(b) *Any differential operator $A \in \mathcal{D}iff_{Z/X}(Q; P)$ induces a map*

$$\mathcal{D}iff_{Z/X}(\mathcal{O}_Z; Q) \rightarrow \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P) : B \mapsto A \circ B.$$

This provides an identification

$$\begin{aligned} \mathcal{D}iff_{Z/X}(Q; P) &= \mathcal{H}om_{\mathcal{D}_{Z/X}}(\mathcal{D}iff_{Z/X}(\mathcal{O}_Z; Q), \mathcal{D}iff_{Z/X}(\mathcal{O}_Z; P)) \\ &= \mathcal{H}om_{\mathcal{D}_{Z/X}}(Q_{\mathcal{D}/X}, P_{\mathcal{D}/X}). \end{aligned}$$

□

3.2 D-modules on ★-quantizations. As before, let M be a complex manifold. In this section, we consider differential operators on ★-quantizations of M . It makes sense to consider differential operators between \mathcal{O} -modules on different quantizations of M .

DEFINITION 3.12. *Let \widetilde{M}' , \widetilde{M} be two neutralized ★-quantizations of M , and let \widetilde{P} be a $\mathcal{O}_{\widetilde{M}}$ -module. Let $A : \mathcal{O}_{\widetilde{M}'} \rightarrow \widetilde{P}$ be an R -linear morphism. We say that A is a **differential operator** if for every point $x \in M$, there exists a neighborhood U of x and $N \geq 0$ with the following property:*

For any open subset $V \subset U$ and any choice of sections

$$\widetilde{f}'_k \in \Gamma(V, \mathcal{O}_{\widetilde{M}'}), \quad \widetilde{f}_k \in \Gamma(V, \mathcal{O}_{\widetilde{M}}) \quad \text{such that} \quad \widetilde{f}'_k = \widetilde{f}_k \pmod{\hbar} \quad (k = 0, \dots, N),$$

the sequence of maps $A_k : \mathcal{O}_{\widetilde{M}'}|_V \rightarrow \widetilde{P}|_V$ defined recursively by

$$A_{-1} := A|_V, \quad A_k := \widetilde{f}'_k A_{k-1} - A_{k-1} \widetilde{f}'_k$$

satisfies $A_N = 0$.

*Let \widetilde{P}' be an $\mathcal{O}_{\widetilde{M}'}$ -module. An R -linear map $A : \widetilde{P}' \rightarrow \widetilde{P}$ is a **differential operator** if for any local section $\widetilde{s} \in \widetilde{P}'$, the map*

$$\mathcal{O}_{\widetilde{M}'} \rightarrow \widetilde{P} : \widetilde{f} \mapsto A(\widetilde{f}\widetilde{s})$$

is a differential operator. Differential operators form a sheaf that we denote by $\mathcal{D}iff_M(\widetilde{P}'; \widetilde{P})$.

3.2.1. Let us describe explicitly the sheaf $\mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$. Recall that locally, the structure sheaf of a neutralized ★-quantization of M is isomorphic to $\mathcal{O}_M \otimes_{\mathbb{C}} R$ equipped with a ★-product. Choose such isomorphisms

$$\theta' : \mathcal{O}_M \otimes_{\mathbb{C}} R \rightarrow \mathcal{O}_{\widetilde{M}'}, \quad \text{and} \quad \theta : \mathcal{O}_M \otimes_{\mathbb{C}} R \rightarrow \mathcal{O}_{\widetilde{M}}$$

for \widetilde{M} and for \widetilde{M}' . For simplicity, we assume that θ and θ' exist globally to avoid passing to an open cover of M . Let us also suppose that M is isomorphic to an open subset of \mathbb{C}^n ; let us fix global coordinates (x_1, \dots, x_n) on M .

We claim that Definition 3.12 reduces to the following description.

LEMMA 3.13. *A map $A : \mathcal{O}_{\widetilde{M}'} \rightarrow \widetilde{P}$ is a differential operator if and only if it can be written in the form*

$$\widetilde{g}' \mapsto \sum_{\alpha=(\alpha_1, \dots, \alpha_n)} \theta \left(\frac{\partial^{\alpha_1} \dots \partial^{\alpha_n} ((\theta')^{-1} \widetilde{g}')}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right) a_\alpha, \tag{3}$$

for sections $a_\alpha \in \widetilde{P}$ such that locally only finitely many of a_α 's are non-zero.

Proof. Denote by $\mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$ the sheaf of operators $A : \mathcal{O}_{\widetilde{M}'} \rightarrow \widetilde{P}$ of the form (3). Let us prove that $\mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}) \subset \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$.

Fix $A \in \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$, and consider its order given by

$$\text{ord}(A) = \max\{\alpha_1 + \dots + \alpha_n : a_\alpha \neq 0\}.$$

Passing to an open cover of M , we may assume that $\text{ord}(A)$ is finite. Let us choose a sequence of sections $\widetilde{f}'_k \in \mathcal{O}_{\widetilde{M}'}, \widetilde{f}_k \in \mathcal{O}_{\widetilde{M}}$ ($k \geq 0$) such that $\widetilde{f}'_k = \widetilde{f}_k \pmod{\hbar}$, and define A_k recursively by

$$A_{-1} := A|_V, \quad A_k := \widetilde{f}_k A_{k-1} - A_{k-1} \widetilde{f}'_k,$$

as in Definition 3.12. We now prove that $A_N = 0$ for some N that depends only on $\text{ord}(A)$, the orders of bidifferential operators giving the \star -products on $\mathcal{O}_{\widetilde{M}}$ and $\mathcal{O}_{\widetilde{M}'}$, and the number r such that $\hbar^r \widetilde{P} = 0$. The proof proceeds by induction on r . For $r = 0$, we have $\widetilde{P} = 0$ and we can take $N = 0$ (or $N = -1$).

Suppose now $r > 0$. Let $\rho(A)$ be the ‘reduction of A modulo \hbar ’ defined to be the composition

$$\mathcal{O}_{\widetilde{M}'} \rightarrow \widetilde{P} \rightarrow \widetilde{P}/\hbar\widetilde{P}.$$

Note that $\widetilde{P}/\hbar\widetilde{P}$ is a \mathcal{O}_M -module, and that $\rho(A)$ factors through \mathcal{O}_M . In other words, $\rho(A) \in \mathcal{D}iff_M(\mathcal{O}_M; \widetilde{P}/\hbar\widetilde{P})$. It is now easy to see that

$$\text{ord}(\rho(A)) > \text{ord}(\rho(A_0)) > \text{ord}(\rho(A_1)) > \dots,$$

and therefore $\rho(A_{\text{ord}(A)}) = 0$. In other words, the image of $A_{\text{ord}(A)}$ is contained in $\hbar\widetilde{P}$.

Now notice that $A_0 \in \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$, and moreover, $\text{ord}(A_0) \leq \text{ord}(A) + C$ for some constant C . We now see that $A_{\text{ord}(A)}$ is a section of $\mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \hbar\widetilde{P})$ and that its order is bounded by $\text{ord}(A)(C + 1) + C$. By the induction hypothesis $A_{\text{ord}(A)} \in \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \hbar\widetilde{P})$ so $A \in \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \hbar\widetilde{P})$.

To complete the proof of the lemma, we need to check that the embedding $\mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}) \hookrightarrow \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}}; \widetilde{P})$ is bijective. Again, we proceed by induction on r such that $\hbar^r \widetilde{P} = 0$. The statement is clear if $r = 1$. For $r > 1$, it follows from the exact sequences

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \hbar\widetilde{P}) & \rightarrow & \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}) & \rightarrow & \mathcal{D}iff'_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}/\hbar\widetilde{P}) & \\ & \downarrow & & \downarrow & & \parallel & \\ 0 \rightarrow & \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \hbar\widetilde{P}) & \rightarrow & \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}) & \rightarrow & \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}/\hbar\widetilde{P}). & \square \end{array}$$

REMARK 3.14. From the proof of Lemma 3.13, we see that the notion of order is not very useful for differential operators on ★-quantizations. The definition of $\text{ord}(A)$ used in the proof is quite artificial and depends on the choice of trivializations θ and θ' . Although this ‘order’ is related to the value of N appearing in Definition 3.12 (which is another candidate for order of differential operator), the relation is very indirect.

To give another example, consider differential operators $\mathcal{O}_{\widetilde{M}} \rightarrow \mathcal{O}_{\widetilde{M}}$. Besides viewing $\mathcal{O}_{\widetilde{M}}$ as a left module over itself, we can view it as a right module over itself (or, equivalently, as an $\mathcal{O}_{\widetilde{M}}^{op}$ -module). It follows from Lemma 3.13 that we obtain the same sheaf of differential operators in these two cases. However, it seems impossible to define orders in compatible way: for instance, right multiplication by a function is $\mathcal{O}_{\widetilde{M}}$ -linear with respect to the left $\mathcal{O}_{\widetilde{M}}$ -module structure (‘has order zero’) and non-linear with respect to the right $\mathcal{O}_{\widetilde{M}}$ -module structure (‘has positive order’).

3.2.2. Consider the sheaf $\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}} := \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \mathcal{O}_{\widetilde{M}})$. It follows from Lemma 3.13 that $\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}}$ is a flat R -module equipped with an identification $\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}}/\hbar\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}} = \mathcal{D}_M$. Moreover, $\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}}$ has a right action of $\mathcal{O}_{\widetilde{M}}$ coming from its action on itself that turns it into a locally free right $\mathcal{O}_{\widetilde{M}}$ -module of infinite rank. (We ignore the commuting left action of $\mathcal{O}_{\widetilde{M}}$ and the two actions of $\mathcal{O}_{\widetilde{M}'}$.) We now get a natural identification

$$\mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P}) = \mathcal{D}_{\widetilde{M}'}^{\widetilde{M}} \otimes_{\mathcal{O}_{\widetilde{M}}} \widetilde{P}.$$

Moreover, this identification respects the structure of a right module over

$$\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}} := \mathcal{D}_{\widetilde{M}'}^{\widetilde{M}} = \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \mathcal{O}_{\widetilde{M}'}).$$

It is easy to see that the functor

$$\widetilde{P} \mapsto \mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}'}; \widetilde{P})$$

has properties similar to those of the induction $P \mapsto P_{\mathcal{D}}$ studied in Section 3.1. Clearly, the functor is exact, because $\mathcal{D}_{\widetilde{M}'}^{\widetilde{M}}$ is flat over $\mathcal{O}_{\widetilde{M}}$. Moreover, given an $\mathcal{O}_{\widetilde{M}_1}$ -

module \tilde{P}_1 and an $\mathcal{O}_{\tilde{M}_2}$ -module \tilde{P}_2 on two neutralized \star -quantizations \tilde{M}_1, \tilde{M}_2 of M , there is a natural identification

$$\mathcal{D}iff_M(\tilde{P}_1; \tilde{P}_2) = \mathcal{H}om_{\mathcal{D}_{\tilde{M}'}}(\mathcal{D}iff_M(\mathcal{O}_{\tilde{M}'}; \tilde{P}_1), \mathcal{D}iff_M(\mathcal{O}_{\tilde{M}'}; \tilde{P}_2)) \tag{4}$$

defined as in Lemma 3.4(b).

REMARK 3.15. It may seem strange that we assign to an $\mathcal{O}_{\tilde{M}}$ -module \tilde{P} on a quantization \tilde{M} a \mathcal{D} -module on a different quantization \tilde{M}' . Here is a different approach to this assignment that should clarify the situation.

Let \tilde{P} be an $\mathcal{O}_{\tilde{M}}$ -module on a neutralized \star -quantization \tilde{M} of M . It is most natural to assign to \tilde{P} the right $\mathcal{D}_{\tilde{M}}$ -module $\mathcal{D}iff_M(\mathcal{O}_{\tilde{M}}; \tilde{P})$ (corresponding to the choice $\tilde{M}' = \tilde{M}$).

On the other hand, we have the following

PROPOSITION 3.16. *For any two neutralized \star -quantizations \tilde{M} and \tilde{M}' of M , the categories of right modules over $\mathcal{D}_{\tilde{M}}$ and over $\mathcal{D}_{\tilde{M}'}$ are canonically equivalent (the same applies to the categories of left \mathcal{D} -modules, but this is irrelevant for our purposes).*

Proof. Indeed, the sheaf $\mathcal{D}_{\tilde{M}'}^{\tilde{M}} = \mathcal{D}iff_M(\mathcal{O}_{\tilde{M}'}; \mathcal{O}_{\tilde{M}})$ is naturally a $\mathcal{D}_{\tilde{M}}\text{-}\mathcal{D}_{\tilde{M}'}$ -bimodule, and Lemma 3.13 shows that it is locally free of rank one as a left $\mathcal{D}_{\tilde{M}}$ -module and as a right $\mathcal{D}_{\tilde{M}'}$ -module. Therefore, it provides a Morita equivalence

$$F \mapsto F \otimes_{\mathcal{D}_{\tilde{M}}} \mathcal{D}_{\tilde{M}'}^{\tilde{M}} \quad (F \text{ is a right } \mathcal{D}_{\tilde{M}}\text{-module})$$

between the two categories of \mathcal{D} -modules. The inverse equivalence is provided by $\mathcal{D}_{\tilde{M}}^{\tilde{M}'}$. □

Note that the equivalence can be viewed as a version of ‘isomonodromic transformation’: since \tilde{M} and \tilde{M}' are quantizations of the same manifold, they differ only ‘infinitesimally’, and it is well known that the notion of a \mathcal{D} -module is ‘topological’ in the sense that it is invariant under infinitesimal deformations.

We can now use the equivalence to pass between \mathcal{D} -modules on different neutralized \star -quantizations of M . It is easy to see that

$$\mathcal{D}iff_M(\mathcal{O}_{\tilde{M}}; \tilde{P}) \otimes_{\mathcal{D}_{\tilde{M}}} \mathcal{D}_{\tilde{M}'}^{\tilde{M}} = \mathcal{D}iff_M(\mathcal{O}_{\tilde{M}'}; \tilde{P}).$$

This allows us to interpret the identification (4) as follows: the sheaves \tilde{P}_i ($i = 1, 2$) give rise to induced $\mathcal{D}_{\tilde{M}_i}$ -modules. Using the ‘isomonodromic transformation’, we can put this \mathcal{D} -modules into a single category (of $\mathcal{D}_{\tilde{M}'}$ -modules), so that it makes sense to talk about homomorphisms between them. One can thus say that the ‘isomonodromic transformation’ is the reason why differential operators between \mathcal{O} -modules on different quantizations make sense.

It should be clear from this discussion that, when one works with \mathcal{D} -modules on a neutralized \star -quantization \widetilde{M}' of M , the choice of \widetilde{M}' is largely irrelevant. For this reason, in most situations we can (and will) take \widetilde{M}' to be the trivial deformation of M : $\mathcal{O}_{\widetilde{M}'} = \mathcal{O}_M \otimes_{\mathbb{C}} R$. The advantage of this choice is that $\mathcal{D}_{\widetilde{M}'}$ -modules then become simply modules over $\mathcal{D}_M \otimes_{\mathbb{C}} R$. On the other hand, in the relative situation (discussed below) it is no longer true that the category of relative \mathcal{D} -modules on a quantization is independent of the choice of quantization, see Remark 3.21. However, relative \mathcal{D} -modules on quantizations appear only in Section 4.4, which is independent from the rest of the paper.

DEFINITION 3.17. *Let $\widetilde{M}_1, \dots, \widetilde{M}_n, \widetilde{M}$ be neutralized \star -quantizations of M (for some $n \geq 1$), and let \widetilde{P} be a $\mathcal{O}_{\widetilde{M}}$ -module. Let $A : \mathcal{O}_{\widetilde{M}_1} \times \dots \times \mathcal{O}_{\widetilde{M}_n} \rightarrow \widetilde{P}$ be an R -polylinear morphism. We say that A is a **polydifferential operator** if locally on M , there exists $N \geq 0$ with the following property:*

For any $j = 1, \dots, n$ and any choice of (local) sections

$$\widetilde{f}'_k \in \mathcal{O}_{\widetilde{M}_j}, \quad \widetilde{f}_k \in \mathcal{O}_{\widetilde{M}} \quad \text{such that} \quad \widetilde{f}'_k = \widetilde{f}_k \pmod{\hbar} \quad (k = 0, \dots, N),$$

the sequence of maps $A_k : \mathcal{O}_{\widetilde{M}_1} \times \dots \times \mathcal{O}_{\widetilde{M}_n} \rightarrow \widetilde{P}$ defined recursively by

$$A_{-1} := A, \quad A_k := \widetilde{f}_k A_{k-1} - A_{k-1}(\text{id} \times \dots \times \widetilde{f}'_k \times \dots \times \text{id})$$

satisfies $A_N = 0$. Here $(\text{id} \times \dots \times \widetilde{f}'_k \times \dots \times \text{id})$ (with \widetilde{f}'_k in the j -th position) refers to the map from $\mathcal{O}_{\widetilde{M}_1} \times \dots \times \mathcal{O}_{\widetilde{M}_n}$ to itself that multiplies j -th argument by \widetilde{f}'_k and leaves the remaining $n - 1$ arguments unchanged.

Let $\widetilde{P}_1, \dots, \widetilde{P}_n$ be \mathcal{O} -modules on $\widetilde{M}_1, \dots, \widetilde{M}_n, \widetilde{M}$. An R -polylinear morphism

$$A : \widetilde{P}_1 \times \dots \times \widetilde{P}_n \rightarrow \widetilde{P}$$

*is a **polydifferential operator** if for any choice of local sections $\widetilde{s}_1 \in \widetilde{P}_1, \dots, \widetilde{s}_n \in \widetilde{P}_n$, the map*

$$\mathcal{O}_{\widetilde{M}_1} \times \dots \times \mathcal{O}_{\widetilde{M}_n} \rightarrow \widetilde{P} : (\widetilde{g}_1, \dots, \widetilde{g}_n) \mapsto A(\widetilde{g}_1 \widetilde{s}_1, \dots, \widetilde{g}_n \widetilde{s}_n)$$

is a polydifferential operator. Polydifferential operators form a sheaf, which we denote by $\mathcal{D}\text{iff}_M(\widetilde{P}_1, \dots, \widetilde{P}_n; \widetilde{P})$.

3.2.3. Let $p_X : Z \rightarrow X$ be a morphism of complex manifolds. Suppose \widetilde{Z} is a neutralized \star -quantization of Z and \widetilde{X} is a neutralized \star -quantization of X ; \widetilde{Z} and \widetilde{X} do not need to be compatible in any way. Let \widetilde{Q} be an $\mathcal{O}_{\widetilde{X}}$ -module and let \widetilde{P} be an $\mathcal{O}_{\widetilde{Z}}$ -module. It is easy to extend Definition 3.12 to construct the sheaf of differential operators $p_X^{-1} \widetilde{Q} \rightarrow \widetilde{P}$, which we denote by $\mathcal{D}\text{iff}_Z(p_X^{-1} \widetilde{Q}; \widetilde{P})$.

Explicitly, an R -linear map $A : p_X^{-1} \widetilde{Q} \rightarrow \widetilde{P}$ is a differential operator if for every point $z \in Z$ and every section $s \in p_X^{-1} \widetilde{Q}$ defined at z , there exists a neighborhood U of z and $N \geq 0$ with the following property:

For any open subset $V \subset U$ and for any choice of sections $\tilde{f}'_k \in \Gamma(V, p_X^{-1}\mathcal{O}_{\tilde{X}})$, $\tilde{f}_k \in \Gamma(V, \mathcal{O}_{\tilde{Z}})$ such that $\tilde{f}'_k = \tilde{f}_k \pmod{\hbar}$ ($k = 0, \dots, N$), the sequence of maps $A_k : p_X^{-1}\tilde{P}|_V \rightarrow \tilde{Q}|_V$ defined by

$$A_{-1} := A|_V, \quad A_k := \tilde{f}_k A_{k-1} - A_{k-1} \tilde{f}'_k$$

satisfies $A_N(s|_V) = 0$.

3.2.4. Finally, let us consider relative differential operators on quantizations. Let $p_X : Z \rightarrow X$ be a submersive morphism of complex manifolds, and let \tilde{Z} and \tilde{X} be neutralized \star -quantizations of Z and X , respectively. Suppose that $p_X : Z \rightarrow X$ extends to a morphism of quantizations $\tilde{p}_X : \tilde{Z} \rightarrow \tilde{X}$. In other words, \tilde{p}_X is an R -linear morphism of ringed spaces that acts on underlying sets as p_X and such that the reduction of $\tilde{p}_X^{-1} : p_X^{-1}\mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{Z}}$ modulo \hbar equals p_X^{-1} . We are mostly interested in the case when $\tilde{Z} = \tilde{X} \times \tilde{Y}$ for a neutralized \star -quantization Y , and \tilde{p}_X is the natural projection.

Now suppose \tilde{Z}' is another neutralized \star -quantization of \tilde{Z} and suppose that $p_X : Z \rightarrow X$ also extends to a morphism $\tilde{p}'_X : \tilde{Z}' \rightarrow \tilde{X}$. Now given \mathcal{O} -modules \tilde{P}' , \tilde{P} on \tilde{Z}' and \tilde{Z} , respectively, we define the sheaf of **relative differential operators** by essentially the same formula as before:

$$\mathcal{D}iff_{Z/X}(\tilde{P}'; \tilde{P}) := \{A \in \mathcal{D}iff_Z(\tilde{P}'; \tilde{P}) : A((\tilde{p}'_X)^{-1}\tilde{f}) = (\tilde{p}_X^{-1}\tilde{f})A \text{ for all } \tilde{f} \in \mathcal{O}_{\tilde{X}}\}.$$

The above properties of differential operators remain true in the setting of relative quantizations.

PROPOSITION 3.18. *The sheaf $\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}})$ is flat over R , and the obvious ‘reduction modulo \hbar ’ map*

$$\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}}) / \hbar \mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}}) \rightarrow \mathcal{D}_{Z/X}$$

is an isomorphism.

Proof. Looking at the filtration by powers of \hbar , we see that it suffices to prove that the ‘reduction modulo \hbar ’ map is surjective. Let us fix $A \in \mathcal{D}_{Z/X}$, and let us lift it to an operator $\tilde{A} \in \mathcal{D}iff_Z(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}})$ (this can always be done locally). It then leads to a bidifferential operator

$$\tilde{B} : p_X^{-1}\mathcal{O}_{\tilde{X}} \times \mathcal{O}_{\tilde{Z}'} \rightarrow \mathcal{O}_{\tilde{Z}} : (\tilde{f}, \tilde{g}) \mapsto \tilde{f}\tilde{A}(\tilde{g}) - \tilde{A}(\tilde{f}\tilde{g})$$

that vanishes modulo \hbar . Note that \tilde{B} is a cocycle:

$$\tilde{f}_1\tilde{B}(\tilde{f}_2, \tilde{g}) - \tilde{B}(\tilde{f}_1\tilde{f}_2, \tilde{g}) + \tilde{B}(\tilde{f}_1, \tilde{f}_2\tilde{g}) = 0 \quad (\tilde{f}_1, \tilde{f}_2 \in p_X^{-1}\mathcal{O}_{\tilde{X}}; \tilde{g} \in \mathcal{O}_{\tilde{Z}'}).$$

We need to show that it is locally a coboundary: there exists a differential operator

$$\tilde{C} : \mathcal{O}_{\tilde{Z}'} \rightarrow \mathcal{O}_{\tilde{Z}}$$

that vanishes modulo \hbar and satisfies

$$\tilde{B}(\tilde{f}, \tilde{g}) = \tilde{f}\tilde{C}(\tilde{g}) - \tilde{C}(\tilde{f}\tilde{g}) \quad (\tilde{f} \in p_X^{-1}\mathcal{O}_{\tilde{X}}; \tilde{g} \in \mathcal{O}_{\tilde{Z}}).$$

Then $\tilde{A} - \tilde{C}$ would lift A to a relative differential operator in $\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}}; \mathcal{O}_{\tilde{Z}})$.

Arguing by induction in powers of \hbar , we see that it suffices to prove the following statement: If $B \in \mathcal{D}iff_Z(p_X^{-1}\mathcal{O}_X, \mathcal{O}_Z; \mathcal{O}_Z)$ satisfies the cocycle condition

$$f_1B(f_2, g) - B(f_1f_2, g) + B(f_1, f_2g) = 0 \quad (f_1, f_2 \in p_X^{-1}\mathcal{O}_X; g \in \mathcal{O}_Z),$$

then it is locally a coboundary: locally, there exists $C \in \mathcal{D}_Z$ such that

$$B(f, g) = fC(g) - C(fg) \quad (f \in p_X^{-1}\mathcal{O}_X; g \in \mathcal{O}_Z).$$

This can be proved directly.

Indeed, let (x_1, \dots, x_n) be local coordinates on X ; we denote their pullbacks to Z in the same way. Then changing B by a coboundary, we can ensure that $B(x_1, g) = 0$. Since B is a cocycle, this implies that

$$B(f, x_1g) = B(x_1f, g) = x_1B(f, g) \quad (f \in p_X^{-1}\mathcal{O}_X, g \in \mathcal{O}_Z).$$

In other words, B does not include differentiation with respect to x_1 : it is a ‘bidifferential operator relative to x_1 ’. We can now treat x_2 in the same way, and so on. \square

REMARK 3.19. Essentially, the proof of Proposition 3.18 verifies vanishing of the first local Hochschild cohomology of \mathcal{O}_X with coefficients in \mathcal{D}_Z . Here ‘local’ means that the Hochschild complex is formed using cochains that are differential operators. (Confusingly, the complex is also local in a different sense: its terms and cohomology objects are sheaves rather than vector spaces.) The proof can be made more transparent using the framework of comodules introduced in Section 3.3, see Remark 3.37.

COROLLARY 3.20. Consider on $\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}}; \mathcal{O}_{\tilde{Z}})$ the right action of $\mathcal{O}_{\tilde{Z}}$ coming from its right action on itself. Then the map

$$\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}}; \mathcal{O}_{\tilde{Z}}) \otimes_{\mathcal{O}_{\tilde{Z}}} \tilde{P} \rightarrow \mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}}; \tilde{P}) : A \otimes p \mapsto A(\bullet)p$$

is an isomorphism for any $\mathcal{O}_{\tilde{Z}}$ -module \tilde{P} .

Proof. Consider the filtration by powers of \hbar . By Proposition 3.18 and Lemma 3.11, we see that the map induces an isomorphism on the associated graded quotients. This implies the corollary. \square

3.2.5. We now obtain a functor

$$\tilde{P} \mapsto \mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}}; \tilde{P})$$

from the category of $\mathcal{O}_{\tilde{Z}}$ -modules to the category of right modules over

$$\mathcal{D}_{\tilde{Z}'/\tilde{X}} := \mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}}).$$

By Corollary 3.20, this functor can be rewritten as

$$\tilde{P} \mapsto \mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}}) \otimes_{\mathcal{O}_{\tilde{Z}}} \tilde{P}.$$

Note that $\mathcal{D}_{Z/X}$ is a flat right \mathcal{O}_Z -module, and therefore Proposition 3.18 implies that $\mathcal{D}iff_{Z/X}(\mathcal{O}_{\tilde{Z}'}; \mathcal{O}_{\tilde{Z}})$ is a flat right $\mathcal{O}_{\tilde{Z}}$ -module. Hence the functor is exact.

REMARK 3.21. In the relative setting, we have a version of the identification (4), and of Proposition 3.16. However, there is an important difference between relative and absolute differential operators on quantizations. Namely, we can consider absolute differential operators between \mathcal{O} -modules on any two neutralized \star -quantizations of a manifold. On the other hand, given a submersive map $Z \rightarrow X$ and \mathcal{O} -modules on two neutralized \star -quantizations \tilde{Z} and \tilde{Z}' of Z , it is necessary to have a neutralized \star -quantization \tilde{X} of X and quantizations of p_X to morphisms $\tilde{p}_X : \tilde{Z} \rightarrow \tilde{X}$, $\tilde{p}'_X : \tilde{Z}' \rightarrow \tilde{X}$ in order for relative differential operators to make sense.

Thus, a relative version of Proposition 3.16 provides an equivalence between the categories of (right) modules over $\mathcal{D}_{\tilde{Z}/\tilde{X}}$ and over $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$. In this way, the category of $\mathcal{D}_{\tilde{Z}/\tilde{X}}$ does not actually depend on the choice of \tilde{Z} ; however, the category does depend on the quantization \tilde{X} of X . (See 3.4.6 for another explanation.) As an extreme example, suppose $Z = X$ and $\tilde{Z} = \tilde{X}$. Then $\mathcal{D}_{\tilde{Z}/\tilde{X}} = \mathcal{O}_{\tilde{X}}$, and the category of $\mathcal{O}_{\tilde{X}}$ -modules clearly depends on the choice of quantization \tilde{X} .

3.3 Coalgebras in the category of \mathcal{D} -modules

3.3.1. Let M be a complex manifold. The category of left \mathcal{D}_M -modules $\text{Mod}(\mathcal{D}_M)$ is naturally a tensor category: given $F, G \in \text{Mod}(\mathcal{D}_M)$, the tensor product $F \otimes_{\mathcal{O}_M} G$ carries a left action of \mathcal{D}_M (the action of vector fields is given by the Leibnitz rule from the actions on F and on G). Consider coalgebras in this tensor category.

For the rest of this section, let us fix a coalgebra $\mathcal{A} \in \text{Mod}(\mathcal{D}_M)$. Here and in the rest of the paper, all coalgebras are assumed to be coassociative and counital, but not necessarily cocommutative. We also assume that \mathcal{A} is flat over \mathcal{O}_M . It is well known that this assumption is necessary to define kernel of morphisms between comodules.

Consider now the category of right \mathcal{D}_M -modules $\text{Mod}(\mathcal{D}_M^{op})$. It has an action of the tensor category $\text{Mod}(\mathcal{D}_M)$: given $F \in \text{Mod}(\mathcal{D}_M)$ and $G \in \text{Mod}(\mathcal{D}_M^{op})$, the tensor product $F \otimes_{\mathcal{O}_M} G$ carries a right action of \mathcal{D}_M . It makes sense to talk about \mathcal{A} -comodules in the category of right \mathcal{D}_M -modules.

DEFINITION 3.22. Let $\text{Comod}(\mathcal{A})$ be the category of (left) \mathcal{A} -comodules in the category $\text{Mod}(\mathcal{D}_M^{op})$. We call objects of $\text{Comod}(\mathcal{A})$ simply ‘ \mathcal{A} -comodules’.

