(1) $\boxed{\text{10 POINTS}}$ Let $u := u(x,t)$ be the temperature in a one-dimensional rod, and satisfy the following initial and boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \beta x \qquad \text{for } 0 < x < 2,\ t > 0, \\ \partial_x u(0,t) &= 0 \qquad \text{and} \qquad \partial_x u(2,t) = -2, \\ u(x,0) &= \frac{8}{\pi} \cos \frac{\pi}{4} x, \end{aligned}$$

where β is a constant. Denote the total thermal energy in the rod ($0 < x < 2$) by

$$E(t) := \int_0^2 u(x,t)\, dx.$$

Answer the following questions.

(i) (2 Points) What is the physical meaning of the boundary condition $\partial_x u(0,t) = 0$?

(ii) (3 Points) Verify that

$$\frac{dE}{dt} = -2 + 2\beta.$$

(iii) (3 Points) Using part (ii), compute E.

(iv) (2 Points) For which value of β does the limit $\lim_{t\to\infty} E(t)$ exist, and what is the limit?

Answer:

(i) Meaning the left end is insulated.

(ii)

$$\frac{dE}{dt} = \int_0^2 u_t\, dx = \int_0^2 (u_x x + \beta x)\, dx = u_x(2,t) - u_x(0,t) + 2\beta = -2 + 2\beta$$

.

(iii) $E(t) = \frac{32}{\pi^2} + t(2\beta - 2)$

(iv) $\beta = 1$, limit is $\frac{32}{\pi^2}$.

(2) $\boxed{\text{10 POINTS}}$ Consider the following boundary value problem for a heat conduction in a rod:

$$\left|\begin{aligned} u_t &= 4u_{xx} - 2u, \quad \text{on } 0 < x < 1 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}\right.$$

(a) (5 points) Separate variables and determine all product solutions of this problem.

(b) (5 points) Find the solution for which the initial temperature is $u(x,0) = 3\sin(\pi x) + \sin(3\pi x)$.

Answer:

(a)

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{(-4n^2\pi^2-2)t} \sin n\pi x$$

(b)

$$u(x,t) = 3e^{(-4\pi^2-2)t} \sin \pi x + e^{(-36\pi^2-2)t} \sin 3\pi x$$

(3) $\boxed{\text{10 POINTS}}$ Let $f(x)$ be a piecewise smooth function. Denote by $g(x)$ the Fourier series of the function $f(x)$ on the interval $[-\pi, \pi]$, that is,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx =: g(x).$$

Decide whether the following statements are true or false. To obtain full credit, you must justify your answers.

(i) (2.5 points) If f is an odd function, then $b_n = 0$ for all $n = 1, 2, \cdots$. T / F

(ii) (2.5 points) If f is continuous, then g must be continuous. T / F

(iii) (2.5 points) If f is bounded, then g must be bounded. T / F

(iv) (2.5 points) For any given function f, the Fourier coefficient b_n can be computed by $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$. T / F

Answer:
(i) F.
(ii) F.
(iii) T.
(iv) F.

(4) $\boxed{\text{10 POINTS}}$ Solve the damped wave equation

$$u_{tt} = u_{xx} - u_t, \qquad 0 < x < 1,\ t > 0,$$

with initial conditions

$$u(x,0) = \sin(2\pi x)$$
$$u_t(x,0) = 0$$

and boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0.$$

Answer:

$$u(x,t) = \sin 2\pi x \Big(e^{-\frac{t}{2}} \cos \sqrt{16\pi^2 - 1}t + \frac{1}{2\sqrt{16\pi^2 - 1}} e^{-\frac{t}{2}} \sin \sqrt{16\pi^2 - 1}t \Big)$$

(5) $\boxed{\text{10 POINTS}}$ Consider the eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + (\lambda - x^4)\phi = 0 \\ \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(1) = 0. \end{cases}$$

Answer the following questions.

(i) (2 points) Is the above eigenvalue problem a *regular* Sturm-Liouville eigenvalue problem?

(ii) (5 points) Show that all eigenvalues λ are non-negative.

(iii) (3 points) Is $\lambda = 0$ an eigenvalue?

Answer:
(i) Yes.
(ii) Use Rayleigh quotient.
(iii) No.

(6) $\boxed{\text{10 POINTS}}$ The displacement function $u(r, \theta, t)$ describing the vibration of a circular membrane $D:\ 0 < r < 2,\ -\pi < \theta < \pi$ satisfies the boundary value problem

$$\left|\begin{aligned} u_{tt} &= 9\nabla^2 u \text{ on } D \\ u(2, \theta, t) &= 0 \\ u(r, \theta, 0) &= 0 \\ u_t(r, \theta, 0) &= -J_0\left(\frac{z_6}{2} r\right), \end{aligned}\right.$$

where $J_0(z)$ is the 0-th Bessel function of the first kind with roots $z_1,\ z_2,\ \ldots,\ z_n,\ \ldots$.

(a) (7 points) Compute $u(r, \theta, t)$.

(b) (3 points) Find the value $u\left(2, \frac{\pi}{6}, 1\right)$.

Answer:

(a)

$$u(r, \theta, t) = -\frac{2}{3z_6} \sin \frac{3z_6 t}{2} J_0\left(\frac{z_6 r}{2}\right)$$

(b)

$$u(2, \frac{\pi}{6}, 1) = 0$$

(7) $\boxed{\text{10 POINTS}}$ Let $u := u(x,t)$ satisfy the following initial and boundary value problem

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + x + 2e^{-\frac{\pi^2}{4}t}\sin\frac{\pi x}{2} \qquad \text{for } 0 < x < 1,\ t > 0 \\
u(0,t) &= 0 \\
\frac{\partial u}{\partial x}(1,t) &= t \\
u(x,0) &= 5\sin\frac{3\pi x}{2}.
\end{aligned}$$

Answer the following questions.

(ii) (8 Points) Find the solution u to the above initial and boundary value problem.

(iii) (2 Points) Prove or disprove

$$\lim_{t\to\infty} u(1,t) = 0.$$

Answer:

(ii)

$$u(x,t) = 2t\sin\frac{\pi x}{2}e^{-\frac{\pi^2 t}{4}} + 5\sin\frac{3\pi x}{2}e^{-\frac{9\pi^2 t}{4}} + xt$$

(iii) No, the limit is ∞.

Mysterious enough, this problem doesn't have part (i).

(8) $\boxed{\text{10 POINTS}}$ Let $u(x,t)$ be the solution of

$$\left|\begin{aligned} u_t &= 2u_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \sin(3\pi x). \end{aligned}\right.$$

(a) (3 points) Use Euler's formula to rewrite the initial data in terms of exponentials.

(b) (7 points) Use a Fourier transform in x to find $u(x,t)$.

Answer:

(a)

$$\sin 3\pi x = \frac{e^{3\pi i x} - e^{-3\pi i x}}{2i}$$

(b)

$$u(x,t) = \sin(3\pi x) e^{-18\pi^2 t}$$