The Awesome Averaging Power of Heat in Geometry

exemplified by the curve shortening flow

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The Heat Equation

heat in a thin rod

Let $u(x,t)$ be the temperature at position x and time t. Then the function u satisfies:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

If the rod is not homogeneous, then its thermal conductivity may depend on the position x, so we could have an equation like

$$\frac{\partial u}{\partial t} = \alpha(x)\frac{\partial^2 u}{\partial x^2}$$

where $\alpha(x) > 0$ measures the thermal conductivity.

The Heat Equation

At a point where u has a local minimum, u is increasing in time–heat flows into the local minimum.

At a point where u has a local maximum, u is decreasing in time–heat flows away from the local maximum.

The HE doesn't just care about convexity–it cares about how convex the function is.

The Heat Equation

"averaging"

The HE pushes interior local maxima down and interior local minima up. If we insulate the ends of the rod ($u'(a) = u'(b) = 0$), then

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \frac{\partial u}{\partial t}(x,t)dx = \int_a^b \frac{\partial^2 u}{\partial x^2}(x,t)dx = u'(b) - u'(a) = 0$$

So the average temperature does not change over time.

Maximum and Comparison Principles

maximum principle

The maximum of u does not increase over time. The minimum of u does not decrease over time.

comparison principle

If $u(x,t)$ and $v(x,t)$ are two solutions to the HE, and

$$u(x,0) \geq v(x,0)$$

then for all $t > 0$,

$$u(x,t) \geq v(x,t)$$

Normal Motion

We want to apply heat diffusion to a curve. We'll consider a simple, closed curve Γ.

We want to let Γ move, so we'll give it a time parameter t and write $\Gamma(t)$.

If we move the points of Γ around Γ, that won't change the curve, only its parametrisation. So we are really concerned with how much Γ is moving in the perpendicular direction.

We want $\frac{\partial}{\partial t}\Gamma(t)$ to be normal to Γ.

Curvature of a Plane Curve

The RHS of the HE is $\dfrac{\partial^2 u}{\partial x^2}$. We need to make sense of the "second derivative" of a curve Γ.

Curves are parametrised: Γ is really

$$\Gamma(s) = \{(x(s), y(s)) | s \in [a, b]\}$$

$\frac{d^2}{ds^2}\Gamma = (x''(s), y''(s))$ is a good start.

The curvature of a plane curve $\Gamma(s)$ is the part of $\frac{d^2}{ds^2}\Gamma$ which is independent of the parametrisation. We'll call the curvature $\kappa(s)$.

Curvature of a Plane Curve

If $\Gamma = \{(x, f(x))\}$ is the graph of a function f, then the curvature of Γ is

$$\kappa(x) = \frac{f''(x)}{(1 + (f'(x))^2)^{\frac{3}{2}}}$$

If f has a local minimum or a local maximum at x_0,

$$\kappa(x_0) = f''(x_0)$$

Curve Shortening Flow

A family of curves $\Gamma(t)$ is said to move by the curve shortening flow if it satisfies:

$$\frac{\partial}{\partial t}\Gamma(s,t) = \kappa(s,t)$$

Historical note: the name "curve shortening flow" was initially suggested by H. Gluck and W. Ziller.

Grayson's Theorem

Theorem

Let $\Gamma(t)$ be a family of plane curves moving by curve shortening flow, whose initial curve $\Gamma(0)$ is simple and closed.

- *After some time t_0, $\Gamma(t)$ is convex.*
- *After $\Gamma(t)$ becomes convex, it shrinks to a point.*
- *If $\Gamma(0)$ encloses a region of area A_0, then $\Gamma(t)$ shrinks to a point at time $\frac{1}{2\pi}A_0$–no earlier and no later.*
- *$\Gamma(t)$ becomes asymptotically round as it shrinks.*

Outline of the Proof

1. A simple curve stays simple.
2. Area decreases: $\text{area}(\Gamma(t)) = \text{area}(\Gamma(0)) - 2\pi t$.
3. Crushing doesn't happen.
4. The curve gets "more convex" along the flow.

A simple curve stays simple

Lemma

If $\Gamma(t)$ is a family of curves moving by CSF, with $\Gamma(0)$ simple, then $\Gamma(t)$ is simple.

Proof.