LEMMA 3.23. *The category $\text{Comod}(\mathcal{A})$ is an abelian category with enough injectives.*

Proof. Let $f : F \rightarrow G$ be a morphism of \mathcal{A} -comodules. The fact that F is \mathcal{O}_M -flat implies that the kernel of f in the category of right \mathcal{D}_M -modules is automatically an \mathcal{A} -comodule. Thus $\text{Comod}(\mathcal{A})$ is abelian. The forgetful functor $\text{Comod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{D}_M^{op})$ admits a right adjoint: the coinduction functor

$$\text{Mod}(\mathcal{D}_M^{op}) \rightarrow \text{Comod}(\mathcal{A}) : F \mapsto \mathcal{A} \otimes_{\mathcal{O}_M} F.$$

Being the right adjoint functor of an exact functor, it preserves injectivity. Thus we can obtain sufficiently many injective \mathcal{A} -comodules by coinduction. \square

3.3.2. Let us now study deformation of \mathcal{A} -comodules. Let $\text{Mod}(\mathcal{D}_M \otimes_{\mathbb{C}} R)$ be the category of left $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -modules. Alternatively, $\text{Mod}(\mathcal{D}_M \otimes_{\mathbb{C}} R)$ is the category of R -modules in the \mathbb{C} -linear category $\text{Mod}(\mathcal{D}_M)$. The tensor product over $\mathcal{O}_M \otimes_{\mathbb{C}} R$ turns $\text{Mod}(\mathcal{D}_M \otimes_{\mathbb{C}} R)$ into a tensor category.

Let $\tilde{\mathcal{A}}$ be an R -deformation of \mathcal{A} . That is, $\tilde{\mathcal{A}}$ is a coalgebra in $\text{Mod}(\mathcal{D}_M \otimes_{\mathbb{C}} R)$ that is flat over R and equipped with an isomorphism $\mathcal{A} \simeq \tilde{\mathcal{A}}/\hbar\tilde{\mathcal{A}}$.

We extend the conventions of Definition 3.22 to $\tilde{\mathcal{A}}$ -comodules. Thus $\tilde{\mathcal{A}}$ -comodules are by definition left $\tilde{\mathcal{A}}$ -comodules in the category $\text{Mod}(\mathcal{D}_M^{op} \otimes_{\mathbb{C}} R)$ of right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -modules; the category of $\tilde{\mathcal{A}}$ -comodules is denoted by $\text{Comod}(\tilde{\mathcal{A}})$.

Given $Q \in \text{Comod}(\mathcal{A})$, we consider the problem of deforming Q to $\tilde{Q} \in \text{Comod}(\tilde{\mathcal{A}})$. More precisely, let $\text{Def}(Q)$ be the groupoid of comodules $\tilde{Q} \in \text{Comod}(\tilde{\mathcal{A}})$ that are flat over R together with an isomorphism $Q \simeq \tilde{Q}/\hbar\tilde{Q}$. As one would expect, deformations of Q are controlled by Ext's from Q to itself.

- PROPOSITION 3.24. (a) *If $\text{Ext}_{\mathcal{A}}^2(Q, Q) = 0$, the groupoid $\text{Def}(Q)$ is non-empty.*
 (b) *If $\text{Ext}_{\mathcal{A}}^1(Q, Q) = \text{Ext}_{\mathcal{A}}^2(Q, Q) = 0$, the groupoid $\text{Def}(Q)$ is connected (that is, all objects are isomorphic).*
 (c) *If $\text{Ext}_{\mathcal{A}}^1(Q, Q) = 0$ and $\tilde{Q} \in \text{Def}(Q)$, then the algebra $\text{End}_{\tilde{\mathcal{A}}}(\tilde{Q})$ is an R -flat deformation of $\text{End}_{\mathcal{A}}(Q)$.*

Proof. The result is essentially classical: for modules over an algebra, it is sketched in [Lau86, Section 2] (and mentioned in [Lau79, p. 150]); and the case we are interested in is similar. Actually, as stated in [Lau79, p. 150], the deformation theory applies ‘in all cases where we have a good cohomology and an obstruction calculus’.

More rigorously, the proposition follows from general theory of deformation of objects in abelian categories due to Lowen ([Low05]). First note that $\text{Comod}(\tilde{\mathcal{A}})$ is a flat deformation of $\text{Comod}(\mathcal{A})$ in the sense of [LV06, Proposition 3.4]. Since $\text{Comod}(\tilde{\mathcal{A}})$ has enough injectives, we need to check that injective $\tilde{\mathcal{A}}$ -comodules are R -coflat. It suffices to check this for coinduced injective objects, that is, for objects of the form $\tilde{\mathcal{A}} \otimes_{\mathcal{O}_M} I$, where I is an injective \mathcal{D}_M^{op} -module. We need to verify that $\text{Ext}_R^1(-, \tilde{\mathcal{A}} \otimes_{\mathcal{O}_M} I) = 0$. But $\text{Ext}_R^1(-, \tilde{\mathcal{A}} \otimes_{\mathcal{O}_M} I)$ is equal to $\tilde{\mathcal{A}} \otimes_{\mathcal{O}_M} \text{Ext}_R^1(-, I)$ by [LV06, Proposition 2.10] and the desired vanishing follows from properties (1) and (2) in [LV06, page 5441]. Now (a) and (b) follow by iterated application of the

dual version of [Low05, Theorem A]. To prove (c), we notice that the filtration of \tilde{Q} by powers of \hbar induces a filtration of $\text{End}_{\tilde{\mathcal{A}}}(\tilde{Q})$, and the associated graded for this filtration is $\text{End}_{\mathcal{A}}(Q) \otimes_{\mathbb{C}} R$. \square

REMARK 3.25. We consider flat deformations, which could be viewed as lifts along the functor

$$\text{Comod}(\tilde{\mathcal{A}}) \rightarrow \text{Comod}(\mathcal{A}) : \tilde{Q} \mapsto \tilde{Q} \otimes_R \mathbb{C}.$$

On the other hand, [Low05, Theorem A] concerns lifting coflat objects, rather than flat ones. That is, [Low05, Theorem A] gives lifts along the functor

$$\text{Comod}(\tilde{\mathcal{A}}) \rightarrow \text{Comod}(\mathcal{A}) : \tilde{Q} \mapsto \mathcal{H}om_R(\mathbb{C}, \tilde{Q});$$

As long as one considers deformation over $R = \mathbb{C}[\hbar]/\hbar^{n+1}$ (rather than over an arbitrary Artinian ring) the two kinds of deformations coincide, because R is a Gorenstein, zero dimensional ring. Indeed, if P is coflat $\tilde{P} \otimes_R \mathbb{C} \cong \mathcal{H}om_R(R, \tilde{P}) \otimes_R \mathbb{C} \cong \mathcal{H}om_R(\text{Hom}_R(\mathbb{C}, R), \tilde{P})$, see e.g. [LV06, Proposition 2.9(3)]. Since R is a local Gorenstein zero dimensional ring, $\text{Hom}_R(\mathbb{C}, R) = R$, and the statement follows. The general case when R is an arbitrary Artinian ring would follow from a version of [Low05, Theorem A] for flat objects; this ‘dual version’ is left as an exercise in [Low05].

3.4 Neutralized \star -quantizations as \mathcal{D} -coalgebras. We now interpret neutralized \star -quantizations and \mathcal{O} -modules on them using the framework of coalgebras and comodules developed in Section 3.3.

3.4.1. Let M be a complex manifold. Given a neutralized \star -quantization \tilde{M} of M , consider differential operators between $\mathcal{O}_{\tilde{M}}$ and the ‘trivial’ deformation $\mathcal{O}_M \otimes_{\mathbb{C}} R$. Set

$$\mathcal{A}_{\tilde{M}} := \text{Diff}_M(\mathcal{O}_{\tilde{M}}; \mathcal{O}_M \otimes_{\mathbb{C}} R).$$

Composition with differential operators on $\mathcal{O}_M \otimes_{\mathbb{C}} R$ equips $\mathcal{A}_{\tilde{M}}$ with a structure of a left $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module.

EXAMPLE 3.26: \star -PRODUCT. Suppose $\mathcal{O}_{\tilde{M}} = \mathcal{O}_M \otimes_{\mathbb{C}} R$ with multiplication given by some \star -product $f \star g$. By Lemma 3.13, a map $A : \mathcal{O}_{\tilde{M}} \rightarrow \mathcal{O}_M \otimes_{\mathbb{C}} R$ is a differential operator if and only if it is of the form

$$A = A_0 + \hbar A_1 + \dots + \hbar^n A_n \in \text{Diff}_M(\mathcal{O}_M; \mathcal{O}_M) \otimes_{\mathbb{C}} R \quad (A_i \in \text{Diff}_M(\mathcal{O}_M; \mathcal{O}_M)).$$

(Technically, Lemma 3.13 applies only locally on M , but once it is proved that A has the required form locally, it is also true globally.) We now see that $\mathcal{A}_{\tilde{M}}$ is the trivial $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module of rank one.

In general, a neutralized \star -quantization $\mathcal{O}_{\tilde{M}}$ is locally isomorphic to $\mathcal{O}_M \otimes_{\mathbb{C}} R$ equipped with a \star -product. Therefore, $\mathcal{A}_{\tilde{M}} \in \text{Mod}(\mathcal{D}_M \otimes_{\mathbb{C}} R)$ is a rank one locally free $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module.

3.4.2. The \mathcal{D}_M -module $\mathcal{A}_{\widetilde{M}}$ has a natural coproduct. Indeed, consider the tensor product

$$\mathcal{A}_{\widetilde{M}} \otimes_{\mathcal{O}_M \otimes_{\mathbb{C}} R} \mathcal{A}_{\widetilde{M}}$$

with the usual \mathcal{D}_M -module structure. It can be identified with the sheaf of bidifferential operators $\mathcal{D}iff(\mathcal{O}_{\widetilde{M}}, \mathcal{O}_{\widetilde{M}}; \mathcal{O}_M \otimes_{\mathbb{C}} R)$ (cf. Example 3.8). The identification sends $A_1 \otimes A_2$ ($A_1, A_2 \in \mathcal{A}_{\widetilde{M}}$) to the operator

$$(\widetilde{f}_1, \widetilde{f}_2) \mapsto A_1(\widetilde{f}_1)A_2(\widetilde{f}_2) \quad (\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{O}_{\widetilde{M}}).$$

Now define the coproduct $\Delta : \mathcal{A}_{\widetilde{M}} \rightarrow \mathcal{A}_{\widetilde{M}} \otimes \mathcal{A}_{\widetilde{M}}$ by $\Delta(A) = A \circ \text{mult}$, where

$$\text{mult} : \mathcal{O}_{\widetilde{M}} \times \mathcal{O}_{\widetilde{M}} \rightarrow \mathcal{O}_{\widetilde{M}}$$

is the multiplication.

PROPOSITION 3.27. *The functor $\widetilde{M} \mapsto \mathcal{A}_{\widetilde{M}}$ is an equivalence between the groupoid of neutralized ★-quantizations of M and the groupoid of R -flat deformations of \mathcal{D}_M as a coassociative and counital coalgebra in $\text{Mod}(\mathcal{D}_M)$.*

Proof. Let us provide an inverse functor. Let $\widetilde{\mathcal{A}}$ be an R -deformation of the left \mathcal{D}_M -module \mathcal{D}_M . In particular, $\widetilde{\mathcal{A}}$ is a locally free $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module of rank one. Set

$$\mathcal{O}_{\widetilde{M}} := \mathcal{H}om_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\widetilde{\mathcal{A}}, \mathcal{O}_M \otimes_{\mathbb{C}} R).$$

Then $\mathcal{O}_{\widetilde{M}}$ is R -flat and $\mathcal{O}_{\widetilde{M}}/\hbar\mathcal{O}_{\widetilde{M}} = \mathcal{O}_M$.

Suppose now that $\widetilde{\mathcal{A}}$ carries a coassociative coproduct $\Delta : \widetilde{\mathcal{A}} \rightarrow \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ extending the canonical coproduct on \mathcal{D}_M . Then Δ induces an associative product on $\mathcal{O}_{\widetilde{M}}$ via

$$\widetilde{f} \cdot \widetilde{g} := (\widetilde{f} \otimes \widetilde{g}) \circ \Delta \quad (\widetilde{f}, \widetilde{g} \in \mathcal{H}om_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\widetilde{\mathcal{A}}, \mathcal{O}_M \otimes_{\mathbb{C}} R)),$$

and the identification $\mathcal{O}_{\widetilde{M}}/\hbar\mathcal{O}_{\widetilde{M}} = \mathcal{O}_M$ respects this product.

Finally, locally we can lift $1 \in \mathcal{D}_M$ to a section $\widetilde{1} \in \widetilde{\mathcal{A}}$. This yields a local isomorphism

$$\mathcal{O}_{\widetilde{M}} \rightarrow \mathcal{O}_M \otimes_{\mathbb{C}} R : \widetilde{f} \mapsto \widetilde{f}(\widetilde{1})$$

of sheaves of R -modules. Under this isomorphism, the product on $\mathcal{O}_{\widetilde{M}}$ corresponds to the ★-product on $\mathcal{O}_M \otimes_{\mathbb{C}} R$ given by the bidifferential operator B such that

$$\Delta(\widetilde{1}) = B(\widetilde{1} \otimes \widetilde{1}).$$

Therefore, \widetilde{M} is a neutralized ★-quantization of M . Verifying that this is indeed the inverse construction to $\widetilde{M} \mapsto \mathcal{A}_{\widetilde{M}}$ is straightforward. □

EXAMPLES 3.28. The coalgebra $\mathcal{A}_{\widetilde{M}^{op}}$ corresponding to the opposite quantization \widetilde{M}^{op} is the opposite coalgebra of $\mathcal{A}_{\widetilde{M}}$: they coincide as \mathcal{D} -modules, but the coproducts are opposite to each other.

Let \widetilde{M} and \widetilde{N} be neutralized \star -quantizations of complex manifolds M and N , respectively. Consider the quantization $\widetilde{M} \times \widetilde{N}$ of $M \times N$. The corresponding coalgebra $\mathcal{A}_{\widetilde{M} \times \widetilde{N}}$ can be obtained as the tensor product $p_M^* \mathcal{A}_{\widetilde{M}} \otimes p_N^* \mathcal{A}_{\widetilde{N}}$. Here $p_M : M \times N \rightarrow \widetilde{M}$ and $p_N : M \times N \rightarrow \widetilde{N}$ are the projections, p_M^* and p_N^* are pullback functors for left \mathcal{D} -modules, and \otimes is the tensor product over $\mathcal{O}_{M \times N} \otimes_{\mathbb{C}} R$ (which gives the tensor structure on $\text{Mod}(\mathcal{D}_{M \times N} \otimes_{\mathbb{C}} R)$).

REMARK 3.29. Let $p : Z \rightarrow X$ be a morphism of complex manifolds. In this paper, we denote the corresponding pullback functor for left \mathcal{D} -modules by p^* . Note that p^* coincides with the \mathcal{O} -module pullback. In the \mathcal{D} -module literature (see e.g. [BGK⁺87, page 232]), this pullback functor is denoted by $p^!$ since under the Riemann-Hilbert correspondence it corresponds to the right adjoint of the pushforward with compact supports, and p^* refers to a different functor (its Verdier dual). In our setup, we always require that p is a submersive map, which ensures that the two functors coincide up to a cohomological shift: $p^! = p^*[2(\dim(Z) - \dim(X))]$. For this reason, we hope that our notation is not unnecessarily confusing.

3.4.3. Now let us study \mathcal{O} -modules, beginning with the non-quantized complex manifold M . Let P be an \mathcal{O}_M -module. Recall that the induced right \mathcal{D}_M -module $P_{\mathcal{D}}$ is defined by

$$P_{\mathcal{D}} = P \otimes_{\mathcal{O}_M} \mathcal{D}_M = \text{Diff}_M(\mathcal{O}_M; P).$$

We claim that it carries a natural coaction

$$\Delta_P : P_{\mathcal{D}} \rightarrow \mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}}$$

of the coalgebra \mathcal{D}_M . (Recall that \mathcal{O}_M acts on $P_{\mathcal{D}} = \text{Diff}_M(\mathcal{O}_M; P)$ through the action of \mathcal{O}_M on itself, so that \mathcal{O}_M plays the role of Q_1 in Lemma 3.7, and Example 3.8). Moreover, Δ_P is a morphism of right \mathcal{D}_M -module. Let us describe it in concrete terms.

The tensor product $\mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}}$ is identified with the sheaf of bidifferential operators $\text{Diff}(\mathcal{O}_M, \mathcal{O}_M; P)$, see Lemma 3.7 and Example 3.8. The identification sends $A \otimes B$ for $A \in \mathcal{D}_M$, $B \in P_{\mathcal{D}}$ to the operator

$$(f, g) \mapsto B(A(f)g) \quad (f, g \in \mathcal{O}_M).$$

Being the tensor product of a left \mathcal{D}_M -module and a right \mathcal{D}_M -module, the sheaf $\mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}}$ carries a natural structure of a right \mathcal{D}_M -module. Explicitly, it is given by the following formulas

$$\begin{aligned} (A \otimes B) \cdot f &= (fA) \otimes B = A \otimes (Bf), \\ (A \otimes B) \cdot \tau &= -(\tau A) \otimes B + A \otimes (B\tau) \quad (A \in \mathcal{D}_M, B \in P_{\mathcal{D}}, f \in \mathcal{O}_M, \tau \in \mathcal{T}_M), \end{aligned}$$

where \mathcal{T}_M is the sheaf of vector fields on M .

LEMMA 3.30. *Under the identification*

$$\mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}} = \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P),$$

the right action of \mathcal{D}_M on $\mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}}$ corresponds to its right action on $\mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P)$ coming from the left action of differential operators on the second argument.

Proof. Suppose $g, h \in \mathcal{O}_M$, then

$$((A \otimes B) \cdot f)(g, h) = (fA \otimes b)(g, h) = B(A(g)f \cdot h) = B(A(g)(fh)) = (A \otimes B)(g, fh).$$

In addition, if $\tau \in \mathcal{T}_M$, we have $\tau(A(g)h) = (\tau A(g))h + A(g)\tau(h)$, and therefore

$$\begin{aligned} ((A \otimes B) \cdot \tau)(g, h) &= (A \otimes (B\tau) - (\tau A) \otimes B)(g, h) = B\tau(A(g)h) - B((\tau A(g))h) \\ &= B(A(g)\tau(h)) = (A \otimes B)(g, \tau(h)). \end{aligned} \quad \square$$

The coaction Δ_P is a homomorphism of right \mathcal{D}_M -modules that sends $A \in P_{\mathcal{D}}$ to the operator $\Delta_P(A) \in \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P)$ given by

$$\Delta_P(A) : (f, g) \mapsto fA(g), \quad f, g \in \mathcal{O}_M.$$

REMARK 3.31. Note that the sub- \mathcal{O}_M -module $P \subset P_{\mathcal{D}}$ generates the induced right \mathcal{D}_M -module $P_{\mathcal{D}}$. Since the coaction

$$\Delta_P : P_{\mathcal{D}} \rightarrow \mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}}$$

is a morphism of right \mathcal{D}_M -modules, it is uniquely determined by its restriction to $P \subset P_{\mathcal{D}}$. Under the identification $P_{\mathcal{D}} = \mathcal{D}iff_M(\mathcal{O}_M; P)$, a section $p \in P \subset P_{\mathcal{D}}$ corresponds to the differential operator

$$p(f) = fp, \quad f \in \mathcal{O}_M,$$

and therefore the operator $\Delta_P(A) \in \mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P)$ is given by

$$\Delta_P(p) : (f, g) \mapsto fgp, \quad f, g \in \mathcal{O}_M.$$

Under the identification $\mathcal{D}iff_M(\mathcal{O}_M, \mathcal{O}_M; P) = \mathcal{D}_M \otimes P_{\mathcal{D}}$, this corresponds to

$$\Delta_P(p) = 1 \otimes p \in \mathcal{D}_M \otimes P_{\mathcal{D}}, \quad (p \in P \subset P_{\mathcal{D}}).$$

3.4.4. Let now \widetilde{M} be a neutralized \star -quantization. To an $\mathcal{O}_{\widetilde{M}}$ -module \widetilde{P} we assign the right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module $\widetilde{P}_{\mathcal{D}} := \mathcal{D}iff_M(\mathcal{O}_M \otimes_{\mathbb{C}} R; \widetilde{P})$. Note that the right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module $(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} = \mathcal{D}iff_M(\mathcal{O}_M \otimes_{\mathbb{C}} R; \mathcal{O}_{\widetilde{M}})$ also has commuting right and left actions of $\mathcal{O}_{\widetilde{M}}$, and we have a natural identification

$$\widetilde{P}_{\mathcal{D}} = (\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} \otimes_{\mathcal{O}_{\widetilde{M}}} \widetilde{P}.$$

Since $(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}}$ is a locally free right $\mathcal{O}_{\widetilde{M}}$ -module, we see that the assignment $\widetilde{P} \rightarrow \widetilde{P}_{\mathcal{D}}$ is an exact functor. Note also that $(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}}$ is the dual of $\mathcal{A}_{\widetilde{M}}$ in the sense that

$$(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} = \mathcal{H}om_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\mathcal{A}_{\widetilde{M}}, \mathcal{D}_M \otimes_{\mathbb{C}} R).$$

The right \mathcal{D}_M -module $\widetilde{P}_{\mathcal{D}}$ is naturally a comodule over the coalgebra $\mathcal{A}_{\widetilde{M}}$. The coaction can be rewritten as in 3.4.3: the tensor product

$$\mathcal{A}_{\widetilde{M}} \otimes \widetilde{P}_{\mathcal{D}}$$

(taken over $\mathcal{O}_M \otimes_{\mathbb{C}} R$) is identified with the sheaf of bidifferential operators $\mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}}, \mathcal{O}_M \otimes_{\mathbb{C}} R; \widetilde{P})$ via

$$A \otimes B \mapsto \left((\widetilde{f}, g) \mapsto B(A(\widetilde{f})g) \right) \quad (A \in \mathcal{A}_{\widetilde{M}}, B \in \widetilde{P}_{\mathcal{D}}, \widetilde{f} \in \mathcal{O}_{\widetilde{M}}, g \in \mathcal{O}_M \otimes_{\mathbb{C}} R).$$

The coaction

$$\Delta_{\widetilde{P}} : \widetilde{P}_{\mathcal{D}} \rightarrow \mathcal{A}_{\widetilde{M}} \otimes \widetilde{P}_{\mathcal{D}}$$

is a homomorphism of right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -modules that sends $A \in \widetilde{P}_{\mathcal{D}}$ to the operator $\Delta_{\widetilde{P}}(A) \in \mathcal{D}iff(\mathcal{O}_{\widetilde{M}}, \mathcal{O}_M \otimes_{\mathbb{C}} R; \widetilde{P})$ given by

$$\Delta_{\widetilde{P}}(A) : (\widetilde{f}, g) \mapsto \widetilde{f}A(g), \quad \widetilde{f} \in \mathcal{O}_{\widetilde{M}}, g \in \mathcal{O}_M \otimes_{\mathbb{C}} R.$$

PROPOSITION 3.32. (a) *The assignment $P \mapsto (P_{\mathcal{D}}, \Delta_P)$ is an equivalence between the category of \mathcal{O}_M -modules $\text{Mod}(\mathcal{O}_M)$ and the category of \mathcal{D}_M -comodules $\text{Comod}(\mathcal{D}_M)$. (Recall that ‘comodule’ is short for ‘left comodule in the category of right \mathcal{D}_M -modules’.)*

(b) *More generally, the functor*

$$\widetilde{P} \mapsto \left(\widetilde{P}_{\mathcal{D}}, \Delta_{\widetilde{P}} \right) : \text{Mod}(\mathcal{O}_{\widetilde{M}}) \rightarrow \text{Comod}(\mathcal{A}_{\widetilde{M}})$$

is an equivalence.

Proof. We postpone the proof of (a) to avoid repetitions: it is a particular case of Proposition 3.35. To derive (b) from (a) fix an $\mathcal{A}_{\widetilde{M}}$ -comodule F and consider the sheaf of homomorphisms

$$\xi(F) := \mathcal{H}om_{\mathcal{A}_{\widetilde{M}}}((\mathcal{O}_{\widetilde{M}})_{\mathcal{D}}, F).$$

Note that the right action of $\mathcal{O}_{\widetilde{M}}$ on itself induces a right action of $\mathcal{O}_{\widetilde{M}}$ on $(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}}$; therefore, $\xi(F)$ is naturally a $\mathcal{O}_{\widetilde{M}}$ -module. Let us prove that the functors $F \mapsto \xi(F)$ and $\widetilde{P} \mapsto \widetilde{P}_{\mathcal{D}}$ are mutually inverse.

Recall that $\widetilde{P}_{\mathcal{D}} = (\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} \otimes_{\mathcal{O}_{\widetilde{M}}} \widetilde{P}$. We therefore obtain functorial morphisms $\xi(F)_{\mathcal{D}} \rightarrow F$ and $\widetilde{P} \rightarrow \xi(\widetilde{P}_{\mathcal{D}})$ (essentially because the two functors are adjoint). We need to prove that the maps are isomorphisms. By Nakayama’s lemma (see [LV06, Proposition 6.3]) it is enough to prove this when F and \widetilde{P} are annihilated by \hbar , in which case it reduces to statement (a). □

REMARK 3.33. (1) Propositions 3.32 and 3.35 may be viewed as a form of Koszul duality, similar to [Pos11].

(2) It follows from the proof of Proposition 3.32 that the inverse of the functor $\tilde{P} \rightarrow \tilde{P}_{\mathcal{D}}$, which we denoted by ξ , can be also given as

$$\xi(F) = F \otimes_{\mathcal{D}_M} \mathcal{O}_M.$$

However, the description of the $\mathcal{O}_{\tilde{M}}$ -module structure on $\xi(F)$ is more complicated in this approach. This construction shows that the R -module structure on $\xi(F)$ depends only on the $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module structure on F and not on the $\mathcal{A}_{\tilde{M}}$ -comodule structure.

We need the following observation about induced \mathcal{D} -modules.

LEMMA 3.34. *Suppose $P \in \text{Mod}(\mathcal{O}_M)$ and $F \in \text{Mod}(\mathcal{D}_M)$. The natural embedding $P \rightarrow P_{\mathcal{D}}$ induces a morphism $P \otimes_{\mathcal{O}_M} F \rightarrow P_{\mathcal{D}} \otimes_{\mathcal{O}_M} F$, and therefore a morphism of right \mathcal{D}_M -modules*

$$i : (P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}} \rightarrow P_{\mathcal{D}} \otimes_{\mathcal{O}_M} F.$$

We claim that i is an isomorphism in $\text{Comod}(\mathcal{D}_M)$.

Proof. It is easy to check that i is an isomorphism of right \mathcal{D}_M -modules by looking at the order filtrations on $(P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}}$ and on $P_{\mathcal{D}}$. It remains to show that i respects the coaction of \mathcal{D}_M ; that is, we need to check that the two compositions

$$\begin{array}{ccc} (P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}} & \xrightarrow{i} & P_{\mathcal{D}} \otimes_{\mathcal{O}_M} F \\ \downarrow & & \downarrow \\ \mathcal{D}_M \otimes_{\mathcal{O}_M} (P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}} & \rightarrow & \mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}} \otimes_{\mathcal{O}_M} F \end{array}$$

coincide. Since the compositions are \mathcal{D}_M -linear, it suffices to check that they coincide on $P \otimes_{\mathcal{O}_M} F \subset (P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}}$, because this sub- \mathcal{O}_M -module generates the induced \mathcal{D}_M -module $(P \otimes_{\mathcal{O}_M} F)_{\mathcal{D}}$. We claim that both compositions send $A \otimes B \in P \otimes_{\mathcal{O}_M} F$ to

$$1 \otimes A \otimes B \in \mathcal{D}_M \otimes_{\mathcal{O}_M} P \otimes_{\mathcal{O}_M} F \subset \mathcal{D}_M \otimes_{\mathcal{O}_M} P_{\mathcal{D}} \otimes_{\mathcal{O}_M} F.$$

Indeed, the left arrow sends $A \otimes B$ to $1 \otimes A \otimes B$ by Remark 3.31. The right arrow equals $\Delta \otimes id_F$; therefore, it sends $A \otimes B$ to $\Delta(A) \otimes B$, which is also equal to $1 \otimes A \otimes B$ by Remark 3.31. This completes the proof. \square

3.4.5. Consider now a relative version of the assignment from Proposition 3.32. Let $p_X : Z \rightarrow X$ be a submersive morphism of complex manifolds. Recall that we view the sheaf of differential operators \mathcal{D}_X as a coalgebra in $\text{Mod}(\mathcal{D}_X)$. Its inverse image $p_X^* \mathcal{D}_X$ is a coalgebra in $\text{Mod}(\mathcal{D}_Z)$. There is a natural identification

$$p_X^* \mathcal{D}_X = \text{Diff}_Z(p_X^{-1} \mathcal{O}_X; \mathcal{O}_Z),$$

where the coalgebra structure is induced from the product on \mathcal{O}_X . This identification can be used to study $p_X^* \mathcal{D}_X$ -comodules.

For any \mathcal{O}_Z -module P , the induced right \mathcal{D}_Z -module $P_{\mathcal{D}}$ is naturally equipped with a $p_X^* \mathcal{D}_X$ -comodule structure as in 3.4.3. Equivalently, consider the natural map $\mathcal{D}_Z \rightarrow p_X^* \mathcal{D}_X$ given by restricting differential operators $\mathcal{O}_Z \rightarrow \mathcal{O}_Z$ to $p_X^{-1} \mathcal{O}_X$. It is a morphism of coalgebras, and the $p_X^* \mathcal{D}_X$ -comodule structure on $P_{\mathcal{D}}$ is induced by the coaction of \mathcal{D}_Z .

In particular, $\mathcal{D}_Z = (\mathcal{O}_Z)_{\mathcal{D}}$ is a $p_X^* \mathcal{D}_X$ -comodule. Notice that it also carries a commuting left action of the algebra of relative differential operators $\mathcal{D}_{Z/X} \subset \mathcal{D}_Z$. We can therefore factor the functor $P \rightarrow P_{\mathcal{D}}$ as a composition of two functors. First, an \mathcal{O}_Z -module P induces a right $\mathcal{D}_{Z/X}$ -module $P_{\mathcal{D}/X} = \text{Diff}_{Z/X}(\mathcal{O}_X; P)$. Second, a right $\mathcal{D}_{Z/X}$ -module $F (= P_{\mathcal{D}/X})$ induces a right \mathcal{D}_Z -module $F \otimes_{\mathcal{D}_{Z/X}} \mathcal{D}_Z$ that carries a coaction $\Delta_{F/p_X^* \mathcal{D}_X}$ of $p_X^* \mathcal{D}_X$.

