MATH 241 FINAL EXAM
FALL 2014

UNIVERSITY *of* PENNSYLVANIA

NAME:

PENN ID:

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Please *turn off and put away all electronic devices.* You may use both sides of a 8.5″ × 11″ sheet of paper for handwritten notes (in your own handwriting) while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work** except in Question Number 1. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good Luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
1	40	
2	20	
3	20	
4	20	
5	25	
6	25	
7	25	
8	25	

Final Exam Total Score	/200

Part I (40 Points): The following question consists of 10 true or false problems. Each part is 4 Points. *In this question only*, you do **not** have to show any work or explain your answers.

Q1 Decide whether the following statements are true or false. *Circle* the Correct Answers.

(i) Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on a one-dimensional rod of length L subject to the boundary conditions

$$u(0,t) = 5 \quad \text{and} \quad u(L,t) = 8.$$

The sum of any two solutions (to the given heat equation and boundary conditions) is again a solution.

True False

(ii) The method of separation of variables (when it works) solves a partial differential equation by converting it into ordinary differential equations.

True False

(iii) The partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x + t$ can be solved directly by using the method of separation of variables.

True False

(iv) If a square plate has its edges held at $0°C$ and is initially at $10°C$, then at some point in the plate and at some future time, the temperature at that point will be equal to $1°C$.

True False

(v) The complex Fourier coefficients of a real-valued function must be real.

True False

(vi) All Fourier coefficients of x on the interval $-1 \leq x \leq 1$ are non-zero.

True False

(vii) Suppose that the Fourier cosine series of x^2 on the interval $0 \leq x \leq \pi$ is given by

$$x^2 \sim \sum_{n=0}^{\infty} a_n \cos nx.$$

The Fourier cosine coefficient a_n goes to zero as $n \to \infty$.

True False

(viii) The Bessel functions $\{J_m(x)\}_{m=0}^{\infty}$ are orthogonal in the following sense: for any different non-negative integers m and n,

$$\int_0^1 J_m(x) J_n(x) x dx = 0.$$

True False

(ix) The Bessel functions $\{J_m(r)\}_{m=0}^{\infty}$ are used to solve the following initial and boundary value problem inside a unit disk:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \nabla^2 u \quad \text{for } r < 1 \\ u(1, \theta, t) = 0 \\ u(r, \theta, 0) = f(r, \theta) \\ \dfrac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta). \end{cases}$$

True False

(x) If we apply a Fourier transform to the derivative of a function $f(x)$, then we will obtain a constant multiple of $\omega F(\omega)$, where $F(\omega)$ is the Fourier transform of $f(x)$.

True False

Part II (60 Points): In this part you will have 3 multiple choice questions. Each of them is 20 Points. *Circle* the Correct Answers and **show all work**. A correct answer without supporting work receives no credit!

Q2 Suppose that the temperature $u(x,t)$ in a one-dimensional rod ($0 \le x \le 2$) satisfies the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial u}{\partial x}(0,t) = 2 \\ \dfrac{\partial u}{\partial x}(2,t) = 4 \\ u(x,0) = \dfrac{3}{2}x^2 - x. \end{cases}$$

The thermal energy $E(t)$ is defined by

$$E(t) := \int_0^2 u(x,t)\, dx.$$

Compute $E(t)$. (Hint: you may find it useful to compute $\dfrac{dE}{dt}$.)

A) $-8t$ B) $8t$ C) 2 D) $2-8t$ E) $2+8t$.

Q3 The function

$$f(x) := 2\sin x + 3\cos x + \sin 5x$$

is expanded into a complex Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{-ikx}$. Compute the product of all *non-zero* coefficients c_k.

A) $\frac{9}{16}$ B) $\frac{25}{16}$ C) 6 D) 9 E) 30.

Q4 A thermos flask has the form of a right cylinder of bottom radius a and height H. It is perfectly insulated along its circular edge and its bottom disk. Initially the temperature of the liquid inside is $100°C$ and the flask is then opened at the top. It begins to lose heat (from the top) at such a rate that the temperature drop at the top is $5°C/\text{minute}$. The partial differential equation is

$$\frac{\partial u}{\partial t} = k\nabla^2 u.$$

Which of the following best describes the initial and boundary conditions for the given problem? Support your answer with reasons.

A) $u(r, \theta, z, 0) = 100$, $u(r, \theta, 0, t) = 0$, $u(r, \theta, H, t) = -5$ and $u(a, \theta, z, t) = 0$.

B) $u(r, \theta, z, 0) = 100$, $\frac{\partial u}{\partial z}(r, \theta, 0, t) = 0$, $\frac{\partial u}{\partial z}(r, \theta, H, t) = -5$ and $\frac{\partial u}{\partial r}(a, \theta, z, t) = 0$.

C) $u(r, \theta, z, 0) = 100$, $\frac{\partial u}{\partial z}(r, \theta, 0, t) = 0$, $\frac{\partial u}{\partial z}(r, \theta, H, t) = -5$ and $\frac{\partial u}{\partial \theta}(a, \theta, z, t) = 0$.

D) $u(r, \theta, z, 0) = 100$, $\frac{\partial u}{\partial r}(r, \theta, 0, t) = 0$, $\frac{\partial u}{\partial r}(r, \theta, H, t) = -5$ and $\frac{\partial u}{\partial \theta}(a, \theta, z, t) = 0$.

E) $\frac{\partial u}{\partial t}(r, \theta, z, 0) = 100$, $u(r, \theta, 0, t) = 0$, $u(r, \theta, H, t) = -5$ and $u(a, \theta, z, t) = 0$.

Part III (100 Points): In this part you will have 4 long questions. Each of them is 25 Points. **Show all work.** A correct answer without supporting work receives no credit!

Q5 Solve the Laplace's equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a circular annulus ($1 < r < 3$) subject to the boundary conditions

$$u(1,\theta) = 0 \quad \text{and} \quad u(3,\theta) = 8\sin 2\theta$$

by the *method of separation of variables.* Express your final answer without any undetermined coefficients.

Q6 Suppose that the temperature u in a non-uniform rod ($0 \leq x \leq 1$) satisfies the following initial and boundary value problem:

$$\begin{cases} c(x)\rho(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(K_0(x)\dfrac{\partial u}{\partial x}\right) \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \\ u(1,t) = 0 \\ u(x,0) = f(x), \end{cases}$$

where c, ρ, K_0 and f are given positive functions. Solve the initial and boundary value problem.

Q7 Consider the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u$$

inside a disk of radius a with zero temperature around the entire boundary. For physical reasons, we have

$$|u(0, \theta, t)| < \infty.$$

Answer the following two questions.

(i) Solve for u if initially $u(r, \theta, 0) = r$.

(ii) Compute $\lim_{t \to \infty} u(r, \theta, t)$.

Q8 Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + xyt$$

for the temperature distribution in a square plate of side length L ($0 \le x \le L$, $0 \le y \le L$) with a heat source. The edges of the plate are kept at $0°C$. Initially, the temperature of the plate is also $0°C$. A solution has the form

$$u(x, y, t) = \sum_{k,l=1}^{\infty} b_{kl}(t) \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{L}.$$

Compute the terms $b_{kl}(t)$ explicitly for all k and l.

End of Final Exam

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