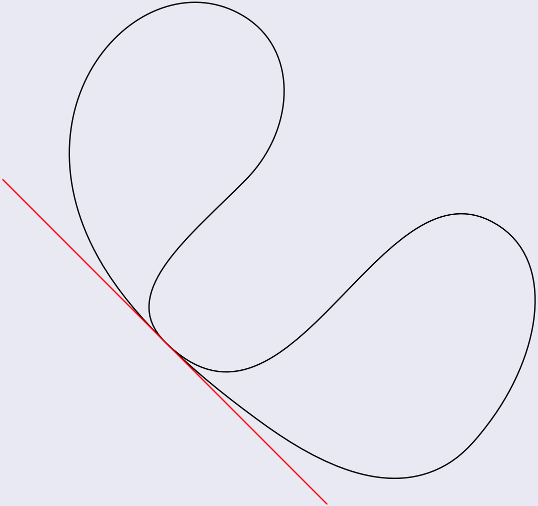
Let t_0 be the first time t that $\Gamma(t)$ intersects itself, i.e. that there are s_0 and s_1 with $\Gamma(s_0, t_0) = \Gamma(s_1, t_0)$.

Because this is the first time, the tangent line to $\Gamma(t_0)$ at s_0 is the same as the tangent line to $\Gamma(t_0)$ at s_1.

So we can write the two pieces of $\Gamma(t_0)$ as graphs over the same tangent line. . . □

A simple curve stays simple

At time t_0



A simple curve stays simple

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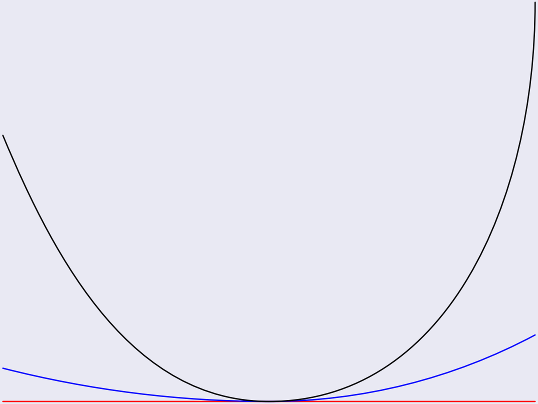
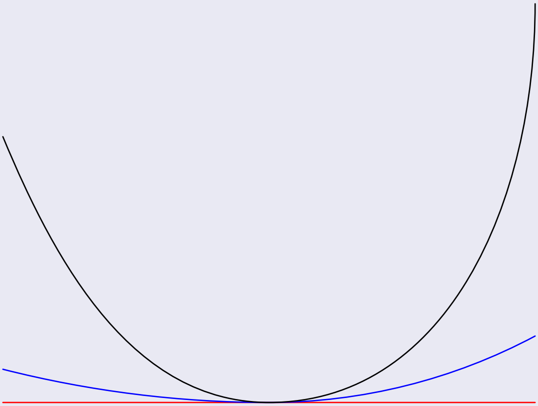


A simple curve stays simple

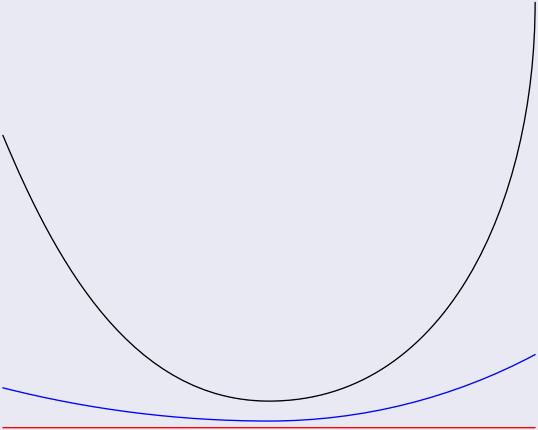
At time t_0



An instant later. . .