PROPOSITION 3.35. *The assignment $F \mapsto (F \otimes_{\mathcal{D}_{Z/X}} \mathcal{D}_Z, \Delta_{F/p_X^* \mathcal{D}_X})$ is an equivalence*

$$\text{Mod}(\mathcal{D}_{Z/X}^{op}) \xrightarrow{\cong} \text{Comod}(p_X^* \mathcal{D}_X)$$

between the category of right $\mathcal{D}_{Z/X}$ -modules and the category $\text{Comod}(p_X^ \mathcal{D}_X)$ of comodules over $p_X^* \mathcal{D}_X$ in the category of right \mathcal{D}_Z -modules.*

REMARK 3.36. Note that Proposition 3.32 is a special case (an absolute version) of Proposition 3.35 (a relative version).

Proof. The question is essentially local, because both categories are global sections of the corresponding stacks of categories, and the functor between them is induced by a 1-morphism of stacks. We may therefore assume that Z is an open subset of \mathbb{C}^{n+m} with coordinates $(x_1, \dots, x_n; y_1, \dots, y_m)$ and that $p_X : Z \rightarrow X$ is given by $p_X(x_1, \dots, x_n; y_1, \dots, y_m) = (x_1, \dots, x_n) \in X \subset \mathbb{C}^n$. Let us verify that the inverse functor sends $G \in \text{Comod}(p_X^* \mathcal{D}_X)$ to $\text{Hom}_{p_X^* \mathcal{D}_X}(\mathcal{D}_Z, G)$.

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, we set

$$\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

A section $A \in p_X^* \mathcal{D}_X$ can be written as

$$A = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} \frac{1}{\alpha!} a_\alpha(x, y) \partial^\alpha,$$

where locally on Z , all but finitely many $a_\alpha(x, y) \in \mathcal{O}_Z$ vanish. (The factor of $\alpha!$ is included to avoid binomial coefficients in the coproduct.) The coproduct $\Delta : p_X^* \mathcal{D}_X \rightarrow p_X^* \mathcal{D}_X \otimes p_X^* \mathcal{D}_X$ is given by

$$\Delta \left(\sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} \frac{1}{\alpha!} a_\alpha(x, y) \partial^\alpha \right) = \sum_{\beta, \gamma \in (\mathbb{Z}_{\geq 0})^n} \frac{1}{\beta! \gamma!} a_{\beta+\gamma}(x, y) \partial^\beta \otimes \partial^\gamma.$$

Let F be a right $\mathcal{D}_{Z/X}$ -module. Then $G = F \otimes_{\mathcal{D}_{Z/X}} \mathcal{D}_Z$ can be described as

$$G = F \otimes_{\mathbb{C}} \mathbb{C} \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right].$$

Note that F is identified with a subsheaf of G , and that F generates G as a right \mathcal{D}_Z -module. Note moreover that the coproduct $\Delta_{F/p_X^* \mathcal{D}_X} : G \rightarrow p_X^* \mathcal{D}_X \otimes G$ has the property $\Delta_{F/p_X^* \mathcal{D}_X}(s) = 1 \otimes s$ for $s \in F \subset G$ (even though $\Delta_{F/p_X^* \mathcal{D}_X}(A) \neq 1 \otimes A$ for general $A \in G$).

This property characterizes $\Delta_{F/p_X^* \mathcal{D}_X}$ uniquely since F generates G as a right \mathcal{D}_Z -module and $\Delta_{F/p_X^* \mathcal{D}_X}$ is \mathcal{D}_Z -linear.

Let us describe explicitly the functor in the opposite direction. Let G be a right \mathcal{D}_Z -module. A morphism $\Delta_G : G \rightarrow p_X^* \mathcal{D}_X \otimes G$ can be written in the form

$$\Delta_G(s) = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} \frac{1}{\alpha!} \partial^\alpha \otimes \Delta_\alpha(s),$$

where $\Delta_\alpha : G \rightarrow G$ are maps such that for given $s \in G$, only finitely many $\Delta_\alpha(s)$ are non-zero locally on Z . Then Δ_G is a coaction if and only if $\Delta_\alpha \circ \Delta_\beta = \Delta_{\alpha+\beta}$ for any $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n$ and Δ_0 is the identity map. In other words, a coaction Δ_G can be written as

$$\Delta_G(s) = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} \frac{1}{\alpha!} \partial^\alpha \otimes (\Delta_1^{\alpha_1} \cdots \Delta_n^{\alpha_n})(s)$$

for some commuting operators $\Delta_1, \dots, \Delta_n : G \rightarrow G$ that are locally nilpotent (that is, every $s \in G$ is annihilated by some power of the operators locally on Z).

It remains to impose the restriction that Δ_G is a morphism of right \mathcal{D}_Z -modules. This results in the following conditions on Δ_i :

$$\begin{aligned} \Delta_i(sf) &= \Delta_i(s)f && (f \in \mathcal{O}_Z, 1 \leq i \leq n, s \in G) \\ \Delta_i \left(s \frac{\partial}{\partial y_j} \right) &= \Delta_i(s) \frac{\partial}{\partial y_j} && (1 \leq i \leq n, 1 \leq j \leq m, s \in G) \\ \Delta_i \left(s \frac{\partial}{\partial x_j} \right) &= \Delta_i(s) \frac{\partial}{\partial x_j} - \begin{cases} s, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} && (1 \leq i \leq n, 1 \leq j \leq n, s \in G). \end{aligned}$$

In other words, the operators Δ_i commute with $\mathcal{D}_{Z/X}$, while the commutativity relations for operators $\Delta_i, \frac{\partial}{\partial x_j}$ are those of the Weyl algebra

$$W = \mathbb{C}[t_1, \dots, t_n] \left\langle \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n} \right\rangle.$$

Since the operators Δ_i are locally nilpotent, Kashiwara's Lemma [Kas03, Theorem 4.30] implies that G decomposes as a tensor product

$$G = F \otimes_{\mathbb{C}} \mathbb{C} \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right],$$

for

$$F := \{s \in G : \Delta_i(s) = 0, \text{ for all } i = 1, \dots, n\} = \{s \in G : \Delta(s) = 1 \otimes s\}.$$

Moreover, F is invariant under $\mathcal{D}_{Z/X}$. Note that F is identified with

$$\text{Hom}_{p_X^* \mathcal{D}_X}(\mathcal{D}_Z, G).$$

It is now clear that the two functors are inverse to each other. □

REMARK 3.37. Proposition 3.35 implies Proposition 3.18. Recall that to prove Proposition 3.18, we had to verify exactness of a sequence

$$\text{Diff}_Z(\mathcal{O}_Z; \mathcal{O}_Z) \rightarrow \text{Diff}_Z(p_X^{-1} \mathcal{O}_X, \mathcal{O}_Z; \mathcal{O}_Z) \rightarrow \text{Diff}_Z(p_X^{-1} \mathcal{O}_X, p_X^{-1} \mathcal{O}_X, \mathcal{O}_Z; \mathcal{O}_Z).$$

It can be seen that this sequence is the beginning of the cobar complex computing the sheaves $\text{Ext}_{p_X^* \mathcal{D}_X}^i(\mathcal{D}_Z, \mathcal{D}_Z)$ of local Ext's in the category $\text{Comod}(p_X^* \mathcal{D}_X)$.

Under the equivalence of Proposition 3.35, $\mathcal{D}_Z \in \text{Comod}(p_X^* \mathcal{D}_X)$ corresponds to $\mathcal{D}_{Z/X} \in \text{Mod}(\mathcal{D}_{Z/X})$. Therefore,

$$\text{Ext}_{p_X^* \mathcal{D}_X}^i(\mathcal{D}_Z, \mathcal{D}_Z) = \text{Ext}_{\mathcal{D}_{Z/X}}^i(\mathcal{D}_{Z/X}, \mathcal{D}_{Z/X}) = \begin{cases} \mathcal{D}_{Z/X}, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Proposition 3.18 follows.

3.4.6. Finally, given a neutralized \star -quantization \tilde{X} of X , the inverse image

$$p_X^* \mathcal{A}_{\tilde{X}} = \text{Diff}_Z(p_X^{-1} \mathcal{O}_{\tilde{X}}; \mathcal{O}_Z \otimes_{\mathbb{C}} R)$$

is a coalgebra in the category of left $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -modules. Let \tilde{Z} be a neutralized \star -quantization of Z such that p extends to a morphism $\tilde{p} : \tilde{Z} \rightarrow \tilde{X}$; in other words, the morphism of algebras $p_X^{-1} \mathcal{O}_X \rightarrow \mathcal{O}_Z$ extends to a morphism $p_X^{-1} \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{Z}}$. Then for any $\mathcal{O}_{\tilde{Z}}$ -module \tilde{P} , the right $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -module $\tilde{P}_{\mathcal{D}}$ is naturally equipped with a $p_X^* \mathcal{A}_{\tilde{X}}$ -comodule structure.

Repeating the proof of Proposition 3.32(b), it is easy to show that the category of $p_X^* \mathcal{A}_{\tilde{X}}$ -comodules is equivalent to the category of right $\mathcal{D}_{\tilde{Z}/\tilde{X}}$ -modules. Moreover, the functor $\tilde{P} \rightarrow \tilde{P}_{\mathcal{D}}$ decomposes as the composition of the induction functor

$$\tilde{P} \rightarrow \mathcal{D}_{\tilde{Z}/\tilde{X}} \otimes_{\mathcal{O}_{\tilde{Z}}} \tilde{P} = \{A \in \text{Diff}_Z(\mathcal{O}_{\tilde{Z}}; \tilde{P}) : A(\tilde{f}\tilde{g}) = \tilde{f}A(\tilde{g}) \text{ for all } \tilde{f} \in p_X^{-1} \mathcal{O}_{\tilde{X}}, \tilde{g} \in \mathcal{O}_{\tilde{Z}}\}$$

(the functor appears in the end of Section 3.2) and this equivalence. We do not need these statements, so we leave the details to the reader. The statements are useful because they allow us to relate two proofs of Theorem 2.18: the proof using comodules (Section 4.1) and the proof using relative differential operators on quantizations (Section 4.4).

REMARK 3.38. As we already mentioned, the results of Section 3.4 are inspired by the ★-pseudotensor structure of [BD04]. For instance, Proposition 3.32 can be extracted from the remark in [BD04, Section 2.1.8]. Let us make the relation more explicit.

Let M be a complex manifold. Recall ([BD04, 2.2.3]) that for right \mathcal{D}_M -modules $F, F^{(1)}, \dots, F^{(n)}$, the space of ★-operations $F^{(1)} \times \dots \times F^{(n)} \rightarrow F$ is defined to be

$$\mathrm{Hom}(\boxtimes_{i=1}^n F^{(i)}, \Delta_* F).$$

Here the homomorphisms are taken in the category of right \mathcal{D} -modules on M^n , $\Delta : M \rightarrow M^n$ is the diagonal embedding, and Δ_* is the \mathcal{D} -module direct image functor. In particular, let $\tilde{P}, \tilde{P}^{(1)}, \dots, \tilde{P}^{(n)}$ be \mathcal{O} -modules on neutralized ★-quantizations $\tilde{M}, \tilde{M}^{(1)}, \dots, \tilde{M}^{(n)}$ of M . Consider the induced right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -modules $\tilde{P}_{\mathcal{D}}^{(1)}, \dots, \tilde{P}_{\mathcal{D}}^{(n)}, \tilde{P}_{\mathcal{D}}$. Then the space of R -linear ★-operations $\tilde{P}_{\mathcal{D}}^{(1)} \times \dots \times \tilde{P}_{\mathcal{D}}^{(n)} \rightarrow \tilde{P}_{\mathcal{D}}$ is identified with the space of polydifferential operators $\tilde{P}^{(1)} \times \dots \times \tilde{P}^{(n)} \rightarrow \tilde{P}$ (see Section 3.4.10).

Let \tilde{M} be a neutralized ★-quantization of M . The induced right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -module $(\mathcal{O}_{\tilde{M}})_{\mathcal{D}}$ is an associative★ algebra (that is, the product is a ★-operation). Moreover, $(\mathcal{O}_{\tilde{M}})_{\mathcal{D}}$ is an R -deformation of $(\mathcal{O}_M)_{\mathcal{D}} = \mathcal{D}_M$; this gives a one-to-one correspondence between neutralized ★-quantizations and deformations of the associative★ algebra \mathcal{D}_M .

Note that the associative★ algebra $(\mathcal{O}_{\tilde{M}})_{\mathcal{D}}$ and the coalgebra $\mathcal{A}_{\tilde{M}}$ are related by the duality (cf. [BD04, 2.5.7]):

$$\mathcal{A}_{\tilde{M}} = \mathrm{Hom}_{\mathcal{D}_M}((\mathcal{O}_{\tilde{M}})_{\mathcal{D}}, \mathcal{D}_M).$$

The correspondence identifies $(\mathcal{O}_{\tilde{M}})_{\mathcal{D}}$ -modules and $\mathcal{A}_{\tilde{M}}$ -comodules.

Similar approach works in the relative situation. Recall that we consider, for a submersive morphism $p_X : Z \rightarrow X$ and a neutralized ★-quantization \tilde{X} of X , the category of comodules over the coalgebra $p_X^* \mathcal{A}_{\tilde{X}}$. Notice however that a pull-back of an associative★ algebra carries a ★-operation only after a cohomological shift: the category of $p_X^* \mathcal{A}_{\tilde{X}}$ -comodules is equivalent to the category of modules over the associative★ algebra $p_X^*(\mathcal{O}_{\tilde{X}})_{\mathcal{D}}[\dim(X) - \dim(Z)]$.

4 Deformation Theory

This section contains the proof of Theorem 2.18. The statement of the theorem is local in Y ; we can therefore assume that Y is a Stein manifold.

4.1 Proof of Theorem 2.18: neutralized case. In this section, we assume that the quantization \mathbb{X} of X is neutralized; this simplifies the argument. In Section 4.2, we explain how to modify the argument for arbitrary ★-quantizations.

4.1.1. Let \mathbb{Y} be a ★-quantization of Y . By definition, the reduction of \mathbb{Y} to \mathbb{C} is neutralized. This neutralization can be lifted to a neutralization of \mathbb{Y} : indeed, once

such a lift is constructed modulo \hbar^k , the obstructions to lifting modulo \hbar^{k+1} lie in $H^2(Y, \mathcal{O}_Y)$, which vanishes because Y is Stein. Similarly, the vanishing of $H^1(Y, \mathcal{O}_Y)$ implies that the lifting is unique up to a non-unique automorphism. Thus, there exists a global neutralization $\alpha \in \mathbb{Y}(Y)$ that agrees with the neutralization of the reduction of \mathbb{Y} to \mathbb{C} . Such α defines a neutralized \star -quantization \tilde{Y}_α of Y .

Moreover, α is unique up to a non-unique isomorphism. Note that automorphisms of α are required to act trivially modulo \hbar . The automorphism group of α is identified with the group

$$\{\tilde{f} \in H^0(Y, \mathcal{O}_{\tilde{Y}_\alpha}) : \tilde{f} = 1 \pmod{\hbar}\},$$

which acts on $\mathcal{O}_{\tilde{Y}_\alpha}$ by conjugation.

We can now restate Theorem 2.18 using neutralized \star -quantizations. In other words, we replace pairs (\mathbb{Y}, \tilde{P}) with triples $(\mathbb{Y}, \tilde{P}, \alpha)$, where α is a neutralization. This leads to the following statement.

PROPOSITION 4.1. *Let X and Y be complex manifolds, P a coherent sheaf on $Z = X \times Y$ whose support is proper over Y . Assume that Y is Stein and that the map $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$ (defined in Section 2.3) is a quasi-isomorphism. Let \tilde{X} be a neutralized \star -quantization of X .*

- (a) *There exists a neutralized \star -quantization \tilde{Y} of Y and a deformation of P to an \mathcal{O} -module \tilde{P} on $\tilde{X} \times \tilde{Y}^{op}$.*
- (b) *The pair (\tilde{Y}, \tilde{P}) is unique up to a non-unique isomorphism.*
- (c) *The automorphism group $\text{Aut}(\tilde{Y}, \tilde{P})$ is identified with the group*

$$\{\tilde{f} \in H^0(Y, \mathcal{O}_{\tilde{Y}}) : \tilde{f} = 1 \pmod{\hbar}\},$$

which acts on $\mathcal{O}_{\tilde{Y}}$ by conjugation and on \tilde{P} using the structure of an \mathcal{O} -module.

4.1.2. Consider the right \mathcal{D}_Z -module $P_{\mathcal{D}}$ induced by P . It carries a natural coaction of \mathcal{D}_Z . Alternatively, we view this coaction as a pair of commuting coactions of $p_X^* \mathcal{D}_X$ and $p_Y^* \mathcal{D}_Y$ using the identification $\mathcal{D}_Z = p_X^* \mathcal{D}_X \otimes p_Y^* \mathcal{D}_Y$. We claim that the coaction morphism

$$\Delta_Y = \Delta_{P,Y} : P_{\mathcal{D}} \rightarrow p_Y^* \mathcal{D}_Y \otimes P_{\mathcal{D}}$$

has the following universal property.

LEMMA 4.2. *In the assumptions of Proposition 4.1, let F be any left \mathcal{D}_Y -module that is flat over \mathcal{O}_Y , and let*

$$\phi : P_{\mathcal{D}} \rightarrow p_Y^* F \otimes P_{\mathcal{D}}$$

be a morphism of right \mathcal{D}_Z -modules that commutes with the coaction of $p_X^ \mathcal{D}_X$, i.e. ϕ is also a morphism in the category $\text{Comod}(p_X^* \mathcal{D}_X)$. Then there exists a unique morphism of left \mathcal{D}_Y -modules $\psi : \mathcal{D}_Y \rightarrow F$ such that $\phi = (id \otimes p_Y^*(\psi)) \circ \Delta_Y$.*

This provides an identification between $\text{Hom}_{p_X^* \mathcal{D}_X}(P_{\mathcal{D}}, p_Y^* F \otimes P_{\mathcal{D}})$ (morphisms in the category $\text{Comod}(p_X^* \mathcal{D}_X)$) and $\text{Hom}_{\mathcal{D}_Y}(\mathcal{D}_Y, F) = H^0(Y, F)$. The identification remains valid in the derived sense:

$$\mathbb{R} \text{Hom}_{p_X^* \mathcal{D}_X}(P_{\mathcal{D}}, p_Y^* F \otimes P_{\mathcal{D}}) \cong R\Gamma(Y, F)$$

and hence

$$\text{Ext}_{p_X^* \mathcal{D}_X}^i(P_{\mathcal{D}}, p_Y^* F \otimes P_{\mathcal{D}}) = H^i(Y, F).$$

Proof. By Proposition 3.35, the category of $p_X^* \mathcal{D}_X$ -comodules is equivalent to the category of right $\mathcal{D}_{Z/X}$ -modules. Under this equivalence, $P_{\mathcal{D}}$ corresponds to $P_{\mathcal{D}/X}$. We claim that for any $G \in \text{Mod}(\mathcal{D}_Z)$, the comodule $G \otimes P_{\mathcal{D}} \in \text{Comod}(p_X^* \mathcal{D}_X)$ corresponds to $G \otimes P_{\mathcal{D}/X} \in \text{Mod}(\mathcal{D}_{Z/X}^{op})$. Indeed, $G \otimes P_{\mathcal{D}} \simeq (G \otimes P)_{\mathcal{D}}$ by Lemma 3.34. The isomorphism respects the coaction of \mathcal{D}_Z , and therefore also the coaction of $p_X^* \mathcal{D}_X$. Similarly, we have an isomorphism of induced right $\mathcal{D}_{Z/X}$ -modules $G \otimes P_{\mathcal{D}/X} \simeq (G \otimes P)_{\mathcal{D}/X}$. This implies the claim.

In particular, the $p_X^* \mathcal{D}_X$ -comodule $p_Y^* F \otimes P_{\mathcal{D}}$ corresponds to $p_Y^* F \otimes P_{\mathcal{D}/X}$. Therefore,

$$\text{Ext}_{p_X^* \mathcal{D}_X}^i(P_{\mathcal{D}}, p_Y^* F \otimes P_{\mathcal{D}}) = \text{Ext}_{\mathcal{D}_{Z/X}}^i(P_{\mathcal{D}/X}, p_Y^* F \otimes P_{\mathcal{D}/X}) = \text{Ext}_{\mathcal{O}_Z}^i(P, p_Y^* F \otimes P_{\mathcal{D}/X}),$$

where the second equality follows from Lemma 3.10. By the projection formula,

$$\begin{aligned} \mathbb{R} \text{Hom}_{\mathcal{O}_Z}(P, p_Y^* F \otimes P_{\mathcal{D}/X}) &= \mathbb{R}\Gamma(Y, F \otimes \mathbb{R}p_{Y*} \mathbb{R} \text{Hom}_{\mathcal{O}_Z}(P, P_{\mathcal{D}/X})) \\ &= \mathbb{R}\Gamma(Y, F \otimes \mathcal{E}(P)) = \mathbb{R}\Gamma(Y, F), \end{aligned}$$

as required. □

4.1.3. Recall that $\mathcal{A}_{\tilde{X}} = \text{Diff}(\mathcal{O}_{\tilde{X}}; \mathcal{O}_X \otimes R)$ is a coalgebra in $\text{Mod}(\mathcal{D}_X \otimes R)$. It is a deformation of the coalgebra $p_X^* \mathcal{D}_X \in \text{Mod}(\mathcal{D}_X)$; that is, it is flat over R and its reduction to \mathbb{C} is identified with $p_X^* \mathcal{D}_X$, see Sections 3.2.2 and 3.3.2. Lemma 4.2 is a universal property of the pair $(P_{\mathcal{D}}, \Delta_Y)$ consisting of an object and a morphism in the category $\text{Comod}(p_X^* \mathcal{D}_X)$. We claim that this pair admits a deformation in the category $\text{Comod}(\mathcal{A}_{\tilde{X}})$, and that the universal property is preserved. Here is a precise statement.

LEMMA 4.3. *In the assumptions of Proposition 4.1, we have the following.*

- (a) *There is a R -flat deformation of $P_{\mathcal{D}} \in \text{Comod}(p_X^* \mathcal{D}_X)$ to $P'_{\mathcal{D}} \in \text{Comod}(p_X^* \mathcal{A}_{\tilde{X}})$. The deformation is unique up to isomorphism.*
- (b) *The coaction Δ_Y extends to a homomorphism of $p_X^* \mathcal{A}_{\tilde{X}}$ -comodules (and, in particular, right \mathcal{D}_Z -modules)*

$$\Delta'_Y : P'_{\mathcal{D}} \rightarrow p_Y^* \mathcal{D}_Y \otimes_{\mathcal{O}_Z} P'_{\mathcal{D}} = (p_Y^* \mathcal{D}_Y \otimes_{\mathbb{C}} R) \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_{\mathcal{D}}.$$

(c) The homomorphism Δ'_Y is universal in the following sense: given any $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ -module F that is flat over $\mathcal{O}_Y \otimes_{\mathbb{C}} R$ and any homomorphism of $p_X^* \mathcal{D}_X$ -comodules

$$\phi : P'_D \rightarrow p_Y^* F \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D,$$

there exists a unique morphism of $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ -modules $\psi : \mathcal{D}_Y \otimes_{\mathbb{C}} R \rightarrow F$ such that $\phi = (id \otimes p_Y^*(\psi)) \circ \Delta'_Y$.

Proof. Since Y is Stein, $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$. By Lemma 4.2, we see that

$$\text{Ext}^1_{p_X^* \mathcal{D}_X}(P_D, P_D) = \text{Ext}^2_{p_X^* \mathcal{D}_X}(P_D, P_D) = 0.$$

Now part (a) follows from Proposition 3.24.

To prove part (b), note that obstructions to extending Δ_Y to Δ'_Y belong to

$$\text{Ext}^1_{p_X^* \mathcal{A}_{\bar{X}}}(P'_D, p_Y^* \mathcal{D}_Y \otimes \hbar P'_D).$$

Using the filtration of $p_Y^* \mathcal{D}_Y \otimes \hbar P'_D$ by powers of \hbar , we see that it suffices to verify the vanishing of

$$\text{Ext}^1_{p_X^* \mathcal{A}_{\bar{X}}}(P'_D, p_Y^* \mathcal{D}_Y \otimes (\hbar^k P'_D / \hbar^{k+1} P'_D)) = \text{Ext}^1_{p_X^* \mathcal{D}_X}(P_D, p_Y^* \mathcal{D}_Y \otimes P_D).$$

By Lemma 4.2, the vanishing is equivalent to the vanishing of $H^1(Y, \mathcal{D}_Y)$, which holds because Y is Stein and \mathcal{D}_Y is a colimit of coherent \mathcal{O}_Y -modules.

In part (c), the correspondence $\psi \mapsto (p_Y^*(\psi) \otimes id) \circ \Delta'_Y$ defines a map

$$H^0(Y, F) = \text{Hom}_{\mathcal{D}_Y \otimes_{\mathbb{C}} R}(\mathcal{D}_Y \otimes_{\mathbb{C}} R, F) \rightarrow \text{Hom}_{p_X^* \mathcal{A}_{\bar{X}}}(P'_D, F \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D),$$

and we need to verify that it is bijective. Using the filtration of F by the submodules $\hbar^k F$, we see that it is enough to verify bijectivity of the map

$$H^i(Y, F/\hbar F) \rightarrow \text{Ext}^i_{p_X^* \mathcal{A}_{\bar{X}}}\left(P'_D, (F/\hbar F) \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D\right).$$

However,

$$\begin{aligned} \text{Ext}^i_{p_X^* \mathcal{A}_{\bar{X}}}\left(P'_D, (F/\hbar F) \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D\right) &= \text{Ext}^i_{p_X^* \mathcal{A}_{\bar{X}}}\left(P'_D, (F \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D) \otimes_R \mathbb{C}\right) \\ &= \text{Ext}^i_{p_X^* \mathcal{D}_X}\left(P'_D \otimes_R \mathbb{C}, (F \otimes_{\mathcal{O}_Z \otimes_{\mathbb{C}} R} P'_D) \otimes_R \mathbb{C}\right) \\ &= \text{Ext}^i_{p_X^* \mathcal{D}_X}\left(P_D, (F/\hbar F) \otimes_{\mathcal{O}_Z} P_D\right), \end{aligned}$$

and the required identity follows from Lemma 4.2. □

4.1.4. Let us prove Proposition 4.1, starting with part (a). Taking into account Propositions 3.27 and 3.32, we need to construct a deformation of the coalgebra \mathcal{D}_Y to a coalgebra $\mathcal{A}_{\tilde{Y}}$ and a deformation of $P_{\mathcal{D}}$ to a right $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -module $P'_{\mathcal{D}}$ equipped with commuting coactions of $p_X^* \mathcal{A}_{\tilde{X}}$ and $p_Y^* \mathcal{A}_{\tilde{Y}}^{op}$.

Lemma 4.3(a) provides a $p_X^* \mathcal{A}_{\tilde{X}}$ -comodule $P'_{\mathcal{D}}$. Consider the morphism Δ'_Y from Lemma 4.3(b). It remains to show that there is a coalgebra structure on $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ such that Δ'_Y becomes a coaction; we can then let $\mathcal{A}_{\tilde{Y}}^{op}$ be $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ with this coalgebra structure. Consider the composition

$$P'_{\mathcal{D}} \rightarrow P'_{\mathcal{D}} \otimes p_Y^*(\mathcal{D}_Y \otimes_{\mathbb{C}} R) \rightarrow P'_{\mathcal{D}} \otimes p_Y^*(\mathcal{D}_Y \otimes_{\mathbb{C}} R) \otimes p_Y^*(\mathcal{D}_Y \otimes_{\mathbb{C}} R)$$

(here and elsewhere in the proof, the tensor products are by default over $\mathcal{O}_Z \otimes_{\mathbb{C}} R$). It is a morphism of $p_X^* \mathcal{A}_{\tilde{X}}$ -comodules, so by Lemma 4.3(c), it corresponds to a morphism

$$\delta' : \mathcal{D}_Y \otimes_{\mathbb{C}} R \rightarrow (\mathcal{D}_Y \otimes_{\mathbb{C}} R) \otimes_{\mathcal{O}_Y \otimes_{\mathbb{C}} R} (\mathcal{D}_Y \otimes_{\mathbb{C}} R).$$

It is easy to check that δ' is indeed a coassociative coproduct on $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ by using the uniqueness part of Lemma 4.3(c).

4.1.5. To prove part (b) of Proposition 4.1, we investigate uniqueness properties of $P'_{\mathcal{D}}$ and $\mathcal{A}_{\tilde{Y}}$. Let $P_{\mathcal{D}}^{(1)}$ be another deformation of $P_{\mathcal{D}}$ to a right $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -module equipped with commuting coactions of $p_X^* \mathcal{A}_{\tilde{X}}$ and $p_Y^* \mathcal{A}_{\tilde{Y}^{(1)}}^{op}$ for some deformation $\mathcal{A}_{\tilde{Y}^{(1)}}$ of the coalgebra \mathcal{D}_Y . Let us construct an isomorphism $(P'_{\mathcal{D}}, \mathcal{A}_{\tilde{Y}}) \simeq (P_{\mathcal{D}}^{(1)}, \mathcal{A}_{\tilde{Y}^{(1)}})$.

Lemma 4.3(a) provides an isomorphism of $p_X^* \mathcal{A}_{\tilde{X}}$ -comodules $\phi : P_{\mathcal{D}}^{(1)} \rightarrow P'_{\mathcal{D}}$. Using ϕ to identify $P_{\mathcal{D}}^{(1)}$ and $P'_{\mathcal{D}}$, we obtain a coaction

$$P'_{\mathcal{D}} \rightarrow P'_{\mathcal{D}} \otimes p_Y^* \mathcal{A}_{\tilde{Y}^{(1)}}^{op}.$$

Now Lemma 4.3(c) provides a morphism of left $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ -modules

$$\phi_Y : \mathcal{D}_Y \otimes_{\mathbb{C}} R \rightarrow p_Y^* \mathcal{A}_{\tilde{Y}^{(1)}}^{op},$$

which is easily seen to be an isomorphism (as it equals identity modulo \hbar). Finally, the uniqueness claim of Lemma 4.3(c) implies that ϕ_Y is a coalgebra homomorphism if $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ is equipped with the coproduct δ' .

