NAME: ______________________ **PENN ID#:** ______________

(201)	(202)	(203)	(204)	Math 241-001	Ron DONAGI/Matthew WIENER
(211)	(212)	(213)	(214)	Math 241-002	Michael PIMSNER/Matthew WIENER

Instructions.

Please write your name and Penn ID in the space provided above, and fill in the oval identifying your recitation. You will have 2 hours to complete this exam.

You are allowed to use one $8\frac{1}{2} \times 11$ sheet, both sides, for notes you wrote yourself. In addition, an extra sheet of notes will be provided with the exam. You are not allowed to use calculators.

Do not detach this sheet from the body of the exam.

This is a multiple-choice test, but you must show your work. Blind guessing will not be credited. No penalties for incorrect answers will be taken.

Please mark your answer on both the front sheet and on the problem itself. If you change an answer, be absolutely clear which choice is your final answer.

Each problem is worth 1 point. No partial credit will be given. No penalties for incorrect answers will be taken.

Questions 1-9							*points*	**Questions 10-18**							*points*
1.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	10.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
2.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	11.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
3.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	12.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
4.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	13.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
5.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	14.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
6.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	15.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
7.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	16.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
8.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	17.	(A)	(B)	(C)	(D)	(E)	(F)	*1*
9.	(A)	(B)	(C)	(D)	(E)	(F)	*1*	18.	(A)	(B)	(C)	(D)	(E)	(F)	*1*

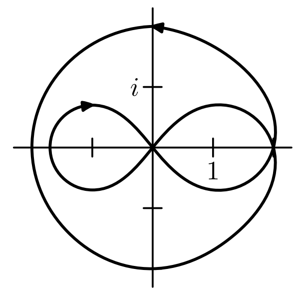
1. (*1 point*) What is the radius of convergence of

$$f(z) = \frac{z^2 - 1}{z^4 + 5z^2 + 4}$$

when expanded in a Taylor series about $z = 2 + i$?

(A) 1 (B) 2 (C) $\sqrt{2}$ (D) $2\sqrt{2}$ (E) $\sqrt{5}$ (F) $\sqrt{13}$

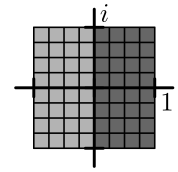
2. (*1 point*) Evaluate $\oint_C \frac{dz}{z^2-1}$ where C is the indicated path.



Ⓐ 0 Ⓑ πi Ⓒ $2\pi i$

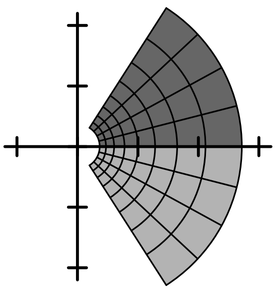
Ⓓ $3\pi i$ Ⓔ $4\pi i$ Ⓕ $5\pi i$

3. (*1 point*) Which region below is the image of $f(z) = e^{iz}$ when applied to the region depicted on the right, sending the light gray portion to the light gray portion, and the dark gray portion to the dark gray portion?

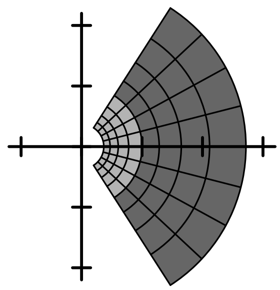


In all the graphs, 1 is located at the first tick mark to the right of the origin, and i is the the first tick mark above the origin.

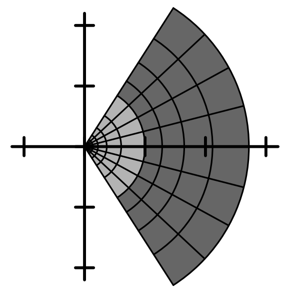
(A)



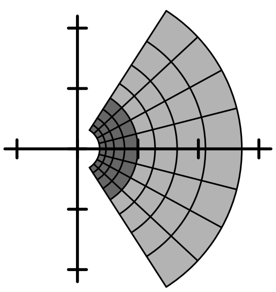
(B)



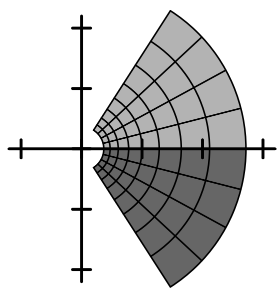
(C)



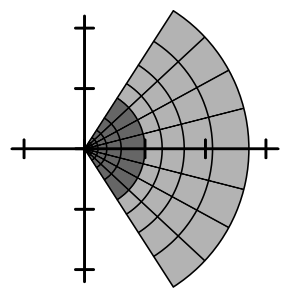
(D)



(E)



(F)



4. (*1 point*) Solve the heat equation $u_{xx} = u_t$ with boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$ and initial condition $u(x,0) = 4\sin x - 7\sin 2x + 10\sin 3x$ for $0 < x < \pi$.