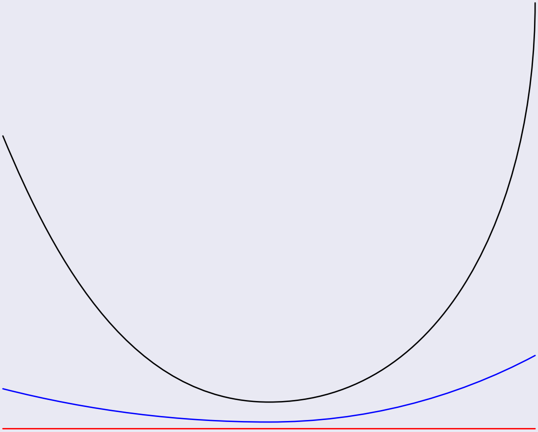
A simple curve stays simple

At time $t_0 + \epsilon$



A simple curve stays simple

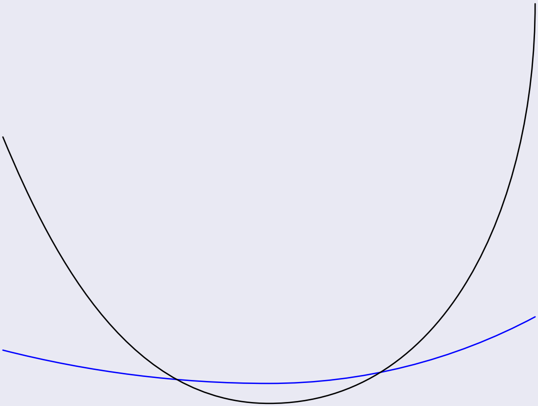
At time $t_0 + \epsilon$



An instant prior. . .

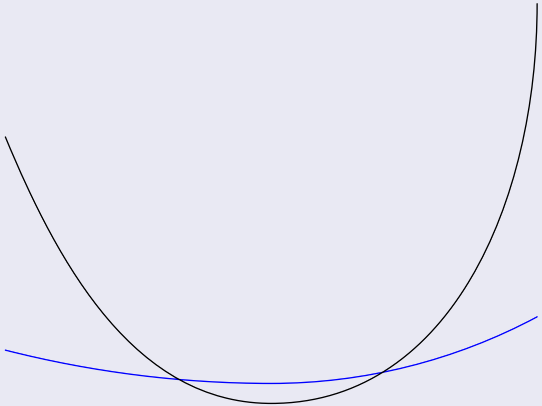
A simple curve stays simple

At time $t_0 - \epsilon$



A simple curve stays simple

At time $t_0 - \epsilon$



But t_0 was the first time $\Gamma(t)$ was nonsimple!

Area under CSF

To compute how the area bounded by $\Gamma(t)$ changes in time, we can break the curve into regions where it is a graph over the x-axis.

For a short time, we can write $y = y(x,t)$, and compute:

Lemma

The evolution of y fixing x is:

$$\frac{\partial}{\partial t} y = \frac{y''}{1 + (y')^2}$$

(Here $' = \frac{\partial}{\partial x}$.)

Area under CSF

So if we consider how much the area bounded by $x = a, x = b, y = x$ and $x = 0$ changes:

$$\begin{aligned}\frac{d}{dt}(\text{area}) &= \frac{d}{dt}\int_a^b y(x,t)dx \\ &= \int_a^b \frac{\partial y}{\partial t}(x,t)dx \\ &= \int_a^b \frac{y''(x,t)}{1+(y'(x,t))^2}dx \\ &= \arctan(y'(x,t))|_{x=a}^{x=b}\end{aligned}$$

Area under CSF

$\arctan(y'(a))$ is the angle the tangent line at a makes with the horizontal.

$\arctan(y'(b))$ is the angle that the tangent line at b makes with the horizontal.

Adding up all the changes in angle, we get that $\frac{d}{dt}(\text{area}(\Gamma(t)))$ is equal to the total change in angle that the tangent line makes as we go once around $\Gamma(t)$.

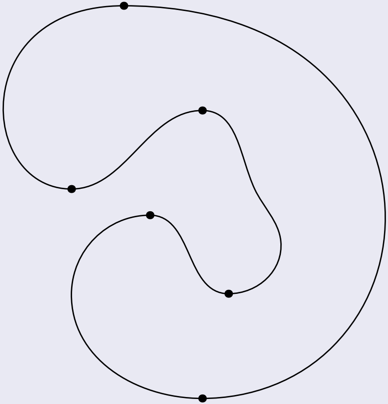
Lemma (Area Lemma)

$$\text{area}(\Gamma(t)) = \text{area}(\Gamma(0)) - 2\pi t$$

The Area Lemma and the fact that we can't introduce crossings tells us that the flow can't possibly continue past time $\frac{1}{2\pi}\,\mathrm{area}(\Gamma(0))$.