4.1.6. It remains to prove part (c). This can be done by analyzing the possible choices for the morphism ϕ . However, it is also easy to give a direct proof.

The group

$$G = \{ \tilde{f} \in H^0(Y, \mathcal{O}_{\tilde{Y}}) : \tilde{f} = 1 \pmod{\hbar} \}$$

acts on (\tilde{Y}, \tilde{P}) ; we want to show that the corresponding morphism of groups $G \rightarrow \text{Aut}(\tilde{Y}, \tilde{P})$ is an isomorphism. Both groups admit filtration by powers of \hbar , and the

claim would follow if we verify that the action map $H^0(Y, \mathcal{O}_Y) \rightarrow \text{End}_{\mathcal{O}_Z}(P)$ is an isomorphism.

By the hypotheses of Proposition 4.1, the map $H^0(Y, \mathcal{O}_Y) \rightarrow H^0(Y, \mathcal{E}(P))$ is an isomorphism. This map factors as

$$H^0(Y, \mathcal{O}_Y) \rightarrow \text{End}_{\mathcal{O}_Z}(P) \rightarrow \text{Hom}_{\mathcal{O}_Z}(P, P_{\mathcal{D}/X}) = H^0(Y, \mathcal{E}(P)) \tag{5}$$

(see 2.3.3). Here the map $\text{End}_{\mathcal{O}_Z}(P) \rightarrow \text{Hom}_{\mathcal{O}_Z}(P, P_{\mathcal{D}/X})$ is the composition with (2). Since the homomorphism (2) is injective, we see that all morphisms in (5) are bijective. This completes the proof of Proposition 4.1. We thus proved Theorem 2.18 in the case when \star -quantization \mathbb{X} is neutralized. \square

REMARK 4.4. We can also define \mathbb{Y} directly (without assuming that Y is a Stein manifold). For an open set $U \subset Y$, let $W = X \times U$. Now let $\mathbb{Y}^{op}(U)$ be the category of triples (l, Q'_D, ν) , where l is a line bundle on U , $Q'_D \in \text{Comod}(p_X^* \mathcal{A}_X|_W)$ is flat over R , and

$$\nu : (P|_W \otimes p_U^* l)_D \rightarrow Q'_D / \hbar Q'_D$$

is an isomorphism of comodules over $p_X^* \mathcal{D}_X|_W$. A morphism of such triples is given by a homomorphism (not necessarily invertible) of line bundles l and a compatible map of the $p_X^* \mathcal{A}_{\tilde{X}}|_W$ -comodules Q'_D . It is not hard to check that this gives a \star -quantization of Y . In particular, if U is Stein, we can construct local sections of $\mathbb{Y}^{op}(U)$ by letting $l = \mathcal{O}_U$ be trivial and using Lemma 4.3 to construct Q'_D . On the other hand, there is no guarantee that \mathbb{Y}^{op} has any global sections.

In fact l and ν are determined by Q'_D , so one can alternatively consider the category of $Q'_D \in \text{Comod}(p_X^* \mathcal{A}_{\tilde{X}}|_W)$ that admit such l and ν . In other words, for a morphism between two such comodules $Q'_D, Q_D^{(1)}$, there is a unique compatible map between the corresponding line bundles $l, l^{(1)}$. Indeed, the claim is local on U , so we may assume that $l = l^{(1)} = \mathcal{O}_U$. A morphism $Q'_D \rightarrow Q_D^{(1)}$ induces a map between their reductions, which is an element of

$$\begin{aligned} \text{End}_{p_X^* \mathcal{D}_X|_W}((P|_W)_D) &= \text{End}_{\mathcal{D}_{W/X}}((P|_W)_{\mathcal{D}/X}) = \text{Hom}_{\mathcal{O}_W}(P|_W, (P|_W)_{\mathcal{D}/X}) \\ &= H^0(U, \mathcal{O}_U), \end{aligned}$$

as required. Here the last identification uses the isomorphism $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$.

4.2 Proof of Theorem 2.18: general case. We now drop the assumption that the \star -quantization \mathbb{X} is neutral. Let us explain the necessary modifications to the argument of Section 4.1.

We begin by extending the approach of Section 3.4 to general (not necessarily neutralized) quantizations.

4.2.1. Let M be a complex manifold, and let \widetilde{M} be a neutralized \star -quantization of M . Recall that to \widetilde{M} , we associate a coalgebra

$$\mathcal{A}_{\widetilde{M}} = \text{Diff}_M(\mathcal{O}_{\widetilde{M}}; \mathcal{O}_M \otimes_{\mathbb{C}} R)$$

in the category $\text{Mod}(\mathcal{D}_M)$, and that $\mathcal{O}_{\widetilde{M}}$ is reconstructed from $\mathcal{A}_{\widetilde{M}}$ as

$$\mathcal{O}_{\widetilde{M}} = \mathcal{H}om_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\mathcal{A}_{\widetilde{M}}, \mathcal{O}_M \otimes_{\mathbb{C}} R),$$

see Proposition 3.27.

For any $\widetilde{f} \in \mathcal{O}_{\widetilde{M}}$, we denote the corresponding morphism by

$$ev(\widetilde{f}) : \mathcal{A}_{\widetilde{M}} \rightarrow \mathcal{O}_M \otimes_{\mathbb{C}} R.$$

It can be viewed as evaluation of differential operators in $\mathcal{D}iff_M(\mathcal{O}_{\widetilde{M}}; \mathcal{O}_M \otimes_{\mathbb{C}} R)$ on \widetilde{f} .

4.2.2. Now suppose $\widetilde{F} \in \text{Comod}(\mathcal{A}_{\widetilde{M}})$. We then let $\widetilde{f} \in \mathcal{O}_{\widetilde{M}}$ act on \widetilde{F} by the composition

$$\widetilde{F} \xrightarrow{\text{coaction}} \mathcal{A}_{\widetilde{M}} \otimes \widetilde{F} \xrightarrow{ev(\widetilde{f})} (\mathcal{O}_M \otimes_{\mathbb{C}} R) \otimes \widetilde{F} = \widetilde{F}.$$

(To simplify notation, in this section all tensor products are assumed to be over $\mathcal{O}_M \otimes_{\mathbb{C}} R$, unless explicitly stated otherwise.) This provides \widetilde{F} with a structure of a $\mathcal{O}_{\widetilde{M}}$ -module. In other words, the action of $\mathcal{O}_{\widetilde{M}}$ on \widetilde{F} is given by the composition

$$\alpha_{\widetilde{F}} : \mathcal{O}_{\widetilde{M}} \otimes_R \widetilde{F} \rightarrow \mathcal{O}_{\widetilde{M}} \otimes_R \mathcal{A}_{\widetilde{M}} \otimes \widetilde{F} \rightarrow \widetilde{F}.$$

Note that the action $\alpha_{\widetilde{F}}$ commutes with the right action of \mathcal{D}_M , but not with the coaction of $\mathcal{A}_{\widetilde{M}}$. Thus, \widetilde{F} naturally acquire a structure of an $\mathcal{O}_{\widetilde{M}} \otimes_{\mathbb{C}} \mathcal{D}_M^{op}$ -module, so we obtain a functor

$$\text{Comod}(\mathcal{A}_{\widetilde{M}}) \rightarrow \text{Mod}(\mathcal{O}_{\widetilde{M}} \otimes_{\mathbb{C}} \mathcal{D}_M^{op}) \quad (6)$$

REMARK 4.5. By Proposition 3.32, \widetilde{F} is induced by $\widetilde{P} \in \text{Mod}(\mathcal{O}_{\widetilde{M}})$, so that we can identify

$$\widetilde{F} = \widetilde{P}_{\mathcal{D}} = \mathcal{D}iff_M(\mathcal{O}_M \otimes_{\mathbb{C}} R; \widetilde{P}).$$

Under this identification, $\alpha_{\widetilde{F}}$ corresponds to the natural action of $\mathcal{O}_{\widetilde{M}}$ on differential operators $\mathcal{O}_M \otimes_{\mathbb{C}} R \rightarrow \widetilde{P}$.

Equivalently, we have an isomorphism

$$\widetilde{F} = \widetilde{P}_{\mathcal{D}} = (\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} \otimes_{\mathcal{O}_{\widetilde{M}}} \widetilde{P},$$

where

$$(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}} = \mathcal{D}iff_M(\mathcal{O}_M \otimes_{\mathbb{C}} R; \mathcal{O}_{\widetilde{M}}).$$

Note that $(\mathcal{O}_{\widetilde{M}})_{\mathcal{D}}$ is a bimodule over $\mathcal{O}_{\widetilde{M}}$ (because $\mathcal{O}_{\widetilde{M}}$ is a bimodule over itself); this induces the action α_F .

LEMMA 4.6. *The functor (6) is fully faithful.*

Proof. Fix $\tilde{F}, \tilde{G} \in \text{Comod}(\mathcal{A}_{\tilde{M}})$, and let $\tilde{\phi} : \tilde{F} \rightarrow \tilde{G}$ be a morphism between the corresponding $\mathcal{O}_{\tilde{M}} \otimes_{\mathbb{C}} \mathcal{D}_M^{op}$ -modules. We need to show that $\tilde{\phi}$ commutes with the coaction of $\mathcal{A}_{\tilde{M}}$. In other words, we need to verify commutativity of the diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\tilde{\phi}} & \tilde{G} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\tilde{M}} \otimes \tilde{F} & \xrightarrow{\text{id} \otimes \tilde{\phi}} & \mathcal{A}_{\tilde{M}} \otimes \tilde{G}. \end{array}$$

Let $\tilde{\phi}_2 : \tilde{F} \rightarrow \mathcal{A}_{\tilde{M}} \otimes \tilde{G}$ be the difference between the two compositions contained in this diagram. Let us check that $\tilde{\phi}_2 = 0$.

Any morphism of right $\mathcal{D}_M \otimes_{\mathbb{C}} R$ -modules $\tilde{\psi} : \tilde{F} \rightarrow \mathcal{A}_{\tilde{M}} \otimes \tilde{G}$ induces a morphism

$$\mathcal{O}_{\tilde{M}} \otimes_R \tilde{F} \rightarrow \mathcal{O}_{\tilde{M}} \otimes_R \mathcal{A}_{\tilde{M}} \otimes \tilde{G} \rightarrow \tilde{G},$$

which we denote by $a(\tilde{\psi})$. It is easy to see that $a(\tilde{\phi}_2) = 0$. Therefore, it suffices to check that the map

$$a : \text{Hom}_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\tilde{F}, \mathcal{A}_{\tilde{M}} \otimes \tilde{G}) \rightarrow \text{Hom}_{\mathcal{D}_M \otimes_{\mathbb{C}} R}(\mathcal{O}_{\tilde{M}} \otimes_R \tilde{F}, \tilde{G})$$

is injective. This is easy to see directly.

Indeed, $\mathcal{A}_{\tilde{M}} \otimes \tilde{G} = \text{Diff}_M(\mathcal{O}_{\tilde{M}}; \tilde{G})$. Under this identification, a sends a morphism $\tilde{\psi} : \tilde{F} \rightarrow \text{Diff}_M(\mathcal{O}_{\tilde{M}}; \tilde{G})$ to the corresponding bilinear operator

$$\mathcal{O}_{\tilde{M}} \otimes_R \tilde{F} \rightarrow \tilde{G} : \tilde{f} \otimes \tilde{s} \mapsto (\tilde{\psi}(\tilde{s}))\tilde{f}.$$

Thus, if $a(\tilde{\psi}) = 0$, then $\tilde{\psi} = 0$. □

REMARK 4.7. The lemma is also easy to prove using Proposition 3.32: indeed, for $\tilde{P} \in \text{Mod}(\mathcal{O}_{\tilde{M}})$, an action of $\mathcal{O}_{\tilde{M}}$ on $\tilde{P}_{\mathcal{D}}$ allows us to reconstruct its action on \tilde{P} using the identification $\tilde{P} = \tilde{P}_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{O}_M$. The advantage of our more complicated argument is that it is easier to generalize to the relative situation.

DEFINITION 4.8. A module $\tilde{F} \in \text{Mod}(\mathcal{O}_{\tilde{M}} \otimes_{\mathbb{C}} \mathcal{D}_M^{op})$ is **of induced type** if it belongs to the essential image of the functor (6).

4.2.3. Let $\underline{\text{Mod}}(\mathcal{D}_M^{op})$ be the stack of right \mathcal{D}_M -modules on M : to an open $U \subset M$, it assigns the category $\underline{\text{Mod}}(\mathcal{D}_M^{op})(U) = \text{Mod}(\mathcal{D}_U^{op})$.

Let \mathbb{M} be a \star -quantization of M (or, more generally, a \star -stack), and let \tilde{P} be an \mathcal{O}_M -module. For every open subset $U \subset M$ and every $\alpha \in \mathbb{M}(U)$, we have a neutralized \star -quantization \tilde{U}_α of U and an $\mathcal{O}_{\tilde{U}_\alpha}$ -module \tilde{P}_α . The corresponding induced right \mathcal{D} -module $(\tilde{P}_\alpha)_{\mathcal{D}}$ carries an action of $\mathcal{O}_{\tilde{U}_\alpha}$. As α varies, this defines a \mathbb{C} -linear functor

$$\tilde{P}_{\mathcal{D}} : \mathbb{M} \rightarrow \underline{\text{Mod}}(\mathcal{D}_M^{op}) : \alpha \mapsto (\tilde{P}_\alpha)_{\mathcal{D}}.$$

DEFINITION 4.9. A \mathbb{C} -linear 1-morphism $\tilde{F} : \mathbb{M} \rightarrow \underline{\mathbf{Mod}}(\mathcal{D}_M^{\text{op}})$ is an $\mathcal{O}_M - \mathcal{D}_M$ -bimodule. The abelian category of $\mathcal{O}_M - \mathcal{D}_M$ -bimodules is denoted by $\mathbf{Mod}(\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{D}_M^{\text{op}})$.

We say that $\tilde{F} \in \mathbf{Mod}(\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{D}_M^{\text{op}})$ is **of induced type** if for every open set $U \subset M$ and every $\alpha \in \mathbb{M}(U)$, the $\mathcal{O}_{\tilde{U}_\alpha} - \mathcal{D}_U$ -bimodule \tilde{F}_α is of induced type. Let

$$\mathbf{Mod}(\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{D}_M^{\text{op}})_{\text{ind}} \subset \mathbf{Mod}(\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{D}_M^{\text{op}})$$

be the full subcategory of bimodules of induced type.

PROPOSITION 4.10. The functor

$$\tilde{P} \mapsto \tilde{P}_D : \mathbf{Mod}(\mathcal{O}_M) \rightarrow \mathbf{Mod}(\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{D}_M^{\text{op}})_{\text{ind}}$$

is an equivalence.

Proof. The statement is local on M , so we may assume that \mathbb{M} is neutral. In this case, the claim reduces to Lemma 4.6. \square

4.2.4. Let us now consider the relative version of the above notions. Let $p_X : Z \rightarrow X$ be a submersive morphism of complex manifolds. A neutralized \star -quantization \tilde{X} of X provides a coalgebra $p_X^* \mathcal{A}_{\tilde{X}} \in \mathbf{Mod}(\mathcal{D}_Z)$. A coaction of $p_X^* \mathcal{A}_{\tilde{X}}$ on a right \mathcal{D}_Z -module \tilde{F} induces an action of $p_X^{-1}(\mathcal{O}_{\tilde{X}})$ on \tilde{F} . Lemma 4.6 remains valid in this situation (with essentially the same proof).

LEMMA 4.11. The resulting functor

$$\mathbf{Comod}(p_X^* \mathcal{A}_{\tilde{X}}) \rightarrow \mathbf{Mod}(p_X^{-1} \mathcal{O}_{\tilde{X}} \otimes_{\mathbb{C}} \mathcal{D}_Z^{\text{op}})$$

is fully faithful.

Proof. As we explained in Remark 4.7, one can prove Lemma 4.6 by using Proposition 3.32. The same reasoning yields a proof of Lemma 4.11 from Proposition 3.35. \square

DEFINITION 4.12. Let \mathbb{X} be a \star -quantization of X . An $\mathcal{O}_{\mathbb{X}} - \mathcal{D}_Z$ -bimodule is a \mathbb{C} -linear 1-morphism of stacks

$$\tilde{F} : \mathbb{X} \rightarrow p_{X*} \underline{\mathbf{Mod}}(\mathcal{D}_Z^{\text{op}}).$$

Here $p_{X*} \underline{\mathbf{Mod}}(\mathcal{D}_Z^{\text{op}})$ is the direct image of the stack $\underline{\mathbf{Mod}}(\mathcal{D}_Z^{\text{op}})$ to X . Thus, for any open subset $U \subset X$,

$$p_{X*} \underline{\mathbf{Mod}}(\mathcal{D}_Z^{\text{op}})(U) = \underline{\mathbf{Mod}}(\mathcal{D}_Z^{\text{op}})(p_X^{-1}(U)) = \mathbf{Mod}(\mathcal{D}_{p_X^{-1}(U)}^{\text{op}}).$$

For any open set $U \subset X$ and any $\alpha \in \mathbb{X}$, the right $\mathcal{D}_{p_X^{-1}(U)}$ -module \tilde{F}_α carries a natural left action of $p_X^{-1}(\mathcal{O}_{\tilde{U}_\alpha})$. We say that \tilde{F} is **of induced type** if for all U and α , the module

$$\tilde{F}_\alpha \in \mathbf{Mod}(p_X^{-1}(\mathcal{O}_{\tilde{U}_\alpha}) \otimes_{\mathbb{C}} \mathcal{D}_{p_X^{-1}(U)}^{\text{op}})$$

is induced from a $p_X^* \mathcal{A}_{\tilde{U}_\alpha}$ -comodule; that is, it belongs to the essential image of the functor of Lemma 4.11. If $V \subset Z$ is an open subset and $W = p_X(V) \subset X$ we will write

$$\mathcal{C}_V = \text{Mod}(p_X^{-1} \mathcal{O}_W \otimes_{\mathbb{C}} \mathcal{D}_V^{op})_{ind}$$

for the category of $\mathcal{O}_W - \mathcal{D}_V$ -bimodules of induced type.

EXAMPLE 4.13. Suppose \mathbb{X} admits a neutralization $\alpha \in \mathbb{X}(X)$. Then $\tilde{F} \in \mathcal{C}_Z$ is uniquely determined by

$$\tilde{F}_\alpha \in \text{Mod}(p_X^{-1} \mathcal{O}_{\tilde{X}_\alpha} \otimes_{\mathbb{C}} \mathcal{D}_Z^{op}).$$

Moreover, \tilde{F}_α must belong to the essential image of the induction functor of Lemma 4.11, so that \tilde{F}_α is induced by a $p_X^*(\mathcal{A}_{\tilde{X}_\alpha})$ -comodules. This defines an equivalence between \mathcal{C} and $\text{Comod}(\mathcal{A}_{\tilde{X}_\alpha})$. See Corollary 6.14 for a more general statement.

PROPOSITION 4.14. \mathcal{C}_Z is an abelian category with enough injective objects.

Proof. The direct image functor $\mathcal{C}_V \rightarrow \mathcal{C}_Z$ preserves injectivity, because its left adjoint is exact. Therefore, the statement of the proposition is local on Z .

Note that the category \mathcal{C}_V depends only on the restriction of the \star -quantization \mathbb{X} to the open subset $W = p_X(V) \subset X$. By choosing V small enough, we may assume that this restriction is neutral. We are thus in the setting of Example 4.13, and Lemma 3.23 implies the statement. \square

4.2.5. Finally, note that \mathcal{C}_Z is equipped with an action of the tensor category of $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -modules: given $\tilde{F} \in \mathcal{C}_Z$ and $G \in \text{Mod}(\mathcal{D}_Z \otimes_{\mathbb{C}} R)$, we define $G \otimes \tilde{F}$ by

$$(G \otimes \tilde{F})_\alpha = G \otimes (\tilde{F}_\alpha) \quad (\alpha \in \mathbb{X}(U), U \subset X).$$

The tensor product is over $\mathcal{O}_Z \otimes_{\mathbb{C}} R$.

4.2.6. We can now repeat the argument of Section 4.1 using the category \mathcal{C}_Z in place of $\text{Comod}(p_X^* \mathcal{A}_X)$. The key step is the following observation.

Let $Z = X \times Y$ be a product of two complex manifolds. Suppose \mathbb{X} is a \star -quantization of X and \tilde{Y} is a neutralized \star -quantization of Y . The inverse image $p_Y^* \mathcal{A}_{\tilde{Y}}$ is a coalgebra in the category $\text{Mod}(\mathcal{D}_Z \otimes_{\mathbb{C}} R)$. Since this category acts on \mathcal{C}_Z , it makes sense to consider $p_Y^* \mathcal{A}_{\tilde{Y}}$ -modules in this category. In particular, for any $\mathcal{O}_{\mathbb{X} \times \tilde{Y}}$ -module \tilde{P} , the induced bimodule $\tilde{P}_{\mathcal{D}} \in \mathcal{C}_Z$ carries a coaction of $p_Y^* \mathcal{A}_{\tilde{Y}}$.

PROPOSITION 4.15. The correspondence $\tilde{P} \mapsto \tilde{P}_{\mathcal{D}}$ is an equivalence between the category of $\mathcal{O}_{\mathbb{X} \times \tilde{Y}}$ -modules and the category of $p_Y^* \mathcal{A}_{\tilde{Y}}$ -comodules in \mathcal{C}_Z .

Proof. The statement is local on Z , so we may assume without losing generality that \mathbb{X} corresponds to a neutralized \star -quantization \tilde{X} of X . Then by Lemma 4.6 we have that $\mathcal{C}_Z = \text{Comod}(p_X^* \mathcal{A}_{\tilde{X}})$, and $p_Y^* \mathcal{A}_{\tilde{Y}}$ -comodules in \mathcal{C}_Z are simply comodules over

$$p_X^* \mathcal{A}_{\tilde{X}} \otimes p_Y^* \mathcal{A}_{\tilde{Y}} = \mathcal{A}_{\tilde{X} \times \tilde{Y}}.$$

The proof now follows from Proposition 3.32. \square

4.2.7. Let us now outline how the argument of Section 4.1 can be adapted to prove Theorem 2.18 in full generality. We are given a coherent sheaf P on $Z = X \times Y$ and a \star -quantization \mathbb{X} of X . As before, we may assume that Y is Stein. One can check that Lemma 4.3 remains true if we replace the category $\text{Comod}(p_X^* \mathcal{A}_{\mathbb{X}})$ with \mathcal{C}_Z ; this requires an analogue of Proposition 3.24 for the category \mathcal{C}_Z , which in turn relies on Proposition 4.14. Such version of Lemma 4.3 provides a deformation of $P_{\mathcal{D}} \in \text{Comod}(p_X^* \mathcal{D}_X)$ to $P'_{\mathcal{D}} \in \mathcal{C}_Z$. Next note that $P'_{\mathcal{D}}$ can be equipped with a homomorphism

$$\Delta'_Y : P'_{\mathcal{D}} \rightarrow (p_Y^* \mathcal{D}_Y \otimes_{\mathbb{C}} R) \otimes P'_{\mathcal{D}},$$

which satisfies the universal property of Lemma 4.3(c). The universal property then implies that there is a unique coproduct δ' on $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ such that Δ'_Y is a coaction. Define the neutralized \star -quantization \tilde{Y} of Y in such a way that the coalgebra $\mathcal{A}_{\tilde{Y}}^{op}$ is identified with $\mathcal{D}_Y \otimes_{\mathbb{C}} R$ equipped with the coproduct δ' . Then $P'_{\mathcal{D}} \in \mathcal{C}$ is a comodule over $p_Y^* \mathcal{A}_{\tilde{Y}}^{op}$, as required.

This proves the existence statement of Theorem 2.18; the proof of uniqueness is completely parallel to the proof of parts (b) and (c) of Proposition 4.1. \square

4.3 Comments on tangible sheaves and \star -deformations. In our approach to Theorem 2.18, we study deformations of \mathcal{O} -modules by deforming induced \mathcal{D} -modules. Under some additional assumptions (which hold, for instance, in the case of the Fourier–Mukai transform on complex tori), one can interpret deformations of induced \mathcal{D} -modules in a more explicit way by looking at ‘ \star -deformations of an \mathcal{O} -module’.

Let M be a complex manifold.

DEFINITION 4.16. A coherent \mathcal{O}_M -module P is **tangible** if $P \simeq i_{\Gamma*} V$, where $i_{\Gamma} : \Gamma \hookrightarrow M$ is a closed analytic submanifold, and V is a holomorphic vector bundle on Γ .

4.3.1. Recall that for an \mathcal{O}_M -module P , we denote by $P_{\mathcal{D}} = \text{Diff}_M(\mathcal{O}_M; P) = P \otimes_{\mathcal{O}_M} \mathcal{D}_M$ the induced right \mathcal{D}_M -module. From Lemma 3.4, we obtain an identification

$$\text{Diff}_M(P; P) = \mathcal{H}om_{\mathcal{O}_M}(P, P_{\mathcal{D}}) = \mathcal{H}om_{\mathcal{D}_M}(P_{\mathcal{D}}, P_{\mathcal{D}}).$$

If we assume that P is tangible, the identification is valid in the derived sense; in other words, there are no ‘higher derived’ differential operators from P to itself.

LEMMA 4.17. Suppose P is tangible. Then

$$\text{Ext}_{\mathcal{O}_M}^i(P, P_{\mathcal{D}}) = \text{Ext}_{\mathcal{D}_M}^i(P_{\mathcal{D}}, P_{\mathcal{D}}) = 0 \quad (i > 0).$$

Proof. Suppose $P = i_{\Gamma*} V$, where $i_{\Gamma} : \Gamma \hookrightarrow M$ is a closed analytic submanifold, and V is a holomorphic vector bundle on Γ . By adjunction, we have

$$\text{Ext}_{\mathcal{O}_M}^i(P, P_{\mathcal{D}}) = i_{\Gamma*} \mathcal{H}om(V, \mathbb{R}^i i_{\Gamma}^! P_{\mathcal{D}}).$$

Recall that

$$\mathbb{R}^i i_{\Gamma}^! (\bullet) = \mathbb{L}_{(\text{codim } \Gamma - i)} i_{\Gamma}^* (\bullet) \otimes \omega_M^{-1} \otimes \omega_{\Gamma},$$

so we need to verify that

$$\mathbb{L}_k i_{\Gamma}^* (P_{\mathcal{D}}) = 0 \quad (k < \text{codim } \Gamma). \tag{7}$$

It is not hard to verify (7) directly. However, it also immediately follows from the Kashiwara Lemma [Kas03, Theorem 4.30], because $P_{\mathcal{D}}$ is a (right) \mathcal{D}_M -module supported by Γ . □

DEFINITION 4.18. A **\star -deformation** of P over R is a pair $\mathfrak{P} := (\tilde{P}, \mathcal{I}_{\tilde{P}})$ where

- \tilde{P} is a sheaf of R -modules on M .
- $\mathcal{I}_{\tilde{P}}$ (a **\star -structure** on \tilde{P}) is a subsheaf in the sheaf of R -module isomorphisms $\mathcal{I}so_m_R(P \otimes_{\mathbb{C}} R, \tilde{P})$ that is a torsor over the sheaf of groups

$$\mathcal{D}iff_0(P \otimes_{\mathbb{C}} R; P \otimes_{\mathbb{C}} R) := \left\{ \sum_{i=0}^n D_i \hbar^i \mid D_0 = 1, D_i \in \mathcal{D}iff_M(P; P) \right\}.$$

The **trivial \star -deformation** $\mathfrak{P}^{\text{triv}}$ of P is defined as the pair

$$\mathfrak{P}^{\text{triv}} := (P \otimes_{\mathbb{C}} R, \mathcal{D}iff_0(P \otimes_{\mathbb{C}} R, P \otimes_{\mathbb{C}} R)).$$

REMARK 4.19. (1) Usually for a deformation quantization one requires flatness over R . In Definition 4.18, \tilde{P} is automatically R -flat. Indeed, $\mathcal{I}_{\tilde{P}}$ has local sections, so that \tilde{P} is locally isomorphic to $P \otimes_{\mathbb{C}} R$.

(2) Since $\mathcal{D}iff_0(P \otimes_{\mathbb{C}} R; P \otimes_{\mathbb{C}} R)$ consists of operators that are normalized to be 1 modulo \hbar , it follows that isomorphisms in $\mathcal{I}_{\tilde{P}}$ induce a canonical isomorphism $P \xrightarrow{\sim} \tilde{P}/\hbar\tilde{P}$.

Next we need the notion of \star -local maps between \star -deformations.

DEFINITION 4.20. Let $\mathfrak{P}_1 = (\tilde{P}_1, \mathcal{I}_{\tilde{P}_1}), \dots, \mathfrak{P}_k = (\tilde{P}_k, \mathcal{I}_{\tilde{P}_k}), \mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}})$ be \star -deformations of some coherent analytic sheaves P_1, \dots, P_k, P on M . We say that an R -linear sheaf map

$$\tilde{f} : \tilde{P}_1 \otimes_R \tilde{P}_2 \otimes_R \dots \otimes_R \tilde{P}_k \rightarrow \tilde{P}$$

is **\star -local** if for every choice of $\tilde{a}_i \in \mathcal{I}_{\tilde{P}_i}, i = 1, \dots, k$, and $\tilde{a} \in \mathcal{I}_{\tilde{P}}$, the map

$$\tilde{a}^{-1} \circ \tilde{f} \circ (\tilde{a}_1 \otimes \dots \otimes \tilde{a}_k) : P_1 \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} P_n \otimes_{\mathbb{C}} R \rightarrow P \otimes_{\mathbb{C}} R$$

is a polydifferential operator, that is, it belongs to $\mathcal{D}iff_M(P_1, \dots, P_k; P) \otimes_{\mathbb{C}} R$. The sheaf of \star -local maps \tilde{f} is denoted by $\mathcal{D}iff_M(\mathfrak{P}_1, \dots, \mathfrak{P}_k; \mathfrak{P})$.