(A) $4e^{-t}\sin x - 7e^{-2t}\sin 2x + 10e^{-3t}\sin 3x$

(B) $4e^{-t}\sin x - 7e^{-2t}\sin 4x + 10e^{-3t}\sin 9x$

(C) $4e^{-t}\sin x - 7e^{-4t}\sin 2x + 10e^{-9t}\sin 3x$

(D) $4e^{-t^2}\sin x - 7e^{-2t^2}\sin 2x + 10e^{-3t^2}\sin 3x$

(E) $16e^{-t}\sin x - 49e^{-4t}\sin 2x + 100e^{-9t}\sin 3x$

(F) $16e^{-t}\sin x - 49e^{-2t}\sin 2x + 100e^{-3t}\sin 3x$

5. (*1 point*) Evaluate

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{(1+z)^{50}}{z^4}dz\,.$$

Ⓐ 0 Ⓑ 1 Ⓒ 50 Ⓓ 2^{50} Ⓔ $50\cdot 49\cdot 48$ Ⓕ $\dfrac{50\cdot 49\cdot 48}{1\cdot 2\cdot 3}$

6. (*1 point*) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}\,.$$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{2\pi}{5}$ (D) $\frac{\pi}{2}$ (E) $\frac{4\pi}{5}$ (F) π

7. (*1 point*) In the Laurent series expansion for $\frac{1}{z}$ centered at 1, convergent for $|z-1|>1$, what is the coefficent of $(z-1)^{-3}$?

(A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1 (F) 2

8. (*1 point*) Consider the Sturm-Liouville problem $y'' + \lambda^2 y = 0$, $y(0) = 0$, $y'(4) = 0$. Which λ and y are the solutions? (In the following, A is an arbitrary coefficient, and $n = 0, 1, 2, \ldots$.)

- (A) $y = A \sin \lambda x, \quad \lambda = \frac{2n+1}{8}\pi$
- (B) $y = A \sin \lambda x, \quad \lambda = \frac{2n+1}{6}\pi$
- (C) $y = A \sin \lambda x, \quad \lambda = \frac{2n+1}{4}\pi$
- (D) $y = A \sin \lambda x, \quad \lambda = \frac{2n+1}{2}\pi$
- (E) $y = A \sin \lambda x, \quad \lambda = \frac{n}{8}\pi$
- (F) $y = A \sin \lambda x, \quad \lambda = \frac{n}{4}\pi$

9. (*1 point*) Give the Fourier series for

$$f(x) = \begin{cases} -\pi, & -\pi < x < -\frac{\pi}{2} \\ 0, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi, & \frac{\pi}{2} < x < \pi \end{cases}$$

(A) $2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x - \frac{2}{3}\sin 6x + \frac{2}{7}\sin 7x + \cdots$

(B) $2\sin x + 2\sin 2x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \frac{2}{3}\sin 6x + \frac{2}{7}\sin 7x + \cdots$

(C) $2\sin x + 2\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{5}\sin 5x - \frac{2}{3}\sin 6x - \frac{2}{7}\sin 7x + \cdots$

(D) $2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x - \frac{2}{3}\sin 6x + \frac{2}{7}\sin 7x + \cdots$

(E) $2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x + \frac{2}{3}\sin 6x + \frac{2}{7}\sin 7x + \cdots$

(F) $2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \frac{1}{2}\sin 4x - \frac{2}{5}\sin 5x - \frac{2}{3}\sin 6x + \frac{2}{7}\sin 7x + \cdots$

10. *(1 point)* Consider the wave equation $u_{xx} = u_{tt}$ with boundary condition $u(0,t) = u(\pi,t) = 0$ for $t > 0$ and initial conditions $u(x,0) = \sin x$ and $u_t(x,0) = \cos x$. Let $U(x,s) = \mathcal{L}\{u(x,t)\}$ be the Laplace transform (in t). Give the ordinary differential equation satisfied by U (in x).