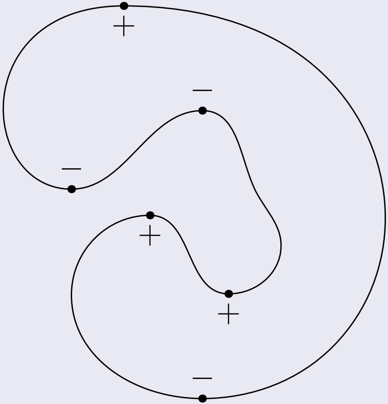
We want to show that, in fact, what happens is that the curve $\Gamma(t)$ collapses to a single point at time $\frac{1}{2\pi}\,\mathrm{area}(\Gamma(0))$.

Critical Points



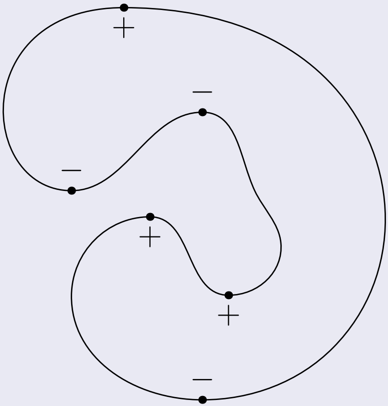
Given a particular time t, let's consider the local minima and local maxima of y on $\Gamma(t)$.

Critical Points



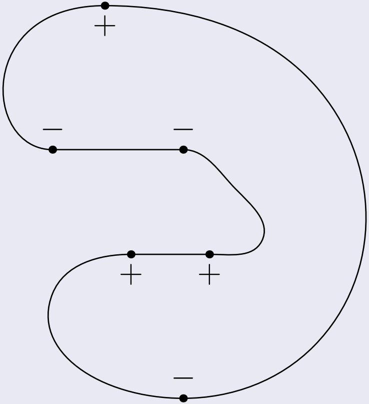
A $+$ followed by a $-$, or vice versa, means the tangent to the curve has turned through an angle of π.

Critical Points



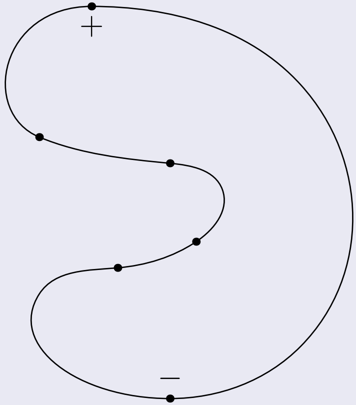
A $+$ followed by a $+$ will flatten out. A $-$ followed by a $-$ will flatten out.

Critical Points



A + followed by a + will flatten out. A − followed by a − will flatten out.

Critical Points



A + followed by a + will flatten out. A − followed by a − will flatten out. "Terraces" will immediately disappear.

More Facts about Critical Points

Critical points can only disappear, not appear.

Three critical points can collapse into one!

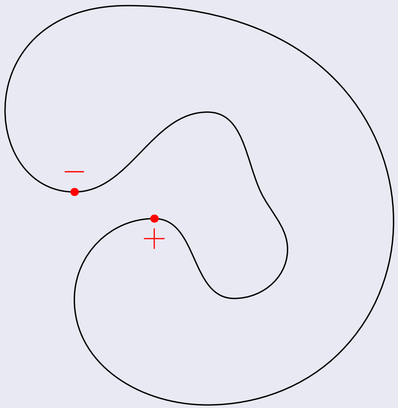
The critical points at time t can be followed backward in time to critical points at time 0. So we get a family of critical points $\{P_1(t), \ldots, P_k(t)\}$, some of which die before the flow stops.

Each critical point $P_k(t)$ lies in a region of the curve which is the graph of some function $f_k(x,t)$.

If $t < s$, then domain $(f_k(x,s)) \subset$ domain $(f_k(x,t))$.

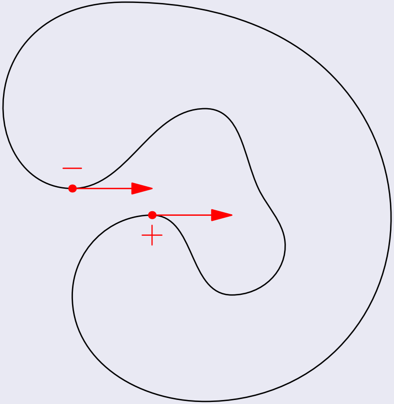
δ-Whiskers

Pick a $+$ and a $-$ critical point, and cut the curve $\Gamma(t)$ at those two points.