4.3.2. Let $\mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}})$ be a ★-deformation of a tangle \mathcal{O}_M -module P . Let $\mathfrak{P}_{\mathcal{D}}$ be the sheaf of ★-local maps

$$\mathcal{D}iff_M(\mathfrak{D}_M^{\text{triv}}; \mathfrak{P}),$$

where $\mathfrak{D}_M^{\text{triv}}$ denotes the trivial ★-deformation of \mathcal{O}_M . Since \mathcal{D}_M acts on $\mathcal{O}_M \otimes_{\mathbb{C}} R$ on the left, we obtain a right action of $\mathcal{D}_M \otimes_{\mathbb{C}} R$ on $\mathfrak{P}_{\mathcal{D}}$. It is clear that $\mathfrak{P}_{\mathcal{D}}$ is R -flat and that $P_{\mathcal{D}} = \mathfrak{P}_{\mathcal{D}}/\hbar\mathfrak{P}_{\mathcal{D}}$, so that $\mathfrak{P}_{\mathcal{D}}$ is an R -deformation of the right \mathcal{D}_M -module $P_{\mathcal{D}}$.

PROPOSITION 4.21. *For any tangle \mathcal{O}_M -module P , the correspondence*

$$\mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}}) \mapsto \mathfrak{P}_{\mathcal{D}}$$

provides an equivalence between the category of ★-deformations of P and that of deformations of the right \mathcal{D}_M -module $P_{\mathcal{D}}$.

Proof. By definition, any ★-deformation \mathfrak{P} of P is locally trivial, that is, \mathfrak{P} is locally isomorphic to $\mathfrak{P}^{\text{triv}}$ with the obvious ★-structure. Therefore, ★-deformations of P are in one-to-one correspondence with torsors over $\mathcal{D}iff_0(P \otimes_{\mathbb{C}} R; P \otimes_{\mathbb{C}} R)$: the correspondence associates $\mathcal{I}_{\tilde{P}}$ to $\mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}})$. On the other hand, for any Stein open set $U \subset M$,

$$\text{Ext}_{\mathcal{D}_U}^1((P_{\mathcal{D}})|_U, (P_{\mathcal{D}})|_U) = 0$$

by Lemma 4.17. Therefore, any deformation of $P_{\mathcal{D}} \in \text{Mod}(\mathcal{D}_M^{\text{op}})$ is locally trivial. Again, we see that deformations of $P_{\mathcal{D}}$ are in one-to-one correspondence with torsors over the same sheaf. This implies the statement.

For the sake of completeness, let us describe the inverse correspondence. Let $P'_{\mathcal{D}}$ be an R -deformation of $P_{\mathcal{D}}$. Set $\tilde{P} = P'_{\mathcal{D}} \otimes_{\mathcal{D}_M} \mathcal{O}_M$. It is easy to see that \tilde{P} is R -flat and that $\tilde{P}/\hbar\tilde{P} = P$, because $\text{Tor}_i^{\mathcal{D}_M}(P_{\mathcal{D}}, \mathcal{O}_M) = 0$ for $i > 0$ by Lemma 3.3. Note that \tilde{P} is equipped with a natural evaluation map $P'_{\mathcal{D}} \rightarrow \tilde{P}$.

Since $\text{Ext}_{\mathcal{O}_M}^i(P, P_{\mathcal{D}}) = 0$ for $i > 0$ by Lemma 4.17, the sheaf

$$\mathcal{H}_{\tilde{P}} = \text{Hom}_{\mathcal{O}_M}(P, P'_{\mathcal{D}}) = \text{Hom}_{\mathcal{D}_M}(P_{\mathcal{D}}, P'_{\mathcal{D}})$$

is R -flat and

$$\mathcal{H}_{\tilde{P}}/\hbar\mathcal{H}_{\tilde{P}} = \text{Hom}_{\mathcal{O}_M}(P, P_{\mathcal{D}}) = \mathcal{D}iff_M(P; P).$$

Let $\mathcal{I}_{\tilde{P}} \subset \mathcal{H}_{\tilde{P}}$ be the preimage of $1 \in \mathcal{D}iff_M(P; P)$.

The evaluation map $P'_{\mathcal{D}} \rightarrow \tilde{P}$ allows us to embed $\mathcal{H}_{\tilde{P}}$ in

$$\text{Hom}_{\mathbb{C}}(P, \tilde{P}) = \text{Hom}_R(P \otimes_{\mathbb{C}} R, \tilde{P}).$$

In this way, we view $\mathcal{I}_{\tilde{P}}$ as a subsheaf of $\text{Isom}_R(P \otimes_{\mathbb{C}} R, \tilde{P}) \subset \text{Hom}_R(P \otimes_{\mathbb{C}} R, \tilde{P})$. It is automatic that $\mathfrak{P} := (\tilde{P}, \mathcal{I}_{\tilde{P}})$ is a ★-deformation of P .

Suppose that $\mathfrak{P}^{(1)}$ and \mathfrak{P} are \star -deformations of two tangible sheaves $P^{(1)}$ and P on M . A \star -local map $\mathfrak{P}^{(1)} \rightarrow \mathfrak{P}$ induces a homomorphism of \mathcal{D}_M -modules

$$\mathfrak{P}_{\mathcal{D}}^{(1)} \rightarrow \mathfrak{P}_{\mathcal{D}},$$

which is simply the composition of \star -local maps

$$\mathcal{O}_M \otimes_{\mathbb{C}} R \rightarrow \tilde{P}^{(1)} \rightarrow \tilde{P}.$$

This provides a one-to-one correspondence between \star -local maps $\mathfrak{P}^{(1)} \rightarrow \mathfrak{P}$ and homomorphisms of \mathcal{D}_M -modules $\mathfrak{P}_{\mathcal{D}}^{(1)} \rightarrow \mathfrak{P}_{\mathcal{D}}$ and completes the proof of the proposition. □

REMARK 4.22. More generally, suppose $\mathfrak{P}^{(1)}, \dots, \mathfrak{P}^{(k)}, \mathfrak{P}$ are \star -deformations of tangible sheaves $P^{(1)}, \dots, P^{(k)}, P$ on M . The argument at the end of the proof of the previous proposition identifies the sheaf $\mathcal{D}iff_M(\mathfrak{P}^{(1)}, \dots, \mathfrak{P}^{(k)}; \mathfrak{P})$ of all \star -local maps with the sheaf of all k -ary \star -operations from $\{\mathfrak{P}_{\mathcal{D}}^{(1)}, \dots, \mathfrak{P}_{\mathcal{D}}^{(k)}\}$ to $\mathfrak{P}_{\mathcal{D}}$. (The definition of \star -operations is recalled in Remark 3.38.)

4.3.3. Let \tilde{M} be a neutralized \star -quantization of M , and let \tilde{P} be an $\mathcal{O}_{\tilde{M}}$ -module that is flat over R and such that $P := \tilde{P}/\hbar\tilde{P}$ is a tangible \mathcal{O}_M -module. Proposition 4.21 and Proposition 3.32 imply that \tilde{P} carries a natural \star -structure. Indeed, consider the sheaf of differential operators $\mathcal{D}iff_M(P \otimes_{\mathbb{C}} R; \tilde{P})$. It can be identified with

$$\mathcal{H}om_{\mathcal{O}_M}(P, \tilde{P}_{\mathcal{D}}),$$

so $\mathcal{D}iff_M(P \otimes_{\mathbb{C}} R; \tilde{P})$ is an R -flat deformation of

$$\mathcal{D}iff_M(P; P) \text{ by Lemma 4.17.}$$

Therefore,

$$\mathcal{I}_{\tilde{P}} := \{A \in \mathcal{D}iff(P \otimes_{\mathbb{C}} R; \tilde{P}) : A = 1 \pmod{\hbar}\}$$

provides a natural \star -structure on \tilde{P} .

Note that if $\mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}})$, then Proposition 4.21 implies that $\mathfrak{P}_{\mathcal{D}} = \tilde{P}_{\mathcal{D}}$ as right \mathcal{D}_M -modules. From now on, we will not make a notational distinction between $\mathfrak{P}_{\mathcal{D}}$ and $\tilde{P}_{\mathcal{D}}$.

Note also that the action map $\mathcal{O}_{\tilde{M}} \times \tilde{P} \rightarrow \tilde{P}$ is \star -local once we equip $\mathcal{O}_{\tilde{M}}$ and \tilde{P} with \star -structures $\mathcal{I}_{\mathcal{O}_{\tilde{M}}}$ and $\mathcal{I}_{\tilde{P}}$. In addition, we have the following

LEMMA 4.23. *Let \tilde{M} be a neutralized \star -quantization of M .*

(a) $\mathcal{I}_{\mathcal{O}_{\tilde{M}}}$ is the only \star -structure on $\mathcal{O}_{\tilde{M}}$ that makes the multiplication map

$$\mathcal{O}_{\tilde{M}} \times \mathcal{O}_{\tilde{M}} \rightarrow \mathcal{O}_{\tilde{M}}$$

\star -local. This provides an equivalence between the category of neutralized \star -quantizations of M and the category of \star -deformations $(\mathcal{O}_{\tilde{M}}, \mathcal{I}_{\mathcal{O}_{\tilde{M}}})$ of \mathcal{O}_M together with a lift of the algebra structure on \mathcal{O}_M to an algebra structure on $\mathcal{O}_{\tilde{M}}$ that is \star -local.

(b) Let \tilde{P} be an $\mathcal{O}_{\tilde{M}}$ -module that is flat over R and such that $P := \tilde{P}/\hbar\tilde{P}$ is a tangible \mathcal{O}_M -module. Then $\mathcal{I}_{\tilde{P}}$ is the only \star -structure on \tilde{P} that makes the action map

$$\mathcal{O}_{\tilde{M}} \times \tilde{P} \rightarrow \tilde{P}$$

\star -local.

Proof. Both statements of the lemma are easily proved directly (for (a), see [KS10, Proposition 2.2.3]). Alternatively they can be reduced to our previous results once we properly interpret them by using Remark 4.22. For part (a), Proposition 4.21 identifies the \star -deformations $(\mathcal{O}_{\tilde{M}}, \mathcal{I}_{\mathcal{O}_{\tilde{M}}})$ with deformations $\widetilde{\mathcal{D}}_M$ of \mathcal{D}_M in the category of right \mathcal{D}_M -modules. By Remark 4.22, a \star -local algebra structure on $\mathcal{O}_{\tilde{M}}$ is interpreted as a \star -operation on $\widetilde{\mathcal{D}}_M$. Dualizing (as in Remark 3.38), we obtain a coalgebra structure on the right \mathcal{D}_M -module

$$\widetilde{\mathcal{A}}_M = \mathcal{H}om_{\mathcal{D}_M}(\widetilde{\mathcal{D}}_M, \mathcal{D}_M).$$

We now see that part (a) is equivalent to Proposition 3.27. In the same way, part (b) is equivalent to Proposition 3.32(a). \square

4.3.4. Consider now the relative situation. Let $p_X : Z \rightarrow X$ be a submersive morphism of complex manifolds. Let $(\tilde{P}, \mathcal{I}_{\tilde{P}})$ be a \star -deformation of a tangible coherent sheaf P on Z , and let $(\tilde{Q}, \mathcal{I}_{\tilde{Q}})$ be a \star -deformation of a tangible coherent sheaf Q on X . Given an R -linear map $\tilde{f} : p_X^{-1}(\tilde{Q}) \rightarrow \tilde{P}$, we can state the property of \tilde{f} being \star -local completely analogously to Definition 4.20.

The notion of \star -locality also makes sense for polylinear maps. In the case that we are particularly interested in we consider a quantization \tilde{X} of X . In this case, the sheaf $\mathcal{O}_{\tilde{X}}$ carries a natural \star -structure that turns it into a \star -deformation of \mathcal{O}_X . Now suppose that we are given an action

$$\tilde{\xi} : p_X^{-1}\mathcal{O}_{\tilde{X}} \otimes_R \tilde{P} \rightarrow \tilde{P}.$$

We then say that $\tilde{\xi}$ is \star -local if for every choice of $\tilde{a} \in \mathcal{I}_{\tilde{P}}$, $\tilde{b} \in \mathcal{I}_{\mathcal{O}_{\tilde{X}}}$, the map

$$\tilde{a}^{-1} \circ \tilde{\xi} \circ (p_X^{-1}(\tilde{b}) \otimes \tilde{a}) : p_X^{-1}(\mathcal{O}_X) \otimes_{\mathbb{C}} P \otimes_{\mathbb{C}} R \rightarrow P \otimes_{\mathbb{C}} R$$

is a polydifferential operator, that is, if it belongs to $\mathcal{D}iff_Z(p_X^{-1}\mathcal{O}_X, P; P) \otimes_{\mathbb{C}} R$.

4.3.5. Recall that the inverse image $p_X^*\mathcal{A}_{\tilde{X}}$ is a coalgebra in the category of left $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -modules. Also recall that to \tilde{P} , we assign a right $\mathcal{D}_Z \otimes_{\mathbb{C}} R$ -module $\tilde{P}_{\mathcal{D}} = \mathcal{D}iff_Z(\mathcal{O}_Z \otimes_{\mathbb{C}} R; \tilde{P})$. It is easy to see that a \star -local action $\tilde{\xi}$ yields a coaction

$$\Delta_{\tilde{\xi}} : \tilde{P}_{\mathcal{D}} \rightarrow p_X^*\mathcal{A}_{\tilde{X}} \otimes \tilde{P}_{\mathcal{D}},$$

where the tensor product is over $\mathcal{O}_Z \otimes_{\mathbb{C}} R$. Here $\Delta_{\tilde{\xi}}$ is given essentially by the same formulas as those in 3.4.3. Namely, let us identify $p_X^* \mathcal{A}_{\tilde{X}} \otimes \tilde{P}_{\mathcal{D}}$ with the sheaf of \star -local maps

$$p_X^{-1} \mathcal{O}_{\tilde{X}} \otimes_{\mathbb{C}} \mathcal{O}_Z \rightarrow \tilde{P}.$$

Then $\Delta_{\tilde{\xi}}$ sends a differential operator

$$\tilde{A} \in \text{Diff}_Z(\mathcal{O}_Z \otimes_{\mathbb{C}} R; \tilde{P}) = \tilde{P}_{\mathcal{D}}$$

to the \star -local map

$$\Delta_{\tilde{\xi}}(\tilde{A}) : \tilde{f} \otimes g \mapsto \tilde{\xi}(\tilde{f} \otimes \tilde{A}(g)).$$

We now have the following straightforward

LEMMA 4.24. *The correspondence $\xi \mapsto \Delta_{\tilde{\xi}}$ is a bijection between \star -local actions $\tilde{\xi}$ and coactions of $p_X^* \mathcal{A}_{\tilde{X}}$ on $\tilde{P}_{\mathcal{D}}$.*

Proof. This follows from Lemma 4.23. Indeed, let $P \in \text{Mod}(\mathcal{O}_M)$ be a tangible sheaf on a manifold M . Then by Proposition 4.21 we have an equivalence of categories

$$\left\{ \begin{array}{l} \star\text{-deformations } \mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}}) \text{ of} \\ P \text{ over } R \end{array} \right\} \cong \left\{ \begin{array}{l} \text{deformations } \mathfrak{P}_{\mathcal{D}} \text{ of } P_{\mathcal{D}} \text{ as a} \\ \text{right } \mathcal{D}_M\text{-module} \end{array} \right\}$$

Furthermore, when we consider $P = \mathcal{O}_M$ and take into account the (co)algebra structure we get equivalences of categories

$$\left\{ \begin{array}{l} \star\text{-deformations } (\widetilde{\mathcal{O}}_M, \mathcal{I}_{\widetilde{\mathcal{O}}_M}) \text{ of} \\ \mathcal{O}_M \text{ over } R \text{ with a lift of} \\ \text{the } \mathcal{O}_M\text{-algebra structure as a} \\ \star\text{-local algebra structure} \end{array} \right\} \stackrel{\text{Lemma 4.23(a)}}{\cong} \left\{ \begin{array}{l} R\text{-deformations } \widetilde{\mathcal{A}}_M \text{ of } \mathcal{D}_M \text{ as} \\ \text{a coalgebra in } \text{Mod}(\mathcal{D}_M) \end{array} \right\} \\ \stackrel{\text{Proposition 3.4.4}}{\cong} \left\{ \star\text{-quantizations } \widetilde{M} \text{ of } M \right\}$$

These equivalences justify the slight abuse of notation $(\widetilde{\mathcal{O}}_M, \mathcal{I}_{\widetilde{\mathcal{O}}_M}) = (\mathcal{O}_{\widetilde{M}}, \mathcal{I}_{\mathcal{O}_{\widetilde{M}}})$ and $\widetilde{\mathcal{A}}_M = \mathcal{A}_{\widetilde{M}}$. Moreover, the coalgebra structure allows us to consider comodules over $\mathcal{A}_{\widetilde{M}}$. Thus we can extend the previous equivalences to:

$$\left\{ \begin{array}{l} \star\text{-deformations } \mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}}) \text{ of} \\ P \text{ over } R \text{ with a lift of the } \mathcal{O}_M\text{-} \\ \text{action to a } \star\text{-local action of } \mathcal{O}_{\widetilde{M}} \end{array} \right\} \stackrel{\text{Lemma 4.23(b)}}{\cong} \left\{ \begin{array}{l} R\text{-deformations of } P_{\mathcal{D}} \text{ to} \\ \text{a comodule } \mathfrak{P}_{\mathcal{D}} \text{ over } \widetilde{\mathcal{A}}_M \end{array} \right\} \\ \stackrel{\text{Proposition 3.32(b)}}{\cong} \left\{ \begin{array}{l} R\text{-deformations of} \\ P \text{ to a module } \tilde{P} \\ \text{over } \mathcal{O}_{\widetilde{M}} \end{array} \right\}.$$

□

4.3.6. Combining Proposition 4.21 and Lemma 4.24, we can now make the argument of Section 4.1 more explicit, at least when P is a tangible coherent sheaf on Z . Let us sketch this reformulation using the notation of Section 4.1. We assume that P is tangible.

The first step is to construct a \star -deformation $\mathfrak{P} = (\tilde{P}, \mathcal{I}_{\tilde{P}})$ of P together with an extension of the action

$$\xi : p_X^{-1}(\mathcal{O}_X) \otimes_{\mathbb{C}} P \rightarrow P$$

to a \star -local action

$$\tilde{\xi} : p_X^{-1}(\mathcal{O}_{\tilde{X}}) \otimes_R \tilde{P} \rightarrow \tilde{P}.$$

Lemma 4.3(a) claims that this can be done in a way that is unique up to a non-canonical isomorphism.

The second step is to extend the action

$$\eta : p_Y^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} P \rightarrow P$$

to a \star -local map (not necessarily an action)

$$\tilde{\eta} : p_Y^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} \tilde{P} \rightarrow \tilde{P}.$$

Such an extension is provided by Lemma 4.3(b).

Finally, by the universal property from Lemma 4.3(c) it follows that there is a unique local product on $p_Y^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} R$ such that $\tilde{\eta}$ is an action of the resulting algebra. Viewing $p_Y^{-1}(\mathcal{O}_Y) \otimes_{\mathbb{C}} R$ with this product, we obtain a neutralized \star -quantization \tilde{Y}^{op} of Y such that \tilde{P} has a structure of a $\mathcal{O}_{\tilde{X} \times \tilde{Y}^{op}}$ -module.

4.4 Another proof of Theorem 2.18. The proof of Theorem 2.18 can be restated using (relative) differential operators on quantizations, thus avoiding coalgebras in the category of comodules. The resulting argument is somewhat more elementary, but less transparent. The approach is also not completely canonical: essentially, one can work with relative differential operators for the morphism $\tilde{Z}' \rightarrow \tilde{X}$, where \tilde{Z}' is any neutralized \star -quantization of $Z = X \times Y$ that admits a map to the given quantization \tilde{X} of X . (The category of $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -modules does not depend on the choice of \tilde{Z}' .) We take $\tilde{Z}' = \tilde{X} \times Y$; more precisely, \tilde{Z}' is the product of \tilde{X} and the trivial neutralized \star -quantization of Y .

4.4.1. As in Section 4.1, we assume that the \star -quantization of X is neutralized, and our goal is to prove Proposition 4.1. One can then extend the proof to all \star -quantizations of X using the approach of Section 4.2, see Section 6.2. This is not done in this paper.

4.4.2. Set $\tilde{Z}' = \tilde{X} \times Y$, and let $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ be the sheaf of differential operators $\mathcal{O}_{\tilde{Z}'} \rightarrow \mathcal{O}_{\tilde{X}}$ that commute with the left action of $p_X^{-1}(\mathcal{O}_{\tilde{X}})$. Recall that $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ is a flat sheaf of

R -algebras by Proposition 3.18 and that

$$\mathcal{D}_{Z/X} = \mathcal{D}_{\tilde{Z}'/\tilde{X}}/\hbar\mathcal{D}_{\tilde{Z}'/\tilde{X}}.$$

Let $P_{\mathcal{D}/X}$ be the right $\mathcal{D}_{Z/X}$ -module induced from P . We derive Proposition 4.1 from the following claim (which is essentially a version of Lemma 4.3).

LEMMA 4.25. *In the assumptions and notation of Proposition 4.1, we have the following.*

- (a) *There is a R -flat deformation of $P_{\mathcal{D}/X}$ to a $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -module $P'_{\mathcal{D}/X}$, unique up to a not necessarily unique isomorphism.*
- (b) *The direct image $\mathcal{E}' := p_{Y*} \text{End}_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}(P'_{\mathcal{D}/X})$ is flat over R . The action of $p_Y^{-1}\mathcal{O}_Y$ on $P_{\mathcal{D}/X}$ (coming from its action on P) induces isomorphisms*

$$\mathcal{O}_Y \rightarrow p_{Y*} \text{End}_{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}) \leftarrow \mathcal{E}'/\hbar\mathcal{E}'.$$

- (c) *Let*

$$m : p_Y^{-1}(\mathcal{O}_Y) \times P_{\mathcal{D}/X} \rightarrow P_{\mathcal{D}/X}$$

be the (\mathbb{C} -bilinear) morphism induced by the action of $p_Y^{-1}(\mathcal{O}_Y) \subset \mathcal{O}_Z$ on P . Then m can be lifted to a \mathbb{C} -bilinear morphism

$$m' : p_Y^{-1}(\mathcal{O}_Y) \times P'_{\mathcal{D}/X} \rightarrow P'_{\mathcal{D}/X}$$

such that $m'(\tilde{f}, A)$ is a morphism of $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -modules for fixed $\tilde{f} \in p_Y^{-1}(\mathcal{O}_Y)$ and a differential operator in \tilde{f} for fixed $A \in P'_{\mathcal{D}/X}$. (To consider differential operators, we use the $\mathcal{O}_{\tilde{Z}'}$ -module structure on $P'_{\mathcal{D}/X}$ induced by the embedding $\mathcal{O}_{\tilde{Z}'} \hookrightarrow \mathcal{D}_{\tilde{Z}'/\tilde{X}}$.)

Proof. Recall that $\mathcal{E}(P) = \mathbb{R}p_{Y,*}\mathbb{R} \text{Hom}_{\mathcal{O}_Z}(P, P_{\mathcal{D}/X})$, and therefore

$$\mathbb{H}^i(\mathcal{E}(P)) = \text{Ext}_{\mathcal{O}_Z}^i(P, P_{\mathcal{D}/X}) = \text{Ext}_{\mathcal{D}_{Z/X}}^i(P_{\mathcal{D}/X}, P_{\mathcal{D}/X})$$

by Lemma 3.10. The hypothesis that $\iota : \mathcal{O}_Y \rightarrow \mathcal{E}(P)$ is a quasi-isomorphism implies that

$$\text{Ext}_{\mathcal{D}_{Z/X}}^i(P_{\mathcal{D}/X}, P_{\mathcal{D}/X}) = \begin{cases} \Gamma(Y, \mathcal{O}_Y) & (i = 0) \\ 0, & (i > 0). \end{cases}$$

This implies part (a) by the usual deformation theory argument (cf. Proposition 3.24). Indeed, the sheaf of algebras $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ is an R -flat deformation of $\mathcal{D}_{Z/X}$ by Proposition 3.18. Therefore, the category $\text{Mod}(\mathcal{D}_{\tilde{Z}'/\tilde{X}})$ is a flat deformation of $\text{Mod}(\mathcal{D}_{Z/X})$ in the sense of [LV06]. Now [Low05, Theorem A] implies that deformations of a $\mathcal{D}_{Z/X}$ -module are controlled by its derived endomorphisms.

To prove **(b)**, consider on $P'_{\mathcal{D}/X}$ the filtration by modules $\hbar^k P'_{\mathcal{D}/X}$; the corresponding quotients are isomorphic to $P_{\mathcal{D}/X}$, so the associated graded module is $P_{\mathcal{D}/X} \otimes_{\mathbb{C}} R$. Since

$$\mathbb{R}p_{Y*} \mathbb{R} \mathcal{H}om_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}(P'_{\mathcal{D}/X}, P_{\mathcal{D}/X}) = \mathbb{R}p_{Y*} \mathbb{R} \mathcal{H}om_{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}, P_{\mathcal{D}/X}) = \mathcal{O}_Y,$$

we see that \mathcal{E}' is filtered by sheaves

$$p_{Y*} \mathcal{H}om_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}(P'_{\mathcal{D}/X}, \hbar^k P'_{\mathcal{D}/X})$$

with the associated graded being $\mathcal{O}_Y \otimes_{\mathbb{C}} R$, as required.

Finally, in **(c)**, let $Q' = \mathcal{D}iff_Z(p_Y^{-1}\mathcal{O}_Y; P'_{\mathcal{D}/X})$ be the sheaf of differential operators $p_Y^{-1}\mathcal{O}_Y \rightarrow P'_{\mathcal{D}/X}$. Note that for $A \in Q'$ and $B \in \mathcal{D}_{\tilde{Z}'/\tilde{X}}$, the product

$$AB : p_Y^{-1}\mathcal{O}_Y \rightarrow P'_{\mathcal{D}/X} : \tilde{f} \mapsto A(\tilde{f})B$$

is again a differential operator. This turns Q' into a right $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -module. Note also that Q' is flat over R and that $Q'/\hbar Q' = \mathcal{D}iff_Z(p_Y^{-1}\mathcal{O}_Y; P_{\mathcal{D}/X})$. We have to show that $m : P_{\mathcal{D}/X} \rightarrow Q'/\hbar Q'$ lifts to a morphism of $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -modules $m' : P'_{\mathcal{D}/X} \rightarrow Q'$.

The obstruction to lifting m lie in

$$\text{Ext}_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}^1(P'_{\mathcal{D}/X}, \hbar Q').$$

Note that $\hbar Q' \subset Q'$ admits a filtration by the modules $\hbar^k Q'$ with quotients isomorphic to $Q'/\hbar Q'$. It therefore suffices to prove the vanishing of

$$\text{Ext}_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}^1(P'_{\mathcal{D}/X}, Q'/\hbar Q') = \text{Ext}_{\mathcal{D}_{Z/X}}^1(P_{\mathcal{D}/X}, Q'/\hbar Q'). \tag{8}$$

Actually, let us prove that

$$\mathbb{R}p_{Y*} \mathbb{R} \mathcal{H}om_{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}, Q'/\hbar Q') = \mathcal{D}_Y.$$

This implies that (8) vanishes, because Y is a Stein manifold, and \mathcal{D}_Y is a direct limit of coherent \mathcal{O}_Y -modules, and therefore $H^1(Y, \mathcal{D}_Y) = 0$.

Indeed, $Q'/\hbar Q'$ is equal to the sheaf of bidifferential operators $p_Y^{-1}(\mathcal{O}_Y) \times \mathcal{O}_Z \rightarrow P$ that agree with the action of $p_X^{-1}(\mathcal{O}_X)$ on \mathcal{O}_Z and on P . Note that even though $P_{\mathcal{D}/X}$ has two structures of an \mathcal{O}_Z -module, both structures lead to the same class of differential operators $p_Y^{-1}(\mathcal{O}_Y) \rightarrow P_{\mathcal{D}/X}$. We can then identify $Q'/\hbar Q'$ with $p_Y^{-1}\mathcal{D}_Y \otimes_{p_Y^{-1}\mathcal{O}_Y} P_{\mathcal{D}/X}$, where \mathcal{O}_Y acts on both \mathcal{D}_Y and $P_{\mathcal{D}/X}$ by left multiplication. We finally get an isomorphism

$$\begin{aligned} \mathbb{R}p_{Y*} \mathbb{R} \mathcal{H}om_{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}, p_Y^{-1}\mathcal{D}_Y \otimes_{p_Y^{-1}\mathcal{O}_Y} P_{\mathcal{D}/X}) \\ = \mathbb{R}p_{Y*} \mathbb{R} \mathcal{H}om_{\mathcal{D}_{Z/X}}(P_{\mathcal{D}/X}, P_{\mathcal{D}/X}) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y. \end{aligned} \quad \square$$

4.4.3. Let us now derive Proposition 4.1, starting with the existence claim. For $P'_{\mathcal{D}/X}$ provided by Lemma 4.25(a), set $\tilde{P} := P'_{\mathcal{D}/X} \otimes_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}} \mathcal{O}_{\tilde{Z}'}$. Then \tilde{P} is a flat sheaf of R -modules equipped with an identification $\tilde{P}/\hbar\tilde{P} = P$.

Lemma 4.25(b) now gives a flat R -deformation of \mathcal{O}_Y to a sheaf of R -algebras \mathcal{E}' on Y . Note that m' from Lemma 4.25(c) can be viewed as a map $p_Y^{-1}(\mathcal{O}_Y) \rightarrow \text{End}_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}(P'_{\mathcal{D}/X})$. By adjunction, we obtain a map $\mathcal{O}_Y \rightarrow \mathcal{E}'$. It follows from Lemma 4.25(b) that the induced map of R -modules $\mu' : \mathcal{O}_Y \otimes_{\mathbb{C}} R \rightarrow \mathcal{E}'$ is an isomorphism.

With respect to μ' , the product on \mathcal{E}' can be written as

$$\mu'(f)\mu'(g) = \mu' \left(\sum_{k=0}^n B_k(f, g)\hbar^k \right) \quad (f, g \in \mathcal{O}_Y \otimes_{\mathbb{C}} R)$$

for some bilinear maps $B_k(f, g) : \mathcal{O}_Y \times \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ with $B_0(f, g) = fg$. We claim that $B_k(f, g)$ is a bidifferential operator for any k (that is, the product on \mathcal{E}' corresponds to a \star -product on $\mathcal{O}_Y \otimes_{\mathbb{C}} R$). Indeed, by construction

$$m'(f, m'(g, A)) = \sum_{k=0}^n m'(B_k(f, g), A)\hbar^k \quad (f, g \in \mathcal{O}_Y, A \in P'_{\mathcal{D}/X}),$$

and the left-hand side is a differential operator in both f and g . Thus by induction in k the B_k 's are bidifferential operators. Define the neutralized \star -quantization \tilde{Y} of Y by setting $\mathcal{O}_{\tilde{Y}}^{op} = \mathcal{E}'$.

