MATH 241-002 FINAL
SPRING 2015

University of Pennsylvania

NAME:

RECITATION NUMBER AND DAY/TIME:

Please *turn off and put away all electronic devices.* You may use both sides of a 3″ × 5″ index card for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from others. **Show all work.** Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
1	8	
2	8	
3	8	
4	8	
5	8	

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
5	15	
6	15	
7	15	
8	15	
10	15	
TOTAL	115	

A PARTIAL TABLE OF INTEGRALS

$$\int_0^x u\cos nu\,du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\,du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\,du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\,du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\,du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\,du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\,du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k}.$$
$J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. Usually z_{nm} refers to the mth positive zero of $J_n(x)$.

- Orthogonality relations:
$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.
$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} \left(\frac{x}{2}\right)^{n+2k}.$$

Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu - \frac{m^2}{\sin\phi}\right)g = 0$. Using the substitution $x = \cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0$. This equation has bounded solutions only when $\mu = n(n+1)$ and $0 \leq m \leq n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:
$$P_n^0(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1 \leq m \leq n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,
$$\text{If } n \neq k \text{ then } \int_{-1}^1 P_n^m(x)P_k^m(x)dx = 0 \text{ and } \int_{-1}^1 \left(P_n^m(x)\right)^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')' + (\alpha^2\rho^2 - n(n+1))f = 0$. If we define the spherical Bessel function $j_n(\rho) = \rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho = 0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:
$$j_n(x) = x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of j_n.
$$\text{If } m \neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx = 0 \quad \text{and} \quad \int_0^1 x^2(j_n(z_{nm}x))^2dx = \frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha > 0$)		Fourier Transform Pairs ($\beta > 0$)	
$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\|x\|}$	$\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\|\omega\|}$
$u(x) = \begin{cases} 0 & \|x\| > \alpha \\ 1 & \|x\| < \alpha \end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega) = \begin{cases} 0 & \|\omega\| > \beta \\ 1 & \|\omega\| < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2 U$
xu	$-i\frac{\partial U}{\partial \omega}$	$x^2 u$	$(-i)^2\frac{\partial^2 U}{\partial \omega^2}$
$u(x - x_0)$	$e^{i\omega x_0}U$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)g(x-s)ds$	FG

1. Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

2. Suppose $u(x,t)$ is a solution of the wave equation $u_{tt} = 9u_{xx}$ for all $-\infty < x < \infty$, with $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Find all the points on the x-axis that can influence the solution at $x = 2$, $t = 4$.

3. Suppose $u(x, y)$ satisfies the Laplace equation $\nabla^2 u = 0$ on the square $0 < x < 1$, $0 < y < 1$ and has boundary values $+1$ on the top and bottom ($y = 0, 1$) and boundary values -1 on the left and right ($x = 0, 1$). What is the value of u at the point $(\frac{1}{3}, \frac{1}{3})$? [Hint: What happens to the PDE and boundary conditions if the square is reflected about the line $y = x$? What can you say about the *sum* of u and its reflection?]

4. Convert the following boundary value problem to Sturm-Liouville form:

$$\varphi'' + (2 - 4x)\varphi' = -\lambda\varphi, \ \varphi(0) = \varphi(1) = 0.$$

Show that $\varphi(x) = x(1 - x)$ is an eigenfunction, finding its eigenvalue exactly. Then briefly explain why the eigenvalue you just computed must be the lowest eigenvalue for this problem.

5. Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$. Suppose $u(x, y, t)$ solves

$$\frac{\partial u}{\partial t} = \nabla^2 u + 2u + e^t \sin(x + 2y)$$

in Ω with $u(x, y, t) = 0$ on $\partial\Omega$ and $u(x, y, 0) = 0$. Suppose also that $v(x, y, t)$ solves

$$\frac{\partial v}{\partial t} = \nabla^2 v + 2v$$

in Ω with $v(x, y, t) = x^2 y$ on $\partial\Omega$ and $v(x, y, 0) = \cos(2x)$. Find a function $w(x, y, t)$ that satisfies

$$\frac{\partial w}{\partial t} = \nabla^2 w + 2w + 3e^t \sin(x + 2y)$$

in Ω with $w(x, y, t) = 5x^2 y$ on $\partial\Omega$ and $v(x, y, 0) = 5\cos(2x)$. You should give a formula for w in terms of u and v.

6. Suppose

$$f(x) := \begin{cases} 1 & 0 \le x < \pi \\ -1 & \pi \le x < 2\pi. \end{cases}$$

(a) Compute the Fourier cosine series of f on the interval $[0, 2\pi)$.

(b) Draw the graph of the Fourier cosine series computed above for x between -2π and 4π. Mark the value of the Fourier series with an "X" at all points of discontinuity.

(c) Using your series above, find the Fourier sine series of

$$g(x) := \begin{cases} x & 0 \le x < \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$$

without using the typical formula to compute the coefficients.

7. Let T be the triangle $0 \leq x \leq 1$ and $0 \leq y \leq x$. The Dirichlet eigenfunctions of the Laplacian on T are given by

$$\phi_{mn}(x,y) := \sin n\pi x \sin m\pi y - \sin m\pi x \sin n\pi y,$$

where $m = 2, 3, \ldots,$ and $n = 1, \ldots, m-1$. Note that these eigenfunctions are mutually orthogonal and satisfy

$$\iint_T |\phi_{mn}(x,y)|^2 dA = \frac{1}{4}.$$

Give a full series solution $u(x, y, t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

in the triangle T with homogeneous Dirichlet boundary conditions and find the formulas you would use to compute the coefficients in the series expansion using the initial data.

8. Consider the solution $u(x,t)$ of the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 1$, subject to the boundary conditions $u(0,t) = 0$, $u(1,t) = t$ and the initial condition $u(x,0) = f(x)$ for some unspecified function f. Solve for $u(x,t)$ as an infinite series and give a formula for the coefficients of your series in terms of the Fourier coefficients of $f(x)$.

9. The Schrödinger equation for a free quantum particle of mass m dictates that the *wave function* $\Psi(x, y, z, t)$ satisfies

$$\frac{\partial \Psi}{\partial t} = \frac{ih}{4\pi m} \nabla^2 \Psi$$

where h is Planck's constant and i is the imaginary unit ($i^2 = -1$). Suppose such a particle is confined to the interior of a spherical cavity of radius $\rho = R$. Give a full series solution for the wave function Ψ assuming homogeneous Dirichlet boundary conditions. You do not need to evaluate coefficients.

10. Solve the initial value problem

$$u_t = u_{xx} + u, \quad u(x,0) = e^{-\pi x^2}$$

for $u(x,t)$ when $t > 0$ and $-\infty < x < \infty$. Fully simplify your answer.