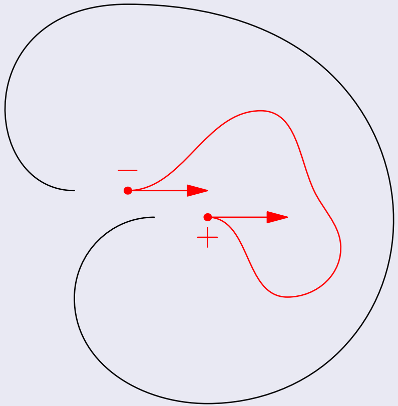
δ-Whiskers

Draw tangent vectors that point "in" to one of the arcs we get from cutting.



δ-Whiskers

If we slide the cut arc in the direction of the arrows, we can go for some distance until it runs into the other arc.



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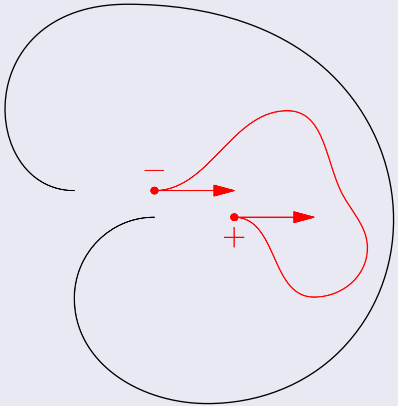
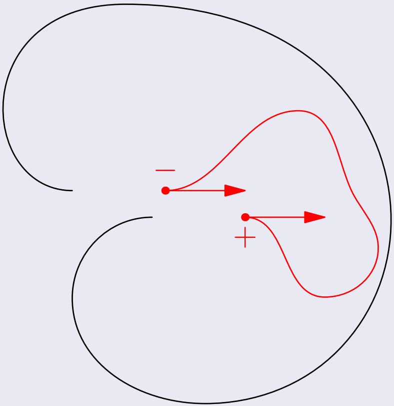


δ-Whiskers

If we slide the cut arc in the direction of the arrows, we can go for some distance until it runs into the other arc.



δ-Whiskers

Call the maximum distance we can slide the arc $\alpha(t)$ before it hits $\beta(t) = \Gamma(t) \setminus \alpha(t)$, $D(\alpha, t)$.

It could be the case that $D(\alpha, t) = \infty$.

We could cut-and-slide, then let time run.

Or we could let time run, then cut-and-slide.

In the former case, notice that the cut-and-slid version of $\alpha(t)$ will immediately push off from $\beta(t)$.

$\frac{d}{dt} D(\alpha, t) \geq 0$

δ-Whiskers

Lemma (δ-Whisker Lemma)

For any initial curve $\Gamma(0)$, there is a $\delta > 0$ depending only on $\Gamma(0)$ so that

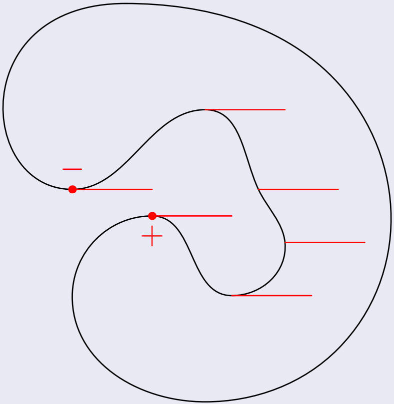
if α is a "nice" subarc of $\Gamma(t)$,
and ℓ is a line segment of length δ, tangent to α at its endpoints,

then the whiskering of α by ℓ is disjoint from $\beta = \Gamma(t) \setminus \alpha$.

The important point is that we can pick a single δ which works for any nice subarc α and any time t.

δ-Whiskers

"Whiskering"



δ-Whiskers

What the δ-Whisker Lemma says is that 'interior' arcs (where $\Gamma(t)$ fails to be convex) run toward convexity faster than the 'exterior' arcs are collapsing.

Lemma (additional facts about δ-whiskers)

δ increases over time.

If α_1 and α_2 are disjoint 'nice' arcs, then their δ-whiskers are disjoint from $\beta_{12} = \Gamma(t) \setminus \{\alpha_1 \cup \alpha_2\}$.

The Rest of the Proof

We've shown, more or less, that the curve gets simpler over time.

The flow removes critical points and pushes out 'interior' nonconvex arcs.

Remaining steps

1. The curve stays smooth until it runs out of area.
2. Quantitative estimate on 'roundness': κ gets closer to being constant.

Other Geometric "Heat Equations"

the heat equation for submanifolds

mean curvature flow

the heat equation for maps between manifolds

harmonic map heat flow

the heat equation for Riemannian metrics

Ricci flow

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