4.4.4. It remains to equip \tilde{P} with the structure of an \mathcal{O} -module on $\tilde{X} \times \tilde{Y}^{op}$. Correcting m' by μ' , we obtain a morphism

$$m_Y = m' \circ (\mu')^{-1} : p_Y^{-1}(\mathcal{O}_{\tilde{Y}}^{op}) \times P'_{\mathcal{D}/X} \rightarrow P'_{\mathcal{D}/X}.$$

It has the same properties as m' : it respects the $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -module structure on $P'_{\mathcal{D}/X}$, and it is a differential operator on $p_Y^{-1}(\mathcal{O}_{\tilde{Y}}^{op})$. Moreover, it defines an action of $p_Y^{-1}(\mathcal{O}_{\tilde{Y}}^{op})$ on $P'_{\mathcal{D}/X}$:

$$m_Y(\tilde{f}, m_Y(\tilde{g}, A)) = m_Y(\tilde{f}\tilde{g}, A) \quad (\tilde{f}, \tilde{g} \in p_Y^{-1}(\mathcal{O}_{\tilde{Y}}^{op}), A \in P'_{\mathcal{D}/X}).$$

Recall now that $P'_{\mathcal{D}/X}$ is a $\mathcal{O}_{\tilde{Z}'}$ -module, in particular, it has a natural action of $p_X^{-1}(\mathcal{O}_{\tilde{X}})$

$$m_X : p_X^{-1}(\mathcal{O}_{\tilde{X}}) \times P'_{\mathcal{D}/X} \rightarrow P'_{\mathcal{D}/X}.$$

m_X does not preserve the \mathcal{D} -module (or even \mathcal{O} -module) structure on $P'_{\mathcal{D}/X}$, but it is still a differential operator in the $p_X^{-1}(\mathcal{O}_{\tilde{X}})$ -variable.

4.4.5. Set $\tilde{Z} = \tilde{X} \times \tilde{Y}^{op}$. The maps m_X and m_Y define an R -bilinear morphism

$$m_Z : \mathcal{O}_{\tilde{Z}} \times P'_{\mathcal{D}/X} \rightarrow P'_{\mathcal{D}/X}$$

that is a differential operator on $\mathcal{O}_{\tilde{Z}}$ by the formula

$$m_Z(\tilde{f}\tilde{g}, A) = m_X(\tilde{f}, m_Y(\tilde{g}, A)) \quad (\tilde{f} \in p_X^{-1}(\mathcal{O}_{\tilde{X}}), \tilde{g} \in p_Y^{-1}(\mathcal{O}_{\tilde{Y}}^{op}), A \in P'_{\mathcal{D}/X}).$$

The map m_Z is an action of $\mathcal{O}_{\tilde{Z}}$, because the actions m_X and m_Y commute. Finally, if we set $I := \{A \in \mathcal{D}_{\tilde{Z}'/\tilde{X}} : A(1) = 0\}$, we see that the action of $\mathcal{O}_{\tilde{Z}}$ on $P'_{\mathcal{D}/X}$ descends to an action on

$$\tilde{P} = P'_{\mathcal{D}/X} \otimes_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}} \mathcal{O}_{\tilde{Z}'} = P'_{\mathcal{D}/X}/P'_{\mathcal{D}/X}I.$$

Thus \tilde{P} is an $\mathcal{O}_{\tilde{Z}}$ -module. This proves part **(a)** of Proposition 4.1

4.4.6. Let us now prove part **(b)**. Suppose $\tilde{Y}^{(1)}$ is a neutralized \star -quantization of Y , and $\tilde{P}^{(1)}$ is an \mathcal{O} -module on $\tilde{X} \times (\tilde{Y}^{(1)})^{op}$ with $\tilde{P}^{(1)}/\hbar\tilde{P}^{(1)} = P$. Then the sheaf of differential operators

$$\tilde{P}_{\mathcal{D}/X}^{(1)} := \text{Diff}_{\tilde{Z}'/\tilde{X}}(\mathcal{O}_{\tilde{Z}'}, \tilde{P}^{(1)})$$

is a flat right $\mathcal{D}_{\tilde{Z}'/\tilde{X}}$ -module that is a flat R -deformation of $P_{\mathcal{D}/X}$. By Lemma 4.25(a), there is an isomorphism $\phi : \tilde{P}_{\mathcal{D}/X}^{(1)} \rightarrow P'_{\mathcal{D}/X}$. Choose ϕ in such a way that it reduces to the identity automorphism of $P_{\mathcal{D}/X}$. Note that ϕ induces an isomorphism of R -modules

$$\tilde{P}^{(1)} = P_{\mathcal{D}/X}^{(1)} \otimes_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}} \mathcal{O}_{\tilde{Z}'} \rightarrow \tilde{P}.$$

For any (local) section $\tilde{f} \in \mathcal{O}_{\tilde{Y}^{(1)}}$, the action of $p_Y^{-1}(\tilde{f})$ on $\tilde{P}^{(1)}$ commutes with the action of $p_X^{-1}(\mathcal{O}_{\tilde{X}})$; therefore, $p_Y^{-1}(\tilde{f})$ acts on $\tilde{P}_{\mathcal{D}/X}^{(1)}$, so that we obtain an action

$$m_Y^{(1)} : p_Y^{-1}(\mathcal{O}_{\tilde{Y}^{(1)}}^{op}) \times \tilde{P}_{\mathcal{D}/X}^{(1)} \rightarrow \tilde{P}_{\mathcal{D}/X}^{(1)}.$$

Under ϕ , it translates into a morphism of sheaves of algebras

$$p_Y^{-1}(\mathcal{O}_{\tilde{Y}^{(1)}}^{op}) \rightarrow \text{End}_{\mathcal{D}_{\tilde{Z}'/\tilde{X}}}(P'_{\mathcal{D}/X}),$$

which by adjunction induces a homomorphism $\mathcal{O}_{\tilde{Y}}^{op} \rightarrow \mathcal{E}' = \mathcal{O}_{\tilde{Y}}^{op}$ (the sheaf \mathcal{E}' is defined in Lemma 4.25). By Lemma 4.25(b), the map is an isomorphism, so it identifies \tilde{Y}^{op} and \tilde{Y} .

4.4.7. It remains to check that the identification $\tilde{P} = \tilde{P}^{(1)}$ is $\mathcal{O}_{\tilde{Z}}$ -linear. Under the isomorphism ϕ , the actions m_X and m_Y correspond to the following actions on differential operators $\tilde{P}_{\mathcal{D}/X}^{(1)}$:

$$\begin{aligned} m_X(\tilde{f}, A)(\tilde{h}) &= A(\tilde{h}\tilde{f}) & (A \in \tilde{P}_{\mathcal{D}/X}^{(1)}, \tilde{f} \in p_X^{-1}(\mathcal{O}_{\tilde{X}}), \tilde{h} \in \mathcal{O}_{\tilde{Z}'}), \\ m_Y(\tilde{g}, A)(\tilde{h}) &= \tilde{g}A(\tilde{h}) & (A \in \tilde{P}_{\mathcal{D}/X}^{(1)}, \tilde{g} \in p_Y^{-1}(\mathcal{O}_{\tilde{Y}}), \tilde{h} \in \mathcal{O}_{\tilde{Z}'}). \end{aligned}$$

The action m_Z is more exotic, but still given by a differential operator on $\mathcal{O}_{\tilde{Z}}$. Hence the induced action of $\mathcal{O}_{\tilde{Z}}$ on

$$\tilde{P}^{(1)} = \tilde{P}_{\mathcal{D}/X}^{(1)} \otimes_{\mathcal{D}_{\tilde{Z}/\tilde{X}}} \mathcal{O}_{\tilde{Z}}, = \tilde{P}_{\mathcal{D}/X}^{(1)} / \tilde{P}_{\mathcal{D}/X}^{(1)} I$$

is again given by a differential operator on $\mathcal{O}_{\tilde{Z}}$. Since $p_X^{-1}(\mathcal{O}_{\tilde{X}})$ and $p_Y^{-1}(\mathcal{O}_Y^{op})$ act in the natural way, so does $\mathcal{O}_{\tilde{Z}}$. This proves part **(b)**.

4.4.8. The proof of part **(c)** given in Section 4.1 is completely straightforward (it does not use \mathcal{D} -modules). For this part, no restatement is required.

This completes the proof of Proposition 4.1. □

5 Computation of Cohomology

In this section, we prove Theorem 2.19.

5.1 An example: The case of complex tori. As a warmup we first consider the case of families of complex tori, where the calculation is explicit. To prove the general case we take a different tack but we hope that seeing a specific calculation first will be illuminating.

5.1.1. Suppose that $f_X : X \rightarrow B$ is a smooth family of complex tori over a complex manifold B . Here and everywhere else in this paper, a family of tori is supposed to admit a holomorphic zero section $B \rightarrow X$, which is fixed. Let $f_Y : Y \rightarrow B$ be the dual family of tori and set $\Gamma := X \times_B Y$. Let $V \rightarrow \Gamma$ be the normalized Poincaré line bundle, and let $P := i_{\Gamma*} V$ denote the corresponding sheaf on $Z = X \times Y$.

Note that P is tangible and that the projection of $\Gamma = \text{supp}(P)$ to X is submersive. This implies

$$\text{Ext}^i(P, P_{\mathcal{D}/X}) = \text{Ext}^i(P, \text{Diff}_{Z/X}(\mathcal{O}_Z; P)) = 0 \quad (i > 0);$$

this is a relative version of Lemma 4.17. Now by Lemma 3.11, we have

$$\mathbb{R} \text{Hom}(P, \text{Diff}_{Z/X}(\mathcal{O}_Z; P)) = \text{Hom}(P, \text{Diff}_{Z/X}(\mathcal{O}_Z; P)) = \text{Diff}_{Z/X}(P; P).$$

Finally,

$$\text{Diff}_{Z/X}(P; P) = i_{\Gamma*} \text{Diff}_{\Gamma/X}(V; V)$$

by definition. Thus Theorem 2.19 reduces to showing that

$$\mathbb{R} \text{pr}_{Y*} \text{Diff}_{\Gamma/X}(V; V) = \mathcal{O}_Y.$$

Here $\text{pr}_Y := (p_Y)|_{\Gamma} : \Gamma \rightarrow Y$.

5.1.2. Consider the filtration of $\text{Diff}_{\Gamma/X}(V; V)$

$$\text{Diff}_{\Gamma/X}^{\leq 0}(V; V) \subset \text{Diff}_{\Gamma/X}^{\leq 1}(V; V) \subset \dots \subset \text{Diff}_{\Gamma/X}^{\leq i}(V; V) \subset \dots \subset \text{Diff}_{\Gamma/X}(V; V)$$

by order of the differential operators on the fibers of $\mathrm{pr}_X := (p_X)|_\Gamma : \Gamma \rightarrow X$. The spectral sequence for the direct image of the filtered sheaf $\mathcal{D}\mathrm{iff}_{\Gamma/X}(V; V)$ along the fibers of pr_Y has E_1 -term

$$E_1^{ij} = \mathbb{R}^i \mathrm{pr}_{Y*} S^{j-i} T_{\Gamma/X},$$

and converges to $\mathbb{R}^{i+j} \mathrm{pr}_{Y*} \mathcal{D}\mathrm{iff}_{\Gamma/X}(V; V)$.

LEMMA 5.1. *The differential*

$$d_1 : \mathbb{R}^i \mathrm{pr}_{Y*} S^j T_{\Gamma/X} \rightarrow \mathbb{R}^{i+1} \mathrm{pr}_{Y*} S^{j-1} T_{\Gamma/X}$$

is given by the cup product with a class $\alpha_{\Gamma/X} \in H^0(Y, \mathbb{R}^1 \mathrm{pr}_{Y*} \Omega_{\Gamma/X}^1)$. The class $\alpha_{\Gamma/X}$ is the image of

$$c_1(V) - \frac{1}{2} c_1(\omega_{\Gamma/X}) \in H^1(\Gamma, \Omega_\Gamma^1)$$

under the natural map

$$H^1(\Gamma, \Omega_\Gamma^1) \rightarrow H^0(Y, \mathbb{R}^1 \mathrm{pr}_{Y*} \Omega_\Gamma^1) \rightarrow H^0(Y, \mathbb{R}^1 \mathrm{pr}_{Y*} \Omega_{\Gamma/X}^1),$$

which we denote by pr_{Y*} .

Proof. This calculation is done in the absolute case in [BB93, Corollary 2.4.6]. The proof of the relative case presents no difficulties; the key step in the calculation is the isomorphism of the graded $\mathbb{C}[\hbar]/\hbar^2$ -Poisson algebra corresponding to the Rees ring of $\mathcal{D}\mathrm{iff}(V; V)$ and the graded Poisson algebra corresponding to the twisted cotangent bundle. Since this isomorphism is written intrinsically in [BB93, Corollary 2.4.5] it carries over immediately to the relative context and so the proof follows. \square

5.1.3. Next note that the relative dualizing sheaf $\omega_{\Gamma/X}$ is trivial locally over B , since $\mathrm{pr}_X : \Gamma \rightarrow X$ is a family of complex tori. So, in order to understand the spectral sequence computing the cohomology of $\mathcal{D}\mathrm{iff}_{\Gamma/X}(V; V)$, we only need to understand the class $\mathrm{pr}_{Y*} c_1(V)$. However the standard Kodaira–Spencer theory identifies the class

$$\mathrm{pr}_{Y*} c_1(V) \in \mathbb{R}^1 \mathrm{pr}_{Y*} \Omega_{\Gamma/Y}^1 \cong (\mathbb{R}^1 \mathrm{pr}_{Y*} \mathcal{O}_\Gamma) \otimes \Omega_{Y/B}^1 = f_Y^* \mathbb{R}^1 f_{X*} \mathcal{O}_X \otimes \Omega_{Y/B}^1$$

with the differential $d\xi_V \in \mathrm{Hom}(T_{Y/B}, T_{\mathrm{Pic}^0(X/B)/B})$ of the classifying map

$$\xi_V : Y \rightarrow \mathrm{Pic}^0(X/B)$$

corresponding to the Poincaré line bundle V . This immediately implies Theorem 2.19 in our case:

COROLLARY 5.2. *If $f_X : X \rightarrow B$ and $f_Y : Y \rightarrow B$ are dual families of complex tori, and if $V \rightarrow \Gamma = X \times_B Y$ is the normalized Poincaré line bundle, then $\mathbb{R} \mathrm{pr}_{Y*} \mathcal{D}\mathrm{iff}_{\Gamma/X}(V; V) = \mathcal{O}_Y$.*

Proof. The differential of the spectral sequence is given by the cup product with

$$\alpha_{\Gamma/X} = \text{pr}_{Y*} \left(c_1(V) - \frac{1}{2} c_1(\omega_{\Gamma/X}) \right) = \text{pr}_{Y*}(c_1(V)) = d\xi_V \in \text{Hom}(T_{Y/B}, f_Y^* \mathbb{R}^1 f_{X*} \mathcal{O}_X)$$

and so

$$\begin{aligned} (\mathbb{R}^i \text{pr}_{Y*} S^j T_{\Gamma/X}, d_1) &= (\mathbb{R}^i \text{pr}_{Y*} \mathcal{O}_\Gamma \otimes S^j T_{Y/B}, d_1) \\ &= (f_Y^* \mathbb{R}^i f_{X*} \mathcal{O}_X \otimes S^j T_{Y/B}, d_1) \\ &= (f_Y^* (\wedge^i \mathbb{R}^1 f_{X*} \mathcal{O}_X) \otimes S^j T_{Y/B}, \bullet \cup d\xi_V) \end{aligned}$$

is just the Koszul complex. □

5.2 Properties of the Fourier–Mukai transform. In this section, we make some observations about the Fourier–Mukai formalism on complex manifolds. They are used to derive Theorem 2.19 in Section 5.3.

LEMMA 5.3. *Let M be a complex manifold. Suppose that $F \in D_{coh}^b(M)$ satisfies $\mathbb{L}i_x^* F = 0$ for all embeddings of points $i_x : \{x\} \hookrightarrow M$, $x \in M$. Then $F = 0$.*

Proof. If $F \neq 0$, let $H^k(F)$ be the top non-vanishing cohomology of F . Then $H^k(\mathbb{L}i_x^* F) \neq 0$ for any $x \in \text{supp}(H^k(F))$. □

Recall that $D_{comp}^b(M) \subset D_{coh}^b(M)$ denotes the full subcategory of compactly-supported complexes.

COROLLARY 5.4. *If $G \in D_{coh}^b(M)$ satisfies $\text{Hom}(F, G) = 0$ for all $F \in D_{comp}^b(M)$, then $G = 0$. In other words, the right orthogonal complement to $D_{comp}^b(M)$ in $D_{coh}^b(M)$ vanishes.*

Proof. For the embedding of a point $i_x : \{x\} \hookrightarrow M$, we have

$$\text{Hom}(i_{x,*} \mathbb{C}[-k], G) \simeq H^{k-\dim M}(\mathbb{L}i_x^* G).$$

Now Lemma 5.3 implies the statement. □

5.2.1. Suppose now that X and Y are two complex manifolds. Fix $P \in D_{coh}^b(X \times Y)$ such that $\text{supp}(P)$ is proper over Y . Let

$$\Phi = \Phi^P : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$$

be the integral transform with respect to P given by

$$\Phi F = \mathbb{R}p_{X*}(p_Y^* F \otimes^{\mathbb{L}} P). \tag{9}$$

The Serre duality implies that Φ has a right adjoint

$$\Psi = \Psi^P : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$$

given by

$$\Psi F = \mathbb{R}p_{Y*}(\mathbb{R}\mathcal{H}om_{X \times Y}(P, p_X^* F)) \otimes \omega_Y[\dim Y]. \quad (10)$$

The fact that Φ and Ψ are adjoint is well known if X and Y are both compact, see e.g. [CS11, Proposition 2.2] for the statement in the framework of algebraic varieties. The analytic version that we use here relies on the duality theory for coherent sheaves on complex manifolds, which is developed in [RR74].

REMARK 5.5. Note that Ψ need not preserve the subcategory of compactly-supported objects. Because of this, the adjunction relation between Φ and Ψ is asymmetric. For instance, a right adjoint of Φ (in this sense) is not always unique.

5.2.2. Set

$$Q := \mathbb{R}p_{23*}(\mathbb{R}\mathcal{H}om_{X \times Y \times Y}(p_{12}^* P, p_{13}^* P) \otimes p_3^* \omega_Y[\dim Y]) \in D_{coh}^b(Y \times Y),$$

where p_{12} , for instance, is the projection of $X \times Y \times Y$ onto the product of the first two factors. Base change implies that Q is the kernel of $\Psi \circ \Phi$:

$$\Psi \circ \Phi(F) = \mathbb{R}p_{1*}(Q \otimes^{\mathbb{L}} p_2^* F), \quad F \in D_{comp}^b(Y).$$

The adjunction between Φ and Ψ yields a morphism of functors

$$\text{Id} \rightarrow \Psi \circ \Phi. \quad (11)$$

Clearly, the kernel of the identity functor is $\Delta_* \mathcal{O}_Y$, where $\Delta : Y \rightarrow Y \times Y$ is the diagonal morphism. We claim that (11) is induced by a morphism of kernels

$$\Delta_* \mathcal{O}_Y \rightarrow Q \quad (12)$$

Let us construct (12).

Set

$$H := \mathbb{R}\mathcal{H}om_{X \times Y \times Y}(p_{12}^* P, p_{13}^* P) \otimes p_3^* \omega_Y[\dim Y] \in D_{coh}^b(X \times Y \times Y),$$

so that $Q = \mathbb{R}p_{23*} H$. The restriction of H to the diagonal $X \times Y \subset X \times Y \times Y$ is easy to describe:

$$\mathbb{L}(\text{id}_X \times \Delta)^* H = \mathbb{R}\mathcal{H}om_{X \times Y}(P, P) \otimes p_Y^* \omega_Y[\dim Y].$$

We therefore have

$$\mathbb{R}(\text{id}_X \times \Delta)^! H = \mathbb{R}\mathcal{H}om_{X \times Y}(P, P).$$

The identity automorphism of P gives a map

$$\mathcal{O}_{X \times Y} \rightarrow \mathbb{R}\mathcal{H}om_{X \times Y}(P, P) = \mathbb{R}(\text{id}_X \times \Delta)^! H,$$

which by adjunction induces a morphism

$$(\text{id}_X \times \Delta)_* \mathcal{O}_{X \times Y} \rightarrow H.$$

We finally give (12) as the composition

$$\Delta_* \mathcal{O}_Y \rightarrow \mathbb{R}p_{23*}(p_{23}^*(\Delta_* \mathcal{O}_Y)) = \mathbb{R}p_{23*}((\text{id}_X \times \Delta)_* \mathcal{O}_{X \times Y}) \rightarrow \mathbb{R}p_{23*} H = Q.$$

PROPOSITION 5.6. *The following conditions are equivalent:*

- (a) $\Phi : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$ is a fully faithful functor;
- (b) The adjunction homomorphism (11) is an isomorphism;
- (c) The morphism of kernels (12) is an isomorphism.

Proof. (a) \Leftrightarrow (b). By definition, Φ is fully faithful if and only if the map

$$\mathrm{Hom}_Y(F, G) \rightarrow \mathrm{Hom}_X(\Phi(F), \Phi(G)) = \mathrm{Hom}_Y(F, \Psi \circ \Phi(G))$$

is an isomorphism for all $F, G \in D_{comp}^b(Y)$. In other words,

$$\mathrm{Hom}_Y(F, \mathrm{Cone}(G \rightarrow \Psi \circ \Phi(G))) = 0.$$

By Corollary 5.4, this is true if and only if the map $G \rightarrow \Psi \circ \Phi(G)$ is an isomorphism for all $G \in D_{comp}^b(Y)$.

(c) \Rightarrow (b) is obvious.

(b) \Rightarrow (c). Consider

$$Q' := \mathrm{Cone}(\Delta_* \mathcal{O}_Y \rightarrow Q) \in D_{coh}^b(X \times Y).$$

The integral transform with respect to Q' is

$$\mathrm{Cone}(\mathrm{Id} \rightarrow \Psi \circ \Phi) : D_{comp}^b(Y) \rightarrow D_{coh}^b(X),$$

which vanishes by (b). Applying the integral transform to sky-scraper sheaves, we see that $Q' = 0$ by Lemma 5.3. \square

5.2.3. Let us now assume that $\mathrm{supp}(P)$ is proper over both X and Y . In this case, the situation is more symmetric: the formula (9) defines a functor

$$\Phi = \Phi^P : D_{coh}^b(Y) \rightarrow D_{coh}^b(X).$$

By Serre duality, $\Psi : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$ is the right adjoint of Φ . Both Φ and Ψ preserve the subcategory of compactly-supported objects.

LEMMA 5.7. *The functor $\Phi = \Phi^P : D_{coh}^b(Y) \rightarrow D_{coh}^b(X)$ is fully faithful if and only if its restriction to $D_{comp}^b(Y)$ is fully faithful (which in turn is equivalent to the other conditions of Proposition 5.6).*

Proof. The ‘only if’ direction is obvious. Let us verify the ‘if’ direction. It suffices to prove that the adjunction morphism (11) is an isomorphism of functors from $D_{coh}^b(Y)$ to itself. This follows because the morphism between their kernels (12) is an isomorphism by Proposition 5.6. \square

5.2.4. Finally, let us make some remarks about the bounded below version of these categories. For any complex manifold M , $D^+(\mathcal{O}_M)$ is the bounded below derived category of $\text{Mod}(\mathcal{O}_M)$. Consider the full subcategories

$$\begin{aligned} D_{coh}^+(M) &= \{\mathcal{F} \in D^+(\mathcal{O}_M) : H^i(\mathcal{F}) \in \text{Coh}(M) \text{ for all } i\} \\ D_{comp}^+(M) &= \{\mathcal{F} \in D^+(\mathcal{O}_M) : H^i(\mathcal{F}) \in \text{Coh}(M) \text{ and } \text{supp}(H^i(\mathcal{F})) \\ &\quad \text{is compact for all } i\}. \end{aligned}$$

As before, suppose the kernel $P \in D_{coh}^b(X \times Y)$ is such that $\text{supp}(P)$ is proper over Y . The same formula (9) defines an extension of Φ to a functor $D_{comp}^+(Y) \rightarrow D_{comp}^+(X)$, essentially because P has finite Tor-dimension. We still denote this extension by Φ . If in addition $\text{supp}(P)$ is proper over X , we obtain a functor $\Phi : D_{coh}^+(Y) \rightarrow D_{coh}^+(X)$.

We need the following simple observation.

LEMMA 5.8. *Suppose the functor $\Phi : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$ is fully faithful.*

(a) *For any $\mathcal{F} \in D_{comp}^b(Y)$, $\mathcal{G} \in D_{comp}^+(Y)$, the action of Φ on homomorphisms*

$$\text{Hom}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}(\Phi(\mathcal{F}), \Phi(\mathcal{G}))$$

is a bijection.

(b) *Assume in addition that $\text{supp}(P)$ is proper over X . Then for any $\mathcal{F} \in D_{coh}^b(Y)$, $\mathcal{G} \in D_{coh}^+(Y)$, the action of Φ on homomorphisms*

$$\text{Hom}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}(\Phi(\mathcal{F}), \Phi(\mathcal{G}))$$

is a bijection.

Proof. Consider the truncation triangle

$$\tau^{\leq N} \mathcal{G} \rightarrow \mathcal{G} \rightarrow \tau^{> N} \mathcal{G} \rightarrow \tau^{\leq N} \mathcal{G}[1].$$

For $N \gg 0$, we have

$$\text{Hom}(\mathcal{F}, \tau^{> N} \mathcal{G}) = \text{Hom}(\Phi(\mathcal{F}), \Phi(\tau^{> N} \mathcal{G})) = 0.$$

Therefore,

$$\text{Hom}(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F}, \tau^{\leq N} \mathcal{G}) = \text{Hom}(\Phi(\mathcal{F}), \Phi(\tau^{\leq N} \mathcal{G})) = \text{Hom}(\Phi(\mathcal{F}), \Phi(\mathcal{G})).$$

This proves (a). The proof of (b) is similar. \square

5.2.5. Similarly, one can consider the bounded above derived category $D^-(\mathcal{O}_M)$ and its full subcategories $D_{coh}^-(M)$ and $D_{comp}^-(M)$ (where M is a complex manifold). Lemma 5.8 remains true if we assume that $\mathcal{F} \in D_{comp}^-(Y)$ and $\mathcal{G} \in D_{coh}^-(Y)$ in parts (a) and (b), respectively.

5.3 Proof of Theorem 2.19

5.3.1. We now turn to the proof of Theorem 2.19. It is convenient to give an alternative description of $\mathcal{D}iff_{X \times Y/X}(P; P)$. Fix $P \in \text{Coh}(X \times Y)$.

Recall that

$$P_{\mathcal{D}/X} = \mathcal{D}_{X \times Y/X} \otimes P.$$

Applying differential operators in $\mathcal{D}_{X \times Y/X}$ to functions on $X \times Y$ that are constant along the fibers of p_Y , we can identify

$$\mathcal{D}_{X \times Y/X} = \mathcal{D}iff_{X \times Y/X}(\mathcal{O}_Y; \mathcal{O}_{X \times Y}) = p_Y^* \mathcal{D}_Y.$$

Recall that \mathcal{D}_Y has two actions of \mathcal{O}_Y (by left and right multiplication), and that $p_Y^* \mathcal{D}_Y = \mathcal{D}_{X \times Y/X}$ has two actions of $\mathcal{O}_{X \times Y}$. Both actions are used: the tensor product $p_Y^* \mathcal{D}_Y \otimes P$ is formed using the left action, but then we consider it as an $\mathcal{O}_{X \times Y}$ -module using the right action.

5.3.2. The two actions of \mathcal{O}_Y on \mathcal{D}_Y allow us to view \mathcal{D}_Y as a sheaf D of $\mathcal{O}_{Y \times Y}$ -modules. Let us agree that the first, respectively second, factor in $Y \times Y$ corresponds to the action of \mathcal{O}_Y on \mathcal{D}_Y by left, respectively right, multiplication. Note that D is not coherent: it is a union of coherent sheaves corresponding to differential operators of bounded order.

In the same way, we view the inverse image $p_Y^* \mathcal{D}_Y$ as the sheaf $p_{23}^* D$ on $X \times Y \times Y$. The tensor product $p_Y^* \mathcal{D}_Y \otimes P$ is interpreted as $p_{23}^* D \otimes p_{12}^* P$. We then consider $p_Y^* \mathcal{D}_Y \otimes P$ as an $\mathcal{O}_{X \times Y}$ -module using the right action of \mathcal{O}_Y on \mathcal{D}_Y ; in other words, we identify

$$P_{\mathcal{D}/X} = p_Y^* \mathcal{D}_Y \otimes P = p_{13*}(p_{23}^* D \otimes p_{12}^* P).$$

Of course, the above formulas make sense for arbitrary $P \in D_{coh}^b(X \times Y)$ that need not be concentrated in a single cohomological degree. Let us prove Theorem 2.19 in this generality.

PROPOSITION 5.9. *Assume that the support of $P \in D_{coh}^b(X \times Y)$ is proper over Y , and that the integral transform*

$$\Phi : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$$

with respect to P is fully faithful. Then

$$\mathbb{R}p_{Y*} \mathbb{R} \text{Hom}_{X \times Y}(P, p_{13*}(p_{23}^* D \otimes p_{12}^* P)) = \mathcal{O}_Y.$$

Proof. By Proposition 5.6, (12) is an isomorphism. It induces an identification

$$\Delta_*(\omega_Y^{-1})[-\dim(Y)] = \mathbb{R}p_{23*}(\mathbb{R} \text{Hom}(p_{13}^* P, p_{12}^* P)).$$

Note that compared to (12), we permuted the two copies of Y .