(A) $\dfrac{d^2U}{dx^2} - s^2U = s\sin x + \cos x$

(B) $\dfrac{d^2U}{dx^2} - s^2U = s\sin x - \cos x$

(C) $\dfrac{d^2U}{dx^2} - s^2U = -s\sin x - \cos x$

(D) $\dfrac{d^2U}{dx^2} - s^2U = \sin x + s\cos x$

(E) $\dfrac{d^2U}{dx^2} - s^2U = \sin x - s\cos x$

(F) $\dfrac{d^2U}{dx^2} - s^2U = -\sin x - s\cos x$

11. (*1 point*) Give a harmonic conjugate for $u = \cos x \sinh y$.

(A) $\cos x \cosh y$ (B) $\cos y \sinh x$ (C) $-\cos x \sinh y$

(D) $\sin x \cosh y$ (E) $\sin x \sinh y$ (F) $-\sin x \cosh y$

12. (*1 point*) Find k such that $x^5 - 10x^3y^2 + kxy^4 + i(5x^4y - 10x^2y^3 + y^5)$ is analytic.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5 (F) There is no such k.

13. (*1 point*) Give a value of $i^{i/\pi}$.

(A) $e^{-1/2}$ (B) $e^{-1/2}(\cos\sqrt{2} + i\sin\sqrt{2})$ (C) $e^{-1} + e^{-1}i$

(D) $e^{-\pi^2/2}$ (E) $e^{-1}(\cos 1 + i\sin 1)$ (F) $e^{-2}(\cos 1 + i\sin 1)$

14. (*1 point*) Consider a semicircle whose diameter is maintained at temperature 0 and whose circular edge is maintained at the temperature $u(1,\theta) = \sin 2\theta - \sin 3\theta$.

The steady-state temperature $u = u(r,\theta)$ satisfies $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. It is given by which of the following?

(A) $J_0(2r)\sin 2\theta - J_0(3r)\sin 3\theta$

(B) $\dfrac{J_0(2r)}{J_0(2)}\sin 2\theta - \dfrac{J_0(3r)}{J_0(3)}\sin 3\theta$

(C) $\dfrac{J_0(2r)}{2}\sin 2\theta - \dfrac{J_0(3r)}{3}\sin 3\theta$

(D) $\dfrac{J_0(2r)}{4}\sin 2\theta - \dfrac{J_0(3r)}{9}\sin 3\theta$

(E) $r^2 \sin 2\theta - r^3 \sin 3\theta$

(F) $r^4 \sin 2\theta - r^6 \sin 3\theta$

15. (*1 point*) Evaluate

$$\oint_{|z|=1} e^{-\frac{1}{z}} \sin\frac{1}{z}\, dz\,.$$

(A) $-2\pi i$ (B) $-\pi i$ (C) 0 (D) πi (E) $2\pi i$ (F) $3\pi i$

16. (*1 point*) Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}\,.$$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{2\pi}{5}$ (D) $\frac{\pi}{2}$ (E) $\frac{\pi}{12}$ (F) $\frac{5\pi}{18}$

17. (*1 point*) Solve

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta = 0$$

subject to the boundary condition $u(1,\theta) = \cos^3\theta$ for $0 < \theta < \pi$, and which is bounded at $r = 0$.

Recall that $\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda\sin\theta\,\Theta = 0$ has a solution $\Theta = P_n(\cos\theta)$ when $\lambda = n(n+1)$, where $P_n(x)$ is the n^{th} Legendre polynomial.

The first four Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, and $P_3(x) = \frac{1}{2}(5x^3-3x)$.

(A) $\frac{3}{5}rP_1(\cos\theta) - \frac{2}{5}r^3P_3(\cos\theta)$

(B) $\frac{3}{5}rP_1(\cos\theta) + \frac{2}{5}r^3P_3(\cos\theta)$

(C) $\frac{1}{5}rP_1(\cos\theta) - \frac{3}{5}r^3P_3(\cos\theta)$

(D) $\frac{1}{5}rP_1(\cos\theta) + \frac{3}{5}r^3P_3(\cos\theta)$

(E) $\frac{3}{5}rP_1(\cos\theta) - \frac{1}{5}r^3P_3(\cos\theta)$

(F) $\frac{3}{5}rP_1(\cos\theta) + \frac{1}{5}r^3P_3(\cos\theta)$

18. (*1 point*) The ordinary differential equation

$$3x^2y'' + (14x + x^2)y' - 4y = 0$$

has a general solution near $x = 0$ of the form $y = c_1F(x) + c_2\sqrt[3]{x}G(x)$, where $F(x)$ has a pole at 0 and $G(x)$ is analytic. What is the order of the pole of F at 0?

Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ 4 Ⓔ 5 Ⓕ 6