UNIVERSITY of PENNSYLVANIA

MATH 241 FINAL EXAM
SPRING 2016

NAME:

INSTRUCTOR (CIRCLE ONE): WIBMER WONG

RECITATION NUMBER AND DAY/TIME:

Please *turn off and put away all electronic devices.* You may use both sides of ONE 8.5″ × 11″ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
1	10	
2	10	
3	10	
4	10	

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
5	10	
6	10	
7	10	
8	10	
TOTAL	80	

FORMULAE INVOLVING BESSEL FUNCTIONS

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k}.$$

$J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha > 0$)		Fourier Transform Pairs ($\beta > 0$)	
$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\lvert x\rvert}$	$\frac{1}{2\pi}\frac{2\alpha}{\omega^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\lvert\omega\rvert}$
$u(x) = \begin{cases} 0 & \lvert x\rvert > \alpha \\ 1 & \lvert x\rvert < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega) = \begin{cases} 0 & \lvert\omega\rvert > \beta \\ 1 & \lvert\omega\rvert < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2 U$
xu	$-i\frac{\partial U}{\partial\omega}$	$x^2 u$	$(-i)^2\frac{\partial^2 U}{\partial\omega^2}$
$u(x - x_0)$	$e^{i\omega x_0}U$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)g(x-s)ds$	FG

A Partial Table of Integrals

$$\int_0^x u\cos nu\, du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\, du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\, du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\, du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\, du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\, du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\, du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Orthogonality Relations for Sines and Cosines

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \end{cases} \qquad \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases} \qquad \int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

Boundary Value Problems for $\phi''(x) = -\lambda\,\phi(x)$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\phi'(0) = 0$ $\phi'(L) = 0$	$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi'(L)$
Eigenvalues	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$ $+ \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$	$A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$	$a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,\mathrm{d}x$ $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$ $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$

Boundary conditions	$\phi(0) = 0$ $\phi'(L) = 0$	$\phi'(0) = 0$ $\phi(L) = 0$
Eigenvalues	$\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{(n+\frac{1}{2})\pi x}{L}$	$\cos\frac{(n+\frac{1}{2})\pi x}{L}$
Series	$f(x) = \sum_{n=0}^{\infty} B_n \sin\frac{(n+\frac{1}{2})\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{(n+\frac{1}{2})\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(n+\frac{1}{2})\pi x}{L}\,\mathrm{d}x$	$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(n+\frac{1}{2})\pi x}{L}\,\mathrm{d}x$

Q1. (10 Points) Let $u(x,t)$ be the temperature in a one-dimensional rod ($0 < x < 4$), and satisfy the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 3\dfrac{\partial^2 u}{\partial x^2} + x^3 \quad 0 < x < 4,\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = 1 \\ \dfrac{\partial u}{\partial x}(4,t) = 5 \\ u(x,0) = x. \end{cases}$$

The total thermal energy is defined by

$$E(t) := \int_0^4 u(x,t)\,dx.$$

Compute $E(t)$.

Q2. (10 Points in Total) Consider the heat equation

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$$

where $0 < x < 1$ and $t > 0$, with boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0$$

and initial condition

$$u(x,0) = 4\sin\left(\frac{3\pi x}{2}\right) + 7\sin\left(\frac{5\pi x}{2}\right).$$

(i) (1 Point) What is the physical meaning of the boundary condition at the left end-point (i.e., $x = 0$)?

(ii) (1 Point) Is there any heat source?

(iii) (8 Points) Solve the above initial and boundary value problem.

You may use without explanation that separating the variables $u(x,t) = \phi(x)G(t)$ leads to the following ordinary differential equations:

$$G'(t) = -3\lambda G(t) \quad \text{and} \quad \phi''(x) = -\lambda\phi(x).$$

Q3. (10 Points in Total)

(i) (8 Points) Compute the Fourier sine series of

$$f(x) := \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

on the interval $0 \leq x \leq 1$.

(ii) (2 Points) For which values of $x \in [0, 1]$, does the Fourier sine series converge to $f(x)$?

Q4. (10 Points) Solve the wave equation for a vibrating rectangular membrane ($0 < x < L$, $0 < y < H$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0,$$

and initial conditions

$$u(x, y, 0) = \alpha(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = 0.$$

You may use without explanation that separating the variables $u(x, y, t) = f(x)g(y)h(t)$ *leads to the following ordinary differential equations:*

$$h''(t) = -\lambda c^2 h(t), \quad f''(x) = -\mu f(x), \quad g''(y) = -(\lambda - \mu) g(y).$$

Q5. (10 Points in Total) Consider the eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \dfrac{2}{x}\dfrac{d\phi}{dx} + \dfrac{\lambda}{x}\phi = 0 & \text{for } 1 < x < 2 \\ \dfrac{d\phi}{dx}(1) - 3\phi(1) = 0 \\ \dfrac{d\phi}{dx}(2) = 0 \end{cases}$$

and answer the following questions:

(i) (3 Points) Rewrite the ordinary differential equation into the Sturm-Liouville form.

(ii) (5 Points) Show that all eigenvalues λ are non-negative.

(iii) (2 Points) Is $\lambda = 0$ an eigenvalue? Justify your answer.

Q6. (10 Points) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Which differential equations are implied when separating the time variable $u(x, y, t) = \phi(x, y)h(t)$?

Q7. (10 Points) Solve the Poisson equation in a unit disk ($r < 1,\ -\pi \leq \theta \leq \pi$):

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^2$$

subject to the boundary condition

$$u(1, \theta) = 5\cos 2\theta.$$

Q8. (10 Points) Use the Fourier transform in x to find the solution $u(x,t)$ to the partial differential equation

$$\frac{\partial u}{\partial t} + 5\frac{\partial u}{\partial x} - 7u = 0,$$

where $-\infty < x < \infty$, $t > 0$ and $u(x,0) = f(x)$.

End of Final Exam

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