One can check that

$$D \otimes^{\mathbb{L}} \Delta_* \mathcal{O}_Y = \Delta_* \omega_Y[\dim(Y)]. \quad (13)$$

(This can be verified by an explicit local calculation using the Koszul resolution; see Remark 5.10 for a more conceptual explanation.) The projection formula gives

$$\Delta_* \mathcal{O}_Y = D \otimes^{\mathbb{L}} \Delta_*(\omega_Y^{-1})[-\dim(Y)] = \mathbb{R}p_{23*}(\mathbb{R}\mathcal{H}om(p_{13}^*P, p_{12}^*P) \otimes^{\mathbb{L}} p_{23}^*D).$$

It remains to take the direct image of both sides with respect to the second projection $Y \times Y \rightarrow Y$. \square

REMARK 5.10. Up to a twist, D equals the $\mathcal{D}_{Y \times Y}$ -module of δ -functions supported on the diagonal. More precisely, $\mathcal{D}_Y \otimes \omega_Y^{-1}$ has two structures of a left \mathcal{D}_Y -module, so $D \otimes p_2^* \omega_Y^{-1}$ is naturally a left $\mathcal{D}_{Y \times Y}$ -module. This \mathcal{D} -module is the direct image of the constant \mathcal{D}_Y -module \mathcal{O}_Y under Δ . From this point of view, (13) simply states that the restriction of the direct image to the diagonal is the constant \mathcal{D}_Y -module.

5.4 Extension of Fourier–Mukai transforms to ★-quantizations. In this section, we study Fourier–Mukai transforms on ★-quantizations, and prove Theorem B. A framework for working with Fourier–Mukai transforms on ★-quantizations was built in [KS10]. Our situation is somewhat simpler, because our ★-quantizations are parametrized by an Artinian ring R , rather than $\mathbb{C}[[\hbar]]$, as in [KS10]. This allows us to give more direct proofs.

5.4.1. First, let M be a complex manifold, and let \mathbb{M} be a ★-quantization of M . The category of $\mathcal{O}_{\mathbb{M}}$ -modules is an abelian category with enough injectives; denote its (bounded) derived category by $D^b(\mathcal{O}_{\mathbb{M}})$. Recall that $D_{coh}^b(\mathbb{M}) \subset D^b(\mathcal{O}_{\mathbb{M}})$ is the full subcategory of complexes with coherent cohomology. Let $D_{comp}^b(\mathbb{M}) \subset D_{coh}^b(\mathbb{M})$ be the full subcategory of complexes whose coherent cohomology has compact support.

5.4.2. The category $\text{Mod}(\mathcal{O}_M)$ is identified with the full subcategory of $\text{Mod}(\mathcal{O}_{\mathbb{M}})$ consisting of modules annihilated by \hbar . Geometrically, this identification is the direct image functor

$$i_* : \text{Mod}(\mathcal{O}_M) \rightarrow \text{Mod}(\mathcal{O}_{\mathbb{M}})$$

corresponding to the embedding $i : M \hookrightarrow \mathbb{M}$. The functor i_* is exact and preserves coherence and support. We therefore obtain a functor

$$i_* : D^b(\mathcal{O}_M) \rightarrow D^b(\mathcal{O}_{\mathbb{M}})$$

such that $i_*(D_{coh}^b(M)) \subset D_{coh}^b(\mathbb{M})$ and $i_*(D_{comp}^b(M)) \subset D_{comp}^b(\mathbb{M})$.

5.4.3. For any $\tilde{F} \in \text{Mod}(\mathcal{O}_{\mathbb{M}})$, the tensor product

$$\tilde{F} \otimes_R \mathbb{C} \in \text{Mod}(\mathcal{O}_{\mathbb{M}})$$

is annihilated by \hbar . We can therefore consider it as an object $i^*(\tilde{F}) \in \text{Mod}(\mathcal{O}_M)$.

This defines a right exact functor $i^* : \text{Mod}(\mathcal{O}_{\mathbb{M}}) \rightarrow \text{Mod}(\mathcal{O}_M)$. Since $\text{Mod}(\mathcal{O}_{\mathbb{M}})$ has enough R -flat objects, we have a derived functor

$$\mathbb{L}i^* : D^b(\mathcal{O}_{\mathbb{M}}) \rightarrow D^-(\mathcal{O}_M).$$

Similarly, the sheaf of homomorphisms

$$\text{Hom}_R(\mathbb{C}, \tilde{F}) \in \text{Mod}(\mathcal{O}_{\mathbb{M}})$$

can be viewed as an object $i^!(\tilde{F}) \in \text{Mod}(\mathcal{O}_M)$, this gives a left exact functor

$$i^! : \text{Mod}(\mathcal{O}_{\mathbb{M}}) \rightarrow \text{Mod}(\mathcal{O}_M).$$

Since $\text{Mod}(\mathcal{O}_{\mathbb{M}})$ has enough injectives, we have a derived functor

$$\mathbb{R}i^! : D^b(\mathcal{O}_{\mathbb{M}}) \rightarrow D^+(\mathcal{O}_M).$$

LEMMA 5.11. *The functors $\mathbb{L}i^*$ and $\mathbb{R}i^!$ are left and right adjoint to i_* , respectively. Both functors preserve the subcategories of coherent and compactly supported coherent sheaves:*

$$\mathbb{L}i^*(D_{coh}^b(\mathbb{M})) \subset D_{coh}^-(M) \quad \mathbb{L}i^*(D_{comp}^b(\mathbb{M})) \subset D_{comp}^-(M)$$

$$\mathbb{R}i^!(D_{coh}^b(\mathbb{M})) \subset D_{coh}^+(M) \quad \mathbb{R}i^!(D_{comp}^b(\mathbb{M})) \subset D_{comp}^+(M).$$

Proof. Let us check the properties of $\mathbb{L}i^*$ (the properties of $\mathbb{R}i^!$ are proved in the same way). The adjunction property is standard; we can construct an identification

$$\text{Hom}(\mathcal{F}, i_*\mathcal{G}) = \text{Hom}(\mathbb{L}i^*\mathcal{F}, \mathcal{G}) \quad (\mathcal{F} \in D^b(\mathcal{O}_{\mathbb{M}}), \mathcal{G} \in D^b(\mathcal{O}_M))$$

by using an R -flat resolution of \mathcal{F} and an injective resolution of \mathcal{G} . To show that $\mathbb{L}i^*\mathcal{F}$ preserves the subcategories $D_{coh}^b(\mathbb{M})$ and $D_{comp}^b(\mathbb{M})$, we note that $\mathbb{L}i^*\mathcal{F}$ can be computed using a free resolution of the R -module \mathbb{C} . □

5.4.4. Now let X and Y be complex manifolds. Let \mathbb{X} and \mathbb{Y} be their \star -quantizations. Set $\mathbb{Z} = \mathbb{X} \times \mathbb{Y}^{op}$; it is a \star -quantization of $Z = X \times Y$.

Fix $\tilde{P} \in D_{coh}^b(\mathbb{Z})$. Suppose that $\text{supp } \tilde{P}$ is proper over Y . Also, let us assume that \tilde{P} has finite Tor-dimension as an R -module. Since R is a local Artinian algebra, this is equivalent to boundedness of the derived tensor product

$$\tilde{P} \otimes_R^{\mathbb{L}} \mathbb{C}.$$

This derived tensor product is naturally an object in the derived category of \mathcal{O}_Z -modules; we can identify it with the derived restriction

$$P := \mathbb{L}i^*\tilde{P}.$$

Thus, we see that $P \in D_{coh}^b(Z)$.

Consider the integral transform functor

$$\Phi = \Phi_P : D^+(\mathcal{O}_Y) \rightarrow D^+(\mathcal{O}_X)$$

with the kernel P . We also have a Fourier–Mukai functor

$$\tilde{\Phi} = \Phi_{\tilde{P}} : D^b(\mathcal{O}_{\mathbb{Y}}) \rightarrow D^b(\mathcal{O}_{\mathbb{X}})$$

with the kernel \tilde{P} . The functor $\tilde{\Phi}$ is a version of the ‘convolutions of kernels’ from [KS10].

REMARK 5.12. Let us quickly review the definition of $\tilde{\Phi}$. Let $\text{Mod}(p_X^{-1}\mathcal{O}_{\mathbb{X}})$ be the category of $p_X^{-1}\mathcal{O}_{\mathbb{X}}$ -modules. Such modules can be defined as R -linear homomorphisms $\mathbb{X} \rightarrow p_{X*}(\mathbf{Sh}_R)$, where $p_{X*}(\mathbf{Sh}_R)$ is the stack that assigns to an open subset $U \subset X$ the category $\mathbf{Sh}_R(p_X^{-1}(U))$ of sheaves of R -modules on $p_X^{-1}(U) \subset Z$. We have a direct image functor

$$\mathbb{R}p_{X*} : D^b(p_X^{-1}\mathcal{O}_{\mathbb{X}}) \rightarrow D^b(\mathcal{O}_{\mathbb{X}}).$$

Similarly, we can consider the category of $p_Y^{-1}\mathcal{O}_{\mathbb{Y}}$ -modules and the inverse image functor

$$p_Y^* : D^b(\mathcal{O}_{\mathbb{Y}}) \rightarrow D^b(p_Y^{-1}\mathcal{O}_{\mathbb{Y}}).$$

Finally, derived tensor product with \tilde{P} makes sense as a functor

$$D^b(p_Y^{-1}\mathcal{O}_{\mathbb{Y}}) \rightarrow D^b(p_X^{-1}\mathcal{O}_{\mathbb{X}}).$$

We can then let $\tilde{\Phi}$ be the composition

$$D^b(\mathcal{O}_{\mathbb{Y}}) \rightarrow D^b(p_Y^{-1}\mathcal{O}_{\mathbb{Y}}) \rightarrow D^b(p_X^{-1}\mathcal{O}_{\mathbb{X}}) \rightarrow D^b(\mathcal{O}_{\mathbb{X}}).$$

LEMMA 5.13. *There is a natural commutative diagram of functors*

$$\begin{array}{ccccc} D^b(\mathcal{O}_Y) & \xrightarrow{i_*} & D^b(\mathcal{O}_{\mathbb{Y}}) & \xrightarrow{\mathbb{R}i^!} & D^+(\mathcal{O}_Y) \\ \downarrow \Phi & & \downarrow \tilde{\Phi} & & \downarrow \Phi \\ D^b(\mathcal{O}_X) & \xrightarrow{i_*} & D^b(\mathcal{O}_{\mathbb{X}}) & \xrightarrow{\mathbb{R}i^!} & D^+(\mathcal{O}_X). \end{array}$$

Proof. The fact that i_* agrees with the Fourier–Mukai transform follows directly from definitions. To see that $\mathbb{R}i^!$ has the same property, we can compute $\mathbb{R}i^!$ using a free resolution of the R -module \mathbb{C} -module. \square

Note that although we consider the integral transform $\tilde{\Phi}$ on \mathcal{O} -modules, some finiteness conditions are necessary in order to ensure that it has reasonable properties, see [KS10, Section 3.2].

LEMMA 5.14. (a) $\tilde{\Phi}(D_{comp}^b(\mathbb{Y})) \subset D_{comp}^b(\mathbb{X})$.

(b) Assume that $\text{supp}(P)$ is proper over X . Then $\tilde{\Phi}(D_{coh}^b(\mathbb{Y})) \subset D_{coh}^b(\mathbb{X})$.

Proof. This claim is essentially a version of [KS10, Theorem 3.2.1]. Note that the essential image $i_*D_{comp}^b(X)$ generates $D_{comp}^b(\mathbb{X})$, in the sense that the latter is the smallest triangulated category containing the former. This observation, together with Lemma 5.13, implies part (a). In the same way, part (b) follows from the observation that $i_*D_{coh}^b(X)$ generates $D_{coh}^b(\mathbb{X})$. \square

We are now ready to prove Theorem B. It is implied by the following statement, which is Theorem 2.16 with relaxed assumptions on \tilde{P} .

PROPOSITION 5.15. *As above, let \mathbb{X}, \mathbb{Y} be \star -quantizations of complex manifolds X and Y , and let $\tilde{P} \in D_{coh}^b(\mathbb{Z})$, where $\mathbb{Z} = \mathbb{X} \times \mathbb{Y}^{op}$. Suppose that \tilde{P} has finite Tor-dimension, and set $P = i^*\tilde{P} \in D^b(Z)$, $Z = X \times Y$. Finally, suppose that $\text{supp}(\tilde{P}) = \text{supp}(P)$ is proper over Y , and that the integral transform functor*

$$\Phi = \Phi_P : D_{comp}^b(Y) \rightarrow D_{comp}^b(X)$$

is fully faithful.

- (a) The integral transform functor $\tilde{\Phi} = \Phi_{\tilde{P}} : D_{comp}^b(\mathbb{Y}) \rightarrow D_{comp}^b(\mathbb{X})$ is fully faithful.
- (b) Let us also assume that $\text{supp}(\tilde{P})$ is proper over X . Then $\tilde{\Phi}$ is also fully faithful as a functor $D_{coh}^b(\mathbb{Y}) \rightarrow D_{coh}^b(\mathbb{X})$.
- (c) Let us assume that $\text{supp}(\tilde{P})$ is proper over X and that Φ provides an equivalence $D_{coh}^b(Y) \rightarrow D_{coh}^b(X)$. Then

$$\tilde{\Phi} : D_{coh}^b(\mathbb{Y}) \rightarrow D_{coh}^b(\mathbb{X})$$

is an equivalence.

Proof. For part (a), it suffices to show that $\tilde{\Phi}$ is fully faithful when restricted to the essential image $i_*(D_{comp}^b(Y))$. Indeed, given $\mathcal{F}, \mathcal{G} \in D_{comp}^b(Y)$, we have

$$\begin{aligned} \text{Hom}(i_*\mathcal{F}, i_*\mathcal{G}) &= \text{Hom}(\mathcal{F}, i^!i_*\mathcal{G}) = \text{Hom}(\Phi(\mathcal{F}), \Phi(i^!i_*\mathcal{G})) = \text{Hom}(\Phi(\mathcal{F}), i^!\tilde{\Phi}(i_*\mathcal{G})) \\ &= \text{Hom}(i_*\Phi(\mathcal{F}), \tilde{\Phi}(i_*\mathcal{G})) = \text{Hom}(\tilde{\Phi}(i_*\mathcal{F}), \tilde{\Phi}(i_*\mathcal{G})) \end{aligned}$$

by Lemmas 5.8, 5.11 and 5.13. The proof of part (b) is similar, except that we take $\mathcal{F}, \mathcal{G} \in D_{coh}^b(Y)$.

Let us prove part (c). By (b), the functor $\tilde{\Phi}$ is fully faithful; therefore, its essential image is a triangulated subcategory of $D_{coh}^b(\mathbb{X})$. It suffices to verify that the image contains $i_*D_{coh}^b(X)$. But this follows from Lemma 5.13. \square

6 Sheaves on \star -Quantizations Via Equivariant Stacks

In this paper, we work with many kinds of geometric objects (mostly sheaves with some additional structure) on \star -quantizations. In this chapter, we suggest a language for defining such objects in a uniform way. To a large extent, this chapter is independent from the rest of the paper; it may be considered an appendix.

6.1 Equivariant stacks over the stack of quantizations

6.1.1. Let M be a complex manifold. Denote by M_{an} the analytic site of M . Let \mathbf{QM} be the stack of local neutralized \star -quantizations of M . In other words, \mathbf{QM} is the stack of groupoids on M_{an} such that for an open subset $U \subset M$, the fiber $\mathbf{QM}(U)$ is the groupoid of neutralized \star -quantizations of U . There is a natural analytic topology on \mathbf{QM} : a family of maps

$$\left\{ \tilde{V}_i \rightarrow \tilde{U} \right\}_{i \in I}, \quad \tilde{V}_i \in \mathbf{QM}(V_i), \quad \tilde{U} \in \mathbf{QM}(U)$$

is a cover if and only if its image in M_{an} is a cover, that is $U = \cup_{i \in I} V_i$. We write \mathbf{QM}_{an} for the analytic site of \mathbf{QM} . The site \mathbf{QM}_{an} comes equipped with a natural sheaf $\mathcal{O}_{\mathbf{QM}}$ of R -algebras, where $\mathcal{O}_{\mathbf{QM}}(\tilde{U}) := \Gamma(U, \mathcal{O}_{\tilde{U}})$. Denote by $\mathcal{O}_{\mathbf{QM}}^\times$ the sheaf of invertible elements in $\mathcal{O}_{\mathbf{QM}}$; it is a sheaf of groups on \mathbf{QM}_{an} .

Let $I_{\mathbf{QM}}$ be the **inertia sheaf** on \mathbf{QM}_{an} : for $\tilde{U} \in \mathbf{QM}(U)$, $I_{\mathbf{QM}}(\tilde{U}) = \text{Aut}_{\mathbf{QM}(U)}(\tilde{U})$ is the group of automorphisms of \tilde{U} that act trivially on U . Equivalently, $\text{Aut}_{\mathbf{QM}(U)}(\tilde{U})$ is the group of automorphisms of the sheaf $\mathcal{O}_{\tilde{U}}$ that act trivially modulo \hbar .

DEFINITION 6.1. *Let G be a sheaf of groups on \mathbf{QM}_{an} . An **inertial action** of G on \mathbf{QM} is a group homomorphism $\rho : G \rightarrow I_{\mathbf{QM}}$ such that*

$$\rho(\alpha)^*(\beta) = \alpha^{-1}\beta\alpha \quad (\alpha, \beta \in G(\tilde{U}), \tilde{U} \in \mathbf{QM}).$$

EXAMPLE 6.2. There is a natural inertial action $\text{Ad} : \mathcal{O}_{\mathbf{QM}}^\times \rightarrow I_{\mathbf{QM}}$, defined as follows. Given $\tilde{U} \in \mathbf{QM}_{\text{an}}$ and $g \in \mathcal{O}_{\mathbf{QM}_{\text{an}}}^\times(\tilde{U}) = \Gamma(U, \mathcal{O}_{\tilde{U}}^\times)$, we let $\text{Ad}(g) \in \text{Aut}_{\mathbf{QM}(U)}(\tilde{U})$ be the automorphism of \tilde{U} that acts trivially on U and acts on $\mathcal{O}_{\tilde{U}}$ by conjugation

$$(\text{Ad}(g))^*(f) := g^{-1}fg, \quad f \in \mathcal{O}_{\tilde{U}}.$$

6.1.2. Let \mathcal{X} be a stack of categories over the site \mathbf{QM}_{an} . Such a stack is the same as a stack over M equipped with a 1-morphism $\iota_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{QM}$.

Since \mathcal{X} can be considered as a stack on two different sites, we have two categories of sections. Namely, given an open $U \subset M$ and $\tilde{U} \in \mathbf{QM}(U)$, we consider the categories $\mathcal{X}(\tilde{U})$ and $\mathcal{X}(U)$: the essential fibers of the projections of \mathcal{X} to \mathbf{QM} and M_{an} , respectively. The natural functor $\mathcal{X}(\tilde{U}) \rightarrow \mathcal{X}(U)$ is faithful, but not full. In particular, for $\alpha \in \mathcal{X}(\tilde{U})$, we have a natural embedding of automorphism groups:

$$\text{Aut}_{\mathcal{X}(\tilde{U})}(\alpha) \hookrightarrow \text{Aut}_{\mathcal{X}(U)}(\alpha) = \text{Aut}_{\mathcal{X}}(\alpha).$$

Explicitly, $\text{Aut}_{\mathcal{X}}(\alpha)$ is the group of automorphisms of α in the total category of \mathcal{X} (note that $\mathcal{X}(U) \subset \mathcal{X}$ is a full subcategory), while $\text{Aut}_{\mathcal{X}(\tilde{U})}(\alpha)$ consists of automorphisms that act trivially on \tilde{U} .

DEFINITION 6.3. Let G be a sheaf of groups on QM_{an} acting inertially on QM and let \mathcal{X} be a stack of categories on QM_{an} . A G -equivariant structure on \mathcal{X} is a collection of group homomorphisms

$$a_{\tilde{U},\alpha} : G(\tilde{U}) \rightarrow \text{Aut}_{\mathcal{X}}(\alpha) \quad (\tilde{U} \in QM, \alpha \in \mathcal{X}(\tilde{U}))$$

satisfying the following conditions:

- (1) For any $\alpha \in \mathcal{X}(\tilde{U})$, $\beta \in \mathcal{X}(\tilde{V})$, any homomorphism $\phi : \alpha \rightarrow \beta$ in \mathcal{X} , and any $g \in G(\tilde{V})$, the diagram

$$\begin{array}{ccc} \alpha & \xrightarrow{\phi} & \beta \\ a_{\tilde{U},\alpha}(g|_{\tilde{U}}) \downarrow & & \downarrow a_{\tilde{V},\beta}(g) \\ \alpha & \xrightarrow{\phi} & \beta \end{array}$$

commutes.

- (2) The composition $G(\tilde{U}) \xrightarrow{a_{\tilde{U},\alpha}} \text{Aut}_{\mathcal{X}}(\alpha) \rightarrow \text{Aut}_{QM}(\tilde{U})$ coincides with the inertial action of G .

REMARK 6.4. Suppose \mathcal{X} is a stack of groupoids. Then Definition 6.3 is naturally reformulated in terms of sheaves on \mathcal{X} . Recall that the inertia sheaf $I_{\mathcal{X}}$ assigns to $\alpha \in \mathcal{X}$ its automorphism group $\text{Aut}_{\mathcal{X}}(\alpha)$. It is obviously a presheaf on \mathcal{X} , which is a sheaf with respect to the natural topology on \mathcal{X} (a family of maps in \mathcal{X} is a cover if and only if its image is a cover in M_{an}). Suppose we are given an inertial action $\rho : G \rightarrow I_{QM}$ of a sheaf of groups G on QM . Then a G -equivariant structure on \mathcal{X} is a homomorphism $a : \iota_{\mathcal{X}}^*G \rightarrow I_{\mathcal{X}}$ of sheaves of groups on \mathcal{X} such that the composition

$$\iota_{\mathcal{X}}^*G \xrightarrow{a} I_{\mathcal{X}} \rightarrow \iota_{\mathcal{X}}^*I_{QM}$$

coincides with $\iota_{\mathcal{X}}^*(\rho) : \iota_{\mathcal{X}}^*G \rightarrow \iota_{\mathcal{X}}^*I_{QM}$.

6.1.3. Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of stacks on QM . If both \mathcal{X} and \mathcal{Y} are equipped with G -equivariant structures, it is clear what it means for F to be G -equivariant.

Now suppose F is faithful. If \mathcal{Y} has a G -equivariant structure, there is at most one equivariant structure on \mathcal{X} that makes F an equivariant 1-morphism. If such a structure exists, we say that \mathcal{X} is a G -invariant. Explicitly, \mathcal{X} is G -invariant if and only if for any $\tilde{U} \in QM$ and any $\alpha \in \mathcal{X}(\tilde{U})$, the image of $\text{Aut}_{\mathcal{X}}(\alpha)$ in $\text{Aut}_{\mathcal{Y}}(F(\alpha))$ contains the image of $G(\tilde{U})$. In particular, if F is fully faithful (so \mathcal{X} is a full substack of \mathcal{Y}), \mathcal{X} is automatically G -invariant.

6.1.4. We are primarily interested in $\mathcal{O}_{QM}^{\times}$ -equivariant stacks, where the inertial action of $\mathcal{O}_{QM}^{\times}$ on QM is the adjoint action from Example 6.2. (Sometimes it is natural to consider stacks equivariant under a ‘congruence subgroup’ of $\mathcal{O}_{QM}^{\times}$, see below.) Most relevant examples arise when we consider stacks of a kind of geometric objects,

such as \mathcal{O} -modules with some additional structure, on neutralized \star -quantizations of open subsets of M . We conclude this section with some examples.

EXAMPLE 6.5. Let $\mathcal{X} = \underline{\mathbf{Mod}}(\mathcal{O}_{QM})$ be the stack of left \mathcal{O}_{QM} -modules. More explicitly, $\mathcal{X}(\tilde{U})$ is the category of left $\mathcal{O}_{\tilde{U}}$ -modules ($\tilde{U} \in QM$). The stack \mathcal{X} is naturally \mathcal{O}_{QM}^\times -equivariant: for a sheaf \tilde{P} of left $\mathcal{O}_{\tilde{U}}$ -modules, we let $g \in \Gamma(U, \mathcal{O}_{\tilde{U}}^\times)$ act on the pair (\tilde{U}, \tilde{P}) by the automorphism $(\text{Ad}(g), \tilde{m}(g))$, where $\tilde{m}(g) : \tilde{P} \rightarrow \tilde{P}$ is the left multiplication by g . Condition (1) in the definition of \mathcal{O}_{QM}^\times -equivariance is satisfied tautologically. To verify condition (2), we must check that for any $f \in \Gamma(U, \mathcal{O}_{\tilde{U}})$ and any $s \in \Gamma(U, \tilde{P})$, we have $\tilde{m}(g)(fs) = (\text{Ad}(g^{-1})f)(\tilde{m}(g)s)$. But by definition,

$$(\text{Ad}(g^{-1})f)(\tilde{m}(g)s) = (gfg^{-1})(gs) = gfs = \tilde{m}(g)(fs).$$

EXAMPLE 6.6. Let Σ be a local property of sheaves of \mathcal{O} -modules on neutralized \star -quantizations. For instance, Σ could be one of the following properties of an \mathcal{O} -module \tilde{P} :

- \tilde{P} is coherent;
- \tilde{P} is R -flat;
- \tilde{P} is R -flat and its reduction $\tilde{P}/\hbar\tilde{P}$ is locally free, or tangible.

Denote by

$$\underline{\mathbf{Mod}}(\mathcal{O}_{QM})^\Sigma \subset \underline{\mathbf{Mod}}(\mathcal{O}_{QM})$$

the substack of \mathcal{O} -modules having property Σ . It is a full substack, so it is automatically \mathcal{O}_{QM}^\times -invariant.

EXAMPLE 6.7. There is also a relative version of these examples. Fix a neutralized \star -quantization \tilde{N} of a complex manifold N , and let $\mathcal{X} = \underline{\mathbf{Mod}}(\mathcal{O}_{QM})/\tilde{N}$ be the stack of left $\mathcal{O}_{QM \times \tilde{N}}$ -modules: its objects are pairs (\tilde{U}, \tilde{P}) , where $\tilde{U} \in QM$, and \tilde{P} is a left $\mathcal{O}_{\tilde{U} \times \tilde{N}}$ -module. Similarly, we can consider $\mathcal{O}_{\tilde{U} \times \tilde{N}}$ -module having some local property Σ .

EXAMPLE 6.8. Another example is given in Section 4.3.4. Let $p_M : Z \rightarrow M$ be a submersive morphism of complex manifolds. Given a neutralized \star -quantization \tilde{U} of an open subset $U \subset M$, let $\mathcal{X}(\tilde{U})$ be the category of triples $(P, \tilde{P}, \tilde{\xi})$, where P is a tangible sheaf on $p_M^{-1}(U)$, \tilde{P} is its \star -deformation, and

$$\tilde{\xi} : p_M^{-1}\mathcal{O}_{\tilde{U}} \otimes_R \tilde{P} \rightarrow \tilde{P}$$

is a \star -local action. The morphisms between $(P, \tilde{P}, \tilde{\xi})$ and $(P', \tilde{P}', \tilde{\xi}')$ are R -linear \star -local maps $\tilde{P} \rightarrow \tilde{P}'$ that commute with the action. The stack $\mathcal{X}(\tilde{U})$ is naturally \mathcal{O}_{QM}^\times -equivariant.

EXAMPLE 6.9. Examples also arise from looking at modules over coalgebras corresponding to quantizations (as in Section 3.4). Recall that a neutralized \star -quantization \tilde{U} of an open subset $U \subset M$ gives rise to a coalgebra $\mathcal{A}_{\tilde{U}} = \text{Diff}(\mathcal{O}_{\tilde{U}}; \mathcal{O}_U \otimes_{\mathbb{C}} R)$ in the category of left $\mathcal{D}_U \otimes_{\mathbb{C}} R$ -modules. We then consider the category $\text{Comod}(\mathcal{A}_{\tilde{U}})$ of $\mathcal{A}_{\tilde{U}}$ -comodules in the category of right $\mathcal{D}_U \otimes_{\mathbb{C}} R$ -modules. This provides a stack \mathcal{X} over QM_{an} defined by $\mathcal{X}(\tilde{U}) = \text{Comod}(\mathcal{A}_{\tilde{U}})$. We claim that the stack is naturally $\mathcal{O}_{\text{QM}}^{\times}$ -equivariant.

Such equivariant structure is provided by Lemma 4.6. Indeed, let us also consider the stack \mathcal{Y} given by

$$\mathcal{Y}(\tilde{U}) = \text{Mod}(\mathcal{O}_{\tilde{U}} \otimes_{\mathbb{C}} \mathcal{D}_U^{op}).$$

This stack has a natural $\mathcal{O}_{\text{QM}}^{\times}$ -equivariant structure. (Note that \mathcal{Y} is a stack of modules over a sheaf of algebras on QM_{an} ; the sheaf of algebras contains \mathcal{O}_{QM} , which allows us to define the equivariant structure.) The functor (6) provides a 1-morphism $\mathcal{X} \rightarrow \mathcal{Y}$, which is fully faithful by Lemma 4.6. Therefore, \mathcal{X} is $\mathcal{O}_{\text{QM}}^{\times}$ -invariant.

Note that by Proposition 3.32, the stack \mathcal{X} is actually identified with the stack of \mathcal{O}_{QM} -modules $\underline{\text{Mod}}(\mathcal{O}_{\text{QM}})$ from Example 6.5.

EXAMPLE 6.10. It is more interesting to consider a relative version of Example 6.9. Let $p_M : Z \rightarrow M$ be a submersive morphism of complex manifolds. Given a neutralized \star -quantization \tilde{U} of an open subset $U \subset M$, the pullback $p_M^* \mathcal{A}_{\tilde{U}}$ is a coalgebra in the category of left \mathcal{D} -modules on $p_M^{-1}(U) \subset Z$. Set $\mathcal{X}(\tilde{U}) = \text{Comod}(p_M^* \mathcal{A}_{\tilde{U}})$. This defines a stack over QM .

The stack \mathcal{X} is naturally $\mathcal{O}_{\text{QM}}^{\times}$ -equivariant. Indeed, Lemma 4.11 gives a full embedding $\mathcal{X} \rightarrow \mathcal{Y}$ into the stack \mathcal{Y} given by

$$\mathcal{Y}(\tilde{U}) = \text{Mod}(p_M^{-1} \mathcal{O}_{\tilde{U}} \otimes_{\mathbb{C}} \mathcal{D}_{p_M^{-1}(U)}^{op}).$$

The stack \mathcal{Y} is naturally $\mathcal{O}_{\text{QM}}^{\times}$ -equivariant, and \mathcal{X} is an invariant substack.

