Name: ______________________ PennID: ______________

Math 241
Final Exam
December 19, 2016

Please *turn off and put away all electronic devices.* You may use one standard letter size sheet of paper with hand-written notes on both sides during this exam. No calculators, no books. **Show your work.** Be as organized as possible. Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Signature Date

Table 1: Boundary value problems for $\phi''(x) = -\lambda\,\phi(x)$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\phi'(0) = 0$ $\phi'(L) = 0$	$\phi(-L) = \phi(L)$ $\phi'(-L) = \phi'(L)$
Eigenvalues	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$	$A_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$ $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$	$a_0 = \frac{1}{2L}\int_{-L}^L f(x)\,\mathrm{d}x$ $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x$ $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$

Table 2: Orthogonality relations for sines and cosines

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L/2, & n = m \neq 0 \\ L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

Problem 1. Solve the heat equation

$$u_t = 3u_{xx},$$

with $0 < x < \pi$ and $t > 0$ subject to the following boundary and initial conditions:

$$u_x(0,t) = u_x(1,t) = 0$$
$$u(x,0) = 2 + 7\cos(4x).$$

Problem 2. Find the solution to the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the region of the plane *outside* the disk of radius 1, which is bounded as $r \to +\infty$, and satisfies $u(1,\theta) = 1 + 3\cos(2\theta)$.

Problem 3. Consider the function $f : [0, 1] \to \mathbb{R}$ given by:

$$f(x) = 1 - x.$$

(a) Sketch the graph of the Fourier Sine Series of $f(x)$ for $-3 \leq x \leq 3$. Mark the points of jump discontinuity.
(b) Compute the Fourier Sine Series of $f(x)$. Simplify the coefficients.
(c) For what values $-1 \leq x \leq 1$ does the Fourier Sine Series converge to $f(x)$?

Problem 4. Solve the wave equation:

$$u_{tt} = u_{xx}, \quad \text{for } 0 < x < 1,\ t > 0$$

subject to the following boundary and initial conditions:

$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 2\sin(3\pi x)$$
$$u_t(x,0) = 0.$$

Problem 5. Consider the eigenvalue problem:

$$\phi'' + (\lambda - x^2)\phi = 0,$$
$$\phi'(0) = 0,\ \phi'(1) = 0.$$

(a) Is this eigenvalue problem a regular Sturm-Liouville problem? What are $p(x), \sigma(x), q(x)$?
(b) Show that $\lambda \geq 0$.
(c) Is $\lambda = 0$ an eigenvalue?

Problem 6. Solve the wave equation

$$u_{tt} = 4u_{xx}, \quad \text{for } -\infty < x < \infty,\ t > 0$$

with the following initial conditions:

$$u(x,0) = x^2,\ u_t(x,0) = 0.$$

Problem 7. Let $u(x,t)$ be a solution of

$$\begin{cases} u_{tt} = u_{xx} - u \text{ for } x \in [0,L], t > 0 \\ u_x(0,t) = u_x(L,t) = 0 \end{cases}$$

Define the *energy* $E(t)$ of this solution by

$$E(t) = \frac{1}{2}\int_0^L u_t^2 + u_x^2 + u^2 \, dx.$$

(a) Show that $E(t)$ is constant;

(b) Use (a) to prove that a solution to the above PDE with initial conditions:

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

is unique. (HINT: Suppose that u and v are two such solutions, and study the energy of $w = u - v$)

Problem 8. Solve the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0$$

for the cylinder $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le z \le H$, together with boundary conditions:

$$\begin{aligned}
\text{on the top} &: \quad u(r,\theta,H)=\sum_{n=1}^{\infty}\frac{1}{n}J_0(z_{0n}r)\sinh(z_{0n}H)\\
\text{on the bottom} &: \quad u(r,\theta,0)=0\\
\text{on the lateral} &: \quad u(1,\theta,z)=0.
\end{aligned}$$

Here $J_0(z)$ be the 0th Bessel function of first kind, and denote its zeroes by $z_{01} < z_{02} < z_{03} < \cdots < z_{0m} < \ldots$.

You may assume without explanations that separating variables $u(r,\theta,z)=f(r)g(\theta)h(z)$ for the above PDE gives the following ODEs:

$$h''=\lambda h, \quad g''+\mu g=0, \quad r(rf')'+(\lambda r^2-\mu)f=0, \quad \lambda,\mu\ge 0.$$

Problem 9. Use the method of eigenfunction expansion to solve the inhomogeneous heat equation:

$$u_t = u_{xx} + e^{-t}\sin(2\pi x) \qquad \text{for } 0 < x < 1,\ t > 0$$

with boundary conditions $u(0,t) = 0$, $u(1,t) = 3$ and initial condition $u(x,0) = x$.

Problem 10. Use the Fourier transform to solve the initial-value problem for the first-order equation:

$$u_t + 5u_x = 0, \quad -\infty < x < \infty,\ t \geq 0$$
$$u(x, 0) = f(x).$$

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