(1) 10 POINTS Derive the heat equation for a one-dimensional rod assuming that the cross-sectional area $A(x)$ is a non-constant function of x, where $0 < x < L$. Assume all other thermal properties are constant, and there is no heat source.

Answer:

$$c\rho A(x)\frac{\partial u}{\partial t}(x,t) = K_0\frac{\partial}{\partial x}\Big(A(x)\frac{\partial u}{\partial x}(x,t)\Big)$$

(2) 10 POINTS The function $u(r, \theta)$ describes the steady state temperature distribution in a thin plate R shaped as an anulus with outer radius 2 and inner radius 1. Suppose that heat flux across the boundary of R is given by $u_r(1, \theta) = 2$ for the inner circle, and $u_r(2, \theta) = c \sin^2(3\theta)$ for the outer circle.

(a) (4 points) What must the value of the constant c? That is: what must the value of c be so that the boundary value problem

$$\left| \begin{array}{l} \nabla^2 u = 0 \\ u_r(1, \theta) = 2 \\ u_r(2, \theta) = c \sin^2(3\theta) \end{array} \right.$$

will have a solution.

(b) (6 points) Find the general solution $u(r, \theta)$, you don't need to compute the coefficients.

Answer:

(a) $c = 2$.
(b)

$$u(r, \theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (A_{m,n} r^n \cos m\theta + B_{m,n} r^{-n} \cos m\theta + C_{m,n} r^n \sin m\theta + D_{m,n} r^{-n} \sin m\theta)$$

(3) 10 POINTS Denote by $g(x)$ the Fourier sine series of the function e^x on the interval $[0, 1]$, that is,

$$e^x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x =: g(x).$$

Decide whether the following statements are true or false. To obtain the full credit, you must justify your reasoning.

(i) (2.5 points) g is an even function. True / False

(ii) (2.5 points) g is periodic. True / False

(iii) (2.5 points) g is bounded. True / False

(iv) (2.5 points) $g(1) = e$. True / False

Answer:
(i) F.
(ii) T.
(iii) T.
(iv) F.

(4) 10 POINTS Let $u(x,t)$ be the vertical displacement of a vibrating string with $1 < x < 3$. The string has free boundaries, and has constant density $\rho = 4$ and tension with constant magnitude $T = 1$ and thus obeys the wave equation $4u_{tt} = u_{xx}$, with boundary conditions $u_x(1,t) = 0$ and $u_x(3,t) = 0$. The initial position and velocity are given by $u(x,0) = 0$ and $u_t(x,0) = \sqrt{2x}$ for all $1 < x < 3$. Calculate the total energy

$$E(t) = \frac{1}{2}\int_1^3 \left(\rho u_t^2 + T u_x^2\right)\, dx,$$

of the string.

Answer:
$E(t) = 16.$

(5) 10 POINTS Estimate the *large* eigenvalues for the following eigenvalue problem:

$$\begin{cases} \dfrac{d}{dx}\left(e^{2x}\dfrac{d\phi}{dx}\right) + \left(\lambda e^{4x} + e^{3x}\right)\phi = 0 \text{ for } 0 < x < 1 \\ \phi(0) = \phi(1) = 0. \end{cases}$$

Answer:

$$\lambda_n \sim \left(\frac{n\pi}{e-1}\right)^2$$

(6) $\boxed{\text{10 POINTS}}$ Let $u(\rho, \theta, \phi)$ be a solution of the Laplace equation $\nabla^2 u = 0$ inside a sphere of radius 2 centered at the origin, subject to the boundary condition $u(2, \theta, \phi) = 5P_2^0(\cos\phi)$, where $P_2^0(x)$ is the associated Legendre function of the first kind.

(a) (5 points) Compute $u(1, \pi, 0)$.

(b) (5 points) Compute $\lim_{\rho\to 0} u(\rho, \pi, 0)$.

Answer:
(a)

$$u(1, \pi, 0) = \frac{5}{4} P_2^0(1) = \frac{5}{4}$$

(b)

$$\lim_{\rho\to 0} u(\rho, \pi, 0) = 0.$$

In fact,

$$u(\rho, \theta, \phi) = \frac{5}{8}\rho^2(3\cos^2\phi - 1).$$

(7) 10 POINTS Solve the Poisson equation in a unit disk ($r < 1$, $-\pi \le \theta \le \pi$):

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^3$$

subject to the boundary condition

$$u(1,\theta) = \sin\theta.$$

Answer:

$$u(r,\theta) = \frac{r^5 - 1}{25} + r\sin\theta$$

(8) 10 POINTS Compute the Fourier transforms of the following functions

(a) (3 points) $2e^{-x^2/9}$,

(b) (4 points) $e^{-(x+1)^2} * e^{-(x-1)^2}$,

(c) (3 points) $\frac{d^3}{dx^3} e^{-x^2}$

Answer:

(a)

$$6\sqrt{\pi}e^{-\frac{9\omega^2}{4}}$$

(b)

$$\frac{1}{2}e^{-\frac{\omega^2}{2}}$$

(c)

$$\frac{i\omega^3}{2\sqrt{\pi}}e^{-\frac{\omega^2}{4}}$$