The stack \mathcal{X} contains the stack of Example 6.8 as a full substack. The embedding between the two stacks is given by Lemma 4.24, which associates a $p_M^* \mathcal{A}_{\tilde{U}}$ -comodule with a \star -deformation of a tangible sheaf equipped with a local action of $p_M^{-1}(\mathcal{O}_{\tilde{U}})$.

Note also that, as we explain in Section 3.4.6, the category of $p_M^* \mathcal{A}_{\tilde{U}}$ -comodules can be identified with the category of right $\mathcal{D}_{\tilde{V}/\tilde{U}}$ -modules, where \tilde{V} is any neutralized \star -quantization of $p_M^{-1}(U) \subset Z$ such that p_M lifts to a map $\tilde{V} \rightarrow \tilde{U}$. Note that such \tilde{V} does not necessarily exist, and if it does, there may be more than one choice, even though the category of right $\mathcal{D}_{\tilde{V}/\tilde{U}}$ -modules does not depend on the choice. At least in the case $Z = M \times N$, it is natural to take $\tilde{V} = \tilde{U} \times N$, and then we get an equivalent description of the stack \mathcal{X} in the language of relative \mathcal{D} -modules on \star -quantizations.

REMARK 6.11. In all of the above examples, the $\mathcal{O}_{\widetilde{M}}^\times$ -equivariant structure on a stack \mathcal{X} actually comes from an action of $\mathcal{O}_{\widetilde{M}}$ on \mathcal{X} , in an appropriate sense. We leave the details of the definition to the reader.

6.2 Sections of equivariant stacks over quantizations

6.2.1. Another source of $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant stacks are \star -quantizations of M . Let \mathbb{M} be a \star -quantization of M , or more generally a \star -stack on M (see Definition 2.5). By definition, \mathbb{M} is a stack of algebroids over M and for any open subset $U \subset M$ and any $\alpha \in \mathbb{M}(U)$, we get a natural neutralized \star -quantization $\widetilde{U}_\alpha \in \widetilde{\mathbb{M}}(U)$:

$$\widetilde{U}_\alpha = (U, \text{Hom}_{\mathbb{M}}(\alpha, \alpha)). \quad (14)$$

The assignment

$$\mathbb{M}(U) \ni \alpha \mapsto \widetilde{U}_\alpha \in \mathbb{Q}M(U)$$

is a morphism $\mathbb{M} \rightarrow \mathbb{Q}M$ of stacks over M , so we can view \mathbb{M} as a stack on $\mathbb{Q}M_{\text{an}}$. It is easy to see that \mathbb{M} is naturally $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant. Indeed, if $U \subset M$ is open and $\alpha \in \mathbb{M}(U)$, then

$$\widetilde{U}_\alpha = (U, \text{End}_{\mathbb{M}}(\alpha)) \quad \text{and} \quad \mathcal{O}_{\mathbb{Q}M}^\times(\widetilde{U}_\alpha) = \Gamma(U, \mathcal{O}_{\widetilde{U}_\alpha}^\times) = \text{Aut}_{\mathbb{M}(U)}(\alpha)$$

acts on α tautologically. It is clear that the action turns \mathbb{M} into an $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant stack.

Similarly we can consider the gerbe \mathbb{M}^\times over M . Thus, for every open subset $U \subset M$, we let $\mathbb{M}^\times(U)$ be the groupoid corresponding to the category $\mathbb{M}(U)$: it has the same objects, and its morphisms are isomorphisms of $\mathbb{M}(U)$. We can view \mathbb{M}^\times as a stack of groupoids on $\mathbb{Q}M_{\text{an}}$. (Note that \mathbb{M}^\times is *not* a gerbe over $\mathbb{Q}M_{\text{an}}$ if $\dim(M) > 1$.) The same construction provides a natural $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant structure on \mathbb{M}^\times . In other words, \mathbb{M}^\times is an $\mathcal{O}_{\mathbb{Q}M}^\times$ -invariant substack of \mathbb{M} .

6.2.2. We now turn to definition of natural objects (such as sheaves of modules with additional structure) on \star -quantizations, or general \star -stacks. Formally, the construction applies to objects that form an $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant stack over $\mathbb{Q}M_{\text{an}}$, such as those considered in Section 6.1. Given an $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant stack, we define its sections over arbitrary \star -stacks using the following as follows.

DEFINITION 6.12. *Let \mathcal{X} be an $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant stack, and let \mathbb{M} be a \star -stack on M . Set $\mathcal{X}(\mathbb{M})$ (the category of **sections** of \mathcal{X} over \mathbb{M}) to be the category of $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant 1-morphisms $\mathbb{M}^\times \rightarrow \mathcal{X}$ of stacks. Here we view \mathbb{M}^\times with the equivariant structure described in 6.2.1.*

6.2.3. Applying Definition 6.12 to equivariant stacks from examples in Section 6.1, we define \mathcal{O} -modules on \mathbb{M} , coherent $\mathcal{O}_{\mathbb{M}}$ -modules, $\mathcal{O}_M - \mathcal{D}_{\mathbb{M}}$ -bimodules, and so on. This provides a ‘uniform’ reformulation of Definitions 2.12, 2.13, 4.9, and 4.12. The equivalence of the two styles of definitions is the subject of Section 6.3.

Note that if \mathbb{M} is neutral, Definition 6.12 is consistent. This follows from simple abstract observations about stacks (Proposition 6.13).

Let \mathcal{X} be a gerbe over M and let \mathcal{Y} be an arbitrary stack. Suppose \mathcal{X} admits a global neutralization $\alpha \in \mathcal{X}(M)$. Any 1-morphism of stacks $F : \mathcal{X} \rightarrow \mathcal{Y}$ yields an object $\beta = F(\alpha) \in \mathcal{Y}(M)$ and a morphism of sheaves $\phi : \text{Aut}_{\mathcal{X}}(\alpha) \rightarrow \text{Aut}_{\mathcal{Y}}(\beta)$.

PROPOSITION 6.13. (a) *The above correspondence provides an equivalence between the category of 1-morphisms $F : \mathcal{X} \rightarrow \mathcal{Y}$ and the category of pairs*

$$(\beta \in \mathcal{Y}(M), \phi : \text{Aut}_{\mathcal{X}}(\alpha) \rightarrow \text{Aut}_{\mathcal{Y}}(\beta)).$$

(b) *Suppose in addition that \mathcal{X} and \mathcal{Y} are G -equivariant stacks on $\mathbb{Q}M$ for a sheaf of groups G that acts inertially on $\mathbb{Q}M$. Assume that the action of G on \mathcal{X} is **simple transitive**: for any $\tilde{U} \in \mathbb{Q}M$ and any $\gamma \in \mathcal{X}(\tilde{U})$, the map $G(\tilde{U}) \rightarrow \text{Aut}_{\mathcal{X}}(\gamma)$ is an isomorphism. Then the correspondence*

$$F \mapsto F(\alpha)$$

yields an equivalence between the category of G -equivariant 1-morphisms $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{Y}(\tilde{M}_\alpha)$. Here $\tilde{M}_\alpha \in \mathbb{Q}M(M)$ is the image of $\alpha \in \mathcal{X}(M)$.

Proof. (a) It is easy to see that the correspondence is faithful. Indeed, we need to show that given two 1-morphisms $F, F' : \mathcal{X} \rightarrow \mathcal{Y}$ and two 2-morphisms $f, f' : F \rightarrow F'$ such that $f(\alpha) = f'(\alpha) : F(\alpha) \rightarrow F'(\alpha)$, we have $f(\gamma) = f'(\gamma) : F(\gamma) \rightarrow F'(\gamma)$ for any $\gamma \in \mathcal{X}$. This follows because γ is locally isomorphic to α .

To see that the correspondence is full, we need to show that for two such 1-morphisms F and F' , any morphism $f : F(\alpha) \rightarrow F'(\alpha)$ that agrees with the action ϕ extends to a morphism of functors. To do this, we need to construct

$$f(\gamma) : F(\gamma) \rightarrow F'(\gamma)$$

for all $\gamma \in \mathcal{X}$. Again, this can be easily done locally: an isomorphism $\alpha|_U \simeq \gamma|_U$ for an open set U allows us to view $f|_U$ as a map

$$f_{\gamma,U} : F(\gamma)|_U \rightarrow F'(\gamma)|_U.$$

Since f agrees with ϕ , the map $f_{\gamma,U}$ does not depend on the choice of isomorphism $\gamma|_U \simeq \alpha|_U$; this allows us to glue the local maps $f_{\gamma,U}$ into a map $f(\gamma)$.

Finally, let us show the correspondence is essentially surjective. Given $\beta \in \mathcal{Y}(M)$ and $\phi : \text{Aut}_{\mathcal{X}}(\alpha) \rightarrow \text{Aut}_{\mathcal{Y}}(\beta)$, let us construct the corresponding functor F . Given $\gamma \in \mathcal{X}(U)$ for open U , we choose an open cover $U = \bigcup U_i$ together with isomorphisms

$$s_i : \alpha|_{U_i} \rightarrow \gamma|_{U_i}.$$

We then obtain a descent datum for \mathcal{Y} : on sets U_i , we have objects $\beta|_{U_i} \in \mathcal{Y}(U_i)$, and on intersections $U_i \cap U_j$, we have isomorphisms $\phi(s_j^{-1}s_i)$ between the restrictions of these objects. Since \mathcal{Y} is a stack, the descent datum gives rise to an object $\delta \in \mathcal{Y}(U)$,

unique up to a unique isomorphism. We set $F(\gamma) = \delta$. It is easy to see that F defined in this way is indeed a 1-morphism of stacks.

Part **(b)** follows from part **(a)**. Indeed, consider a sheaf G_α on M that assigns to an open set $U \subset M$ the group $G(\tilde{U}_\alpha)$, where $\tilde{U}_\alpha \in \mathcal{Q}M(U)$ is the image of $\alpha|_U$ (which is an open subset of \widetilde{M}_α). The G -equivariant structure on \mathcal{X} defines a morphism $a_\alpha : G_\alpha \rightarrow \text{Aut}_{\mathcal{X}}(\alpha)$; similarly, the G -equivariant structure on \mathcal{Y} defines a morphism $a_\beta : G_\beta \rightarrow \text{Aut}_{\mathcal{Y}}(\beta)$ for any $\beta \in \mathcal{Y}(\widetilde{M}_\alpha)$. A pair (β, ϕ) as in part **(a)** corresponds to a G -equivariant 1-morphism $\mathcal{X} \rightarrow \mathcal{Y}$ if and only if $\phi a_\alpha = a_\beta$. Since the action of G on X is simple transitive, a_α is an isomorphism, so if β is given, ϕ is uniquely determined. \square

COROLLARY 6.14. *Suppose \mathbb{M} admits a global neutralization $\alpha \in \mathbb{M}(M)$. Let $\widetilde{M}_\alpha \in \mathcal{Q}M$ be the corresponding global neutralized quantization, as in (14). Then for any $\mathcal{O}_{\mathcal{Q}M}^\times$ -equivariant stack \mathcal{X} , the categories $\mathcal{X}(\mathbb{M})$ and $\mathcal{X}(\widetilde{M}_\alpha)$ are naturally equivalent. The equivalence is given by*

$$F \mapsto F(\alpha) \in \mathcal{X}(\widetilde{M}_\alpha) \quad (F \in \mathcal{X}(\mathbb{M}));$$

recall that $F \in \mathcal{X}(\mathbb{M})$ is an $\mathcal{O}_{\mathcal{Q}M}^\times$ -equivariant 1-morphism $\mathbb{M}^\times \rightarrow \mathcal{X}$.

Proof. Follows from Proposition 6.13(b), because \mathbb{M}^\times is a gerbe on which $\mathcal{O}_{\mathcal{Q}M}^\times$ acts simply transitively. \square

6.2.4. Let us also mention a version of Definition 6.12 that is specifically adapted to \star -quantizations of M , rather than \star -stacks. Recall (Definition 2.9) that a \star -quantization of M is a \star -stack \mathbb{M} whose reduction to \mathbb{C} is neutralized. This additional structure allows us to define a wider class of objects on \mathbb{M} . From the point of view of equivariant stacks, we can define $\mathcal{X}(\mathbb{M})$ for a stack \mathcal{X} that is equivariant with respect to a group smaller than $\mathcal{O}_{\mathcal{Q}M}^\times$.

Let \mathbb{M} be a \star -quantization of M . For any open set $U \subset M$ and any $\alpha \in \mathbb{M}(U)$, let α' be the reduction of α to \mathbb{C} ; then $\alpha' \in \mathbb{M}'(U)$, where \mathbb{M}' is the reduction of \mathbb{M} to \mathbb{C} . Recall that \mathbb{M}' is equipped with a global neutralization $\beta \in \mathbb{M}'(M)$.

We define $\mathbb{M}^{(1)}$ to be the stack of collections (α, s') , where $\alpha \in \mathbb{M}$, and s' is the trivialization of α' . That is, s' is an isomorphism between α' and the restriction $\beta|_U$. Clearly, $\mathbb{M}^{(1)}$ is a stack of groupoids on M .

We have a natural faithful functor $\mathbb{M}^{(1)} \rightarrow \mathbb{M}^\times$; in particular, it allows us to view $\mathbb{M}^{(1)}$ as a stack over $\mathcal{Q}M$. Note that $\mathbb{M}^{(1)}$ is not invariant with respect to the canonical action of $\mathcal{O}_{\mathcal{Q}M}^\times$ on \mathbb{M}^\times (described in 6.2.1). However, $\mathbb{M}^{(1)}$ is invariant for the ‘congruence’ subgroup $\mathcal{O}_{\mathcal{Q}M}^{(1)} \subset \mathcal{O}_{\mathcal{Q}M}^\times$ defined by

$$\mathcal{O}_{\mathcal{Q}M}^{(1)}(\tilde{U}) := \{\tilde{f} \in \Gamma(U, \mathcal{O}_{\tilde{U}}^\times) \mid \tilde{f} = 1 \pmod{\hbar}\} \quad (\tilde{U} \in \mathcal{Q}M).$$

Thus $\mathbb{M}^{(1)}$ is naturally an $\mathcal{O}_{\mathcal{Q}M}^{(1)}$ -equivariant stack. It is easy to see that $\mathbb{M}^{(1)}$ is a gerbe over M on which $\mathcal{O}_{\mathcal{Q}M}^{(1)}$ acts simply and transitively.

REMARK 6.15. Let \mathbb{M} be a \star -stack. To turn \mathbb{M} into a \star -quantization, we need to choose a trivialization of the reduction of \mathbb{M} to \mathbb{C} . Alternatively, one can view this choice as a reduction of the ‘structure group’ of \mathbb{M}^\times from \mathcal{O}_{QM}^\times to $\mathcal{O}_{QM}^{(1)}$. Here we essentially identify \star -stacks with \mathcal{O}_{QM}^\times -gerbes and \star -quantizations with $\mathcal{O}_{QM}^{(1)}$ -gerbes over QM . We come back to this point of view in Section 6.4.

We can now modify Definition 6.12.

DEFINITION 6.16. *Let \mathcal{X} be an $\mathcal{O}_{QM}^{(1)}$ -equivariant stack, and let \mathbb{M} be a \star -quantization of M . Set $\mathcal{X}(\mathbb{M})$ (the category of **sections** of \mathcal{X} over \mathbb{M}) to be the category of $\mathcal{O}_{QM}^{(1)}$ -equivariant 1-morphisms $\mathbb{M}^{(1)} \rightarrow \mathcal{X}$ of stacks.*

6.2.5. Definition 6.16 agrees with other situations when the category of sections is defined. First of all, suppose the \star -quantization \mathbb{M} is neutralized. In other words, we are given $\alpha \in \mathbb{M}(M)$ and an isomorphism between the reduction α' of α to \mathbb{C} and the standard neutralization of \mathbb{M}' (which is part of the \star -quantization structure). Then $\mathcal{X}(\mathbb{M})$ is naturally equivalent to $\mathcal{X}(M_\alpha)$; the statement and its proof are completely parallel to Corollary 6.14.

Also, Definitions 6.16 and 6.12 agree whenever they both make sense. (We give the statement in concrete terms, but the proof is an abstract stack argument. The abstract formulation of the statement is left to the reader.)

LEMMA 6.17. *Let \mathbb{M} be a \star -quantization of M , and let \mathcal{X} be an \mathcal{O}_{QM}^\times -equivariant stack. Given an \mathcal{O}_{QM}^\times -equivariant 1-morphism $\mathbb{M}^\times \rightarrow \mathcal{X}$, the composition*

$$\mathbb{M}^{(1)} \rightarrow \mathbb{M}^\times \rightarrow \mathcal{X}$$

is clearly $\mathcal{O}_{QM}^{(1)}$ -equivariant. We claim that this provides an equivalence between the category of \mathcal{O}_{QM}^\times -equivariant 1-morphisms $\mathbb{M}^\times \rightarrow \mathcal{X}$ and the category of $\mathcal{O}_{QM}^{(1)}$ -equivariant 1-morphisms $\mathbb{M}^{(1)} \rightarrow \mathcal{X}$.

Proof. Given two stacks $\mathcal{Y}_1, \mathcal{Y}_2$ over M , we can consider the stack of 1-morphisms $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$, whose category of global sections over open $U \subset M$ is the category of 1-morphisms $\mathcal{Y}_1|_U \rightarrow \mathcal{Y}_2|_U$. Similarly, we can consider stacks of equivariant 1-morphisms between equivariant stacks. We can thus restate the lemma using stacks: we need to show the natural 1-morphism between the stack of \mathcal{O}_{QM}^\times -equivariant 1-morphisms $\mathbb{M}^\times \rightarrow \mathcal{X}$ and the stack of $\mathcal{O}_{QM}^{(1)}$ -equivariant 1-morphisms $\mathbb{M}^{(1)} \rightarrow \mathcal{X}$ is an equivalence.

In this formulation, the claim becomes local on M , so we can assume without losing generality that \mathbb{M} is a neutralized \star -quantization. Then the claim follows from Proposition 6.13(b) (cf. Corollary 6.14.) □

EXAMPLES 6.18. Obviously, any \mathcal{O}_{QM}^\times -equivariant stack \mathcal{X} , such as those considered in Section 6.1, is also equivariant for $\mathcal{O}_{QM}^{(1)}$ (and any other subgroup of \mathcal{O}_{QM}^\times).

We obtain other examples of $\mathcal{O}_{QM}^{(1)}$ -equivariant stacks by looking at objects whose reduction modulo \hbar is fixed. For instance, let us fix an \mathcal{O}_M -module P . Consider a stack \mathcal{X} over QM such that for any open $U \subset M$ and any $\tilde{U} \in QM(U)$, the category $\mathcal{X}(U)$ consists of pairs (\tilde{P}, i) , where \tilde{P} is a R -flat $\mathcal{O}_{\tilde{U}}$ -module and

$$i : \tilde{P}/\hbar\tilde{P} \rightarrow P|_U$$

is an isomorphism of \mathcal{O}_U -modules. It is not hard to see that for any \star -quantization \mathbb{M} of M , the category of sections $\mathcal{X}(\mathbb{M})$ is the category of deformations of P to a \mathcal{O}_M -module. Similar examples can be constructed for other stacks from Section 6.1.

6.3 Equivariance and R -linearity. Sheaves on quantizations can also be described using representations of stacks of algebroids (as in for instance [KS10]). Let us compare this approach to the framework of inertial actions of \mathcal{O}_{QM}^\times .

For example, Definition 2.12 defines \mathcal{O} -modules on a \star -stack \mathbb{M} to be **representations** of \mathbb{M} , that is, 1-morphisms $\tilde{F} : \mathbb{M} \rightarrow \mathbf{Sh}_R$ of stacks of R -linear categories over M . Here \mathbf{Sh}_R be the stack of sheaves of R -modules on M : to an open $U \subset M$, it assigns the category $\mathbf{Sh}_R(U)$ of sheaves of R -modules on U .

6.3.1. Let $\tilde{F} : \mathbb{M} \rightarrow \mathbf{Sh}_R$ be a representation of \mathbb{M} . For any open $U \subset M$ and any $\alpha \in \mathbb{M}(U)$, $\tilde{F}(\alpha)$ is a sheaf of R -modules on U . Since \tilde{F} is a morphism of stacks of R -linear categories, the action of R on $\tilde{F}(\alpha)$ naturally extends to an action of the sheaf of R -algebras

$$\mathcal{O}_{\tilde{U}_\alpha} = \mathit{End}_{\mathbb{M}}(\alpha).$$

We denote the resulting $\mathcal{O}_{\tilde{U}_\alpha}$ -module by $\tilde{F}_{(\alpha)}$. In particular, if $\alpha \in \mathbb{M}(M)$ is a global section, then we get a functor

$$\begin{aligned} (\text{representations of } \mathbb{M}) &\longrightarrow (\mathcal{O}_{\tilde{M}_\alpha}\text{-modules}) \\ \tilde{F} &\longrightarrow \tilde{F}_{(\alpha)}, \end{aligned}$$

which is an equivalence. For general \mathbb{M} , a representation of \mathbb{M} can be explicitly described as a collection of $\mathcal{O}_{\tilde{U}_\alpha}$ -modules over neutralized open patches together with gluing conditions as in [PS04].

One can check that the two approaches to \mathcal{O}_M -modules lead to the same result.

PROPOSITION 6.19. *Let \mathbb{M} be a \star -stack over M . There is an equivalence between the category of representations of \mathbb{M} and the category of \mathcal{O}_{QM}^\times -equivariant 1-morphisms from \mathbb{M}^\times to the stack $\mathcal{X} = \underline{\mathbf{Mod}}(\mathcal{O}_{QM})$, which we denoted by $\mathcal{X}(\mathbb{M})$. The equivalence assigns to $\tilde{P} \in \mathcal{X}(\mathbb{M})$ the composition*

$$\mathbb{M} \xrightarrow{\tilde{P}} \mathcal{X} = \underline{\mathbf{Mod}}(\mathcal{O}_{QM}) \rightarrow \mathbf{Sh}_R.$$

Here the last arrow is the forgetful 1-morphism.

Proof. Let us describe the inverse correspondence. Suppose that $\tilde{F} : \mathbb{M} \rightarrow \mathbf{Sh}_R$ is a representation of \mathbb{M} . Then the 1-morphism $\tilde{F}|_{\mathbb{M}^\times}$ naturally lifts to a 1-morphism

$$\begin{array}{ccc} \mathbb{M}^\times & \xrightarrow{\tilde{F}_{(\bullet)}} & \mathbf{Mod}(\mathcal{O}_{QM}) \\ & \searrow \tilde{F} & \swarrow \\ & \mathbf{Sh}_R & \end{array}$$

where

$$\begin{aligned} \tilde{F}_{(\bullet)} : \quad \mathbb{M}^\times &\longrightarrow \mathbf{Mod}(\mathcal{O}_{QM}) \\ \alpha &\longmapsto (\tilde{U}_\alpha, \tilde{F}_{(\alpha)}). \end{aligned}$$

Moreover, by construction $\tilde{F}_{(\bullet)}$ is equivariant with respect to the natural $\mathcal{O}_{QM_{\text{an}}}^\times$ -equivariant structures on \mathbb{M}^\times and $\mathbf{Mod}(\mathcal{O}_{QM})$. Thus $\tilde{F}_{(\bullet)}$ is naturally an object of $\mathcal{X}(\mathbb{M})$, as required. \square

6.3.2. Proposition 6.19 can be generalized in the following manner. Let \mathbf{S} be a stack of R -linear categories over M_{an} (above we have $\mathbf{S} = \mathbf{Sh}_R$) and let \mathcal{X} be the stack of $\mathcal{O}_{QM_{\text{an}}}$ -modules in \mathbf{S} . Concretely \mathcal{X} is the stack of categories on QM_{an} such that, for every $\tilde{U} \in \mathbf{S}$, the category $\mathcal{X}(\tilde{U})$ is the category of $\mathcal{O}_{\tilde{U}}$ -modules in $\mathbf{S}|_U$. In other words, $\mathcal{X}(\tilde{U})$ is the category of pairs (α, a) , where $\alpha \in \mathbf{S}(U)$ and $a : \mathcal{O}_{\tilde{U}} \rightarrow \text{End}_{\mathbf{S}}(\alpha)$ is a homomorphism of sheaves of R -algebras on U .

PROPOSITION 6.20. *For any \star -quantization \mathbb{M} of M , there is a natural equivalence between the category of R -linear 1-morphisms $\mathbb{M} \rightarrow \mathbf{S}$ of stacks over M_{an} , and the category of sections $\mathcal{X}(\mathbb{M})$.*

Proof. As in Proposition 6.19, the equivalence is given by the composition with the forgetful 1-morphism $\mathcal{X} \rightarrow \mathbf{S}$ of stacks over M_{an} , and the proof is similar. \square

6.3.3. In particular, Proposition 6.20 provides an equivalence between Definitions 4.9 and 4.12, and the corresponding definitions using \mathcal{O}_{QM}^\times -equivariant stacks.

6.4 The 2-category of \star -quantizations. If we have an action of a group on a space, equivariant objects can be viewed as objects on the quotient, which is usually a stack rather than a space. Similarly, if a sheaf of groups acts on a stack, equivariant objects can be thought of as objects on a quotient 2-stack. In this section, we explore this point of view in our setting.

6.4.1. Let G be a sheaf of groups acting inertially on QM_{an} . Let us define the quotient 2-stack $[QM/G]$. The definition mimics that of the quotient 1-stack $[N/H]$ for an action of a group H on a space N . Recall that $[N/H]$ is the stack of groupoids that assigns to every space S the category $[N/H](S)$ of pairs (T, φ) , where T is a H -torsor on S , and $\varphi : T \rightarrow N$ is a H -equivariant map. In the same spirit, $[QM/G]$ can be defined as the 2-stack of G -gerbes on QM . This leads to the following definition.

DEFINITION 6.21. Let \mathcal{X} be a G -equivariant stack over $\mathbb{Q}M$. We say that \mathcal{X} is a G -gerbe if \mathcal{X} is a gerbe over M and the action of G on \mathcal{X} is simple and transitive (simple transitive actions are defined in Proposition 6.13).

We let $[\mathbb{Q}M/G](M)$ be the 2-category of G -gerbes on M . More generally, let $U \subset M$ be an open subset. Then G restricts to a sheaf of groups $G|_U$ acting inertially on $\mathbb{Q}U$, and we let $[\mathbb{Q}M/G](U)$ be the 2-category of $G|_U$ -gerbes.

This definition gives an abstract description of a quotient 2-stack. In particular cases, the quotient is identified with the 2-stacks of \star -stacks and \star -quantizations.

Theorem 6.22. (a) The correspondence $\mathbb{M} \mapsto \mathbb{M}^\times$ gives an equivalence between the 2-category of \star -stacks on M and the 2-category of $\mathcal{O}_{\mathbb{Q}M}^\times$ -gerbes $[\mathbb{Q}M/\mathcal{O}_{\mathbb{Q}M}^\times](M)$. (Recall that \mathbb{M}^\times carries a natural simple transitive $\mathcal{O}_{\mathbb{Q}M}^\times$ -equivariant structure, which is described in 6.2.1.)
 (b) Similarly, the correspondence $\mathbb{M} \mapsto \mathbb{M}^{(1)}$ gives an equivalence between the 2-category of \star -quantizations of M and the 2-category of $\mathcal{O}_{\mathbb{Q}M}^{(1)}$ -gerbes $[\mathbb{Q}M/\mathcal{O}_{\mathbb{Q}M}^{(1)}](M)$.

Proof. Two parts of the theorem are completely analogous, so we prove only part (a). Let us construct an inverse correspondence. Fix an $\mathcal{O}_{\mathbb{Q}M}^\times$ -gerbe \mathcal{X} , and let us construct the corresponding \star -stack \mathbb{M} .

On the level of objects, $\text{ob}(\mathbb{M}) \cong \text{ob}(\mathcal{X})$. It remains to describe morphisms. Suppose $U_1, U_2 \subset M$ are open sets, $\tilde{U}_i \in \mathbb{Q}M(U_i)$, and $\alpha_i \in \mathcal{X}(\tilde{U}_i)$ for $i = 1, 2$. Let us describe $\text{Hom}_{\mathbb{M}}(\alpha_1, \alpha_2)$. Assume $U_1 \subset U_2$, otherwise the space of morphisms is empty.

Consider the sheaf $\mathcal{H}om_{\mathcal{X}}(\alpha_1, \alpha_2)$ on U_1 . The sheaf is a torsor over the sheaf of groups $\mathcal{H}om_{\mathcal{X}}(\alpha_1, \alpha_1)$ (because \mathcal{X} is a gerbe). For any open subset $V \subset U_1$, the action of $\mathcal{O}_{\mathbb{Q}M}^\times$ on \mathcal{X} induces a map

$$\Gamma(V, \mathcal{O}_{\tilde{U}_1}^\times) \rightarrow \text{Hom}_{\mathcal{X}}(\alpha_1|_V, \alpha_1|_V),$$

which is bijective because the action is simple transitive. This yields an identification

$$\mathcal{O}_{\tilde{U}_1}^\times = \mathcal{H}om_{\mathcal{X}}(\alpha_1, \alpha_1).$$

We finally set

$$\text{Hom}_{\mathbb{M}}(\alpha_1, \alpha_2) := \Gamma\left(U_1, \mathcal{H}om_{\mathcal{X}}(\alpha_1, \alpha_2) \times_{\mathcal{O}_{\tilde{U}_1}^\times} \mathcal{O}_{\tilde{U}_1}\right).$$

Further details (such as construction of the composition of morphisms) are left to the reader. \square

6.4.2. Theorem 6.22 clarifies Definitions 6.12, 6.16. For instance, Definition 6.12 concerns the inertial action of $G = \mathcal{O}_{\mathbb{Q}M}^\times$ on $\mathbb{Q}M$. Naturally, a G -equivariant stack \mathcal{X} over $\mathbb{Q}M$ can be viewed as a stack over the quotient $[\mathbb{Q}M/G]$. We can therefore define sections of \mathcal{X} over objects of $[\mathbb{Q}M/G]$. By Theorem 6.22, the category of global sections $[\mathbb{Q}M/G](M)$ is equivalent to the category of \star -stacks on M . This leads to the definition of sections of \mathcal{X} over a \star -stack \mathbb{M} (Definition 6.12).

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