MATH 241 FINAL EXAM
FALL 2013

NAME:

INSTRUCTOR: **Wong** **Pantev**

RECITATION NUMBER AND DAY/TIME:

Please *turn off and put away all electronic devices.* You may use both sides of a $8.5'' \times 11''$ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck!

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

QUESTION NUMBER	POINTS POSSIBLE	YOUR SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	

TOTAL SCORE	/80

A Partial Table of Integrals

$$\int_0^x u\cos nu\;du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\;du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\;du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\;du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\;du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\;du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\;du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

  $J_0(0) = 1,\; J_n(0) = 0$ if $n > 0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu - \frac{m^2}{\sin\phi}\right) g = 0$. Using the substitution $x = \cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + \left(\mu - \frac{m^2}{1-x^2}\right) g = 0$. This equation has bounded solutions only when $\mu = n(n+1)$ and $0 \le m \le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1 \le m \le n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,

$$\text{If } n \neq k \text{ then } \int_{-1}^{1} P_n^m(x)P_k^m(x)dx = 0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2 dx = \frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')' + (\alpha^2\rho^2 - n(n+1))f = 0$. If we define the spherical Bessel function $j_n(\rho) = \rho^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho = 0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x) = x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of j_m.

$$\text{If } m \neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx = 0 \quad \text{and} \quad \int_0^1 x^2(j_n(z_{nm}x))^2 dx = \frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{i\omega x}dx, \qquad \mathcal{F}^{-1}[U](x) = \int_{-\infty}^{\infty} U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha > 0$) | Fourier Transform Pairs ($\beta > 0$)

$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$	$u(x) = \mathcal{F}^{-1}[U]$	$U(\omega) = \mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}} e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\|x\|}$	$\frac{1}{2\pi} \frac{2\alpha}{x^2 + \alpha^2}$	$\frac{2\beta}{x^2 + \beta^2}$	$e^{-\beta\|\omega\|}$
$u(x) = \begin{cases} 0 & \|x\| > \alpha \\ 1 & \|x\| < \alpha \end{cases}$	$\frac{1}{\pi} \frac{\sin \alpha\omega}{\omega}$	$2\frac{\sin \beta x}{x}$	$U(\omega) = \begin{cases} 0 & \|\omega\| > \beta \\ 1 & \|\omega\| < \beta \end{cases}$
$\delta(x - x_0)$	$\frac{1}{2\pi} e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega - \omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2 U$
xu	$-i\frac{\partial U}{\partial \omega}$	$x^2 u$	$(-i)^2 \frac{\partial^2 U}{\partial \omega^2}$
$u(x - x_0)$	$e^{i\omega x_0} U$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) g(x - s) ds$	FG

(1) $\boxed{\text{10 POINTS}}$ Let $u := u(x,t)$ be the temperature in a one-dimensional rod, and satisfy the following initial and boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \beta x \qquad \text{for } 0 < x < 2,\ t > 0, \\ \partial_x u(0,t) &= 0 \qquad \text{and} \qquad \partial_x u(2,t) = -2, \\ u(x,0) &= \frac{8}{\pi} \cos \frac{\pi}{4} x, \end{aligned}$$

where β is a constant. Denote the total thermal energy in the rod $(0 < x < 2)$ by

$$E(t) := \int_0^2 u(x,t)\, dx.$$

Answer the following questions.

(i) (2 Points) What is the physical meaning of the boundary condition $\partial_x u(0,t) = 0$?

(ii) (3 Points) Verify that

$$\frac{dE}{dt} = -2 + 2\beta.$$

(iii) (3 Points) Using part (ii), compute E.

(iv) (2 Points) For which value of β does the limit $\lim_{t\to\infty} E(t)$ exist, and what is the limit?

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(2) 10 POINTS Consider the following boundary value problem for a heat conduction in a rod:

$$\begin{vmatrix} u_t = 4u_{xx} - 2u, \quad \text{on } 0 < x < 1 \\ u(0,t) = 0 \\ u(1,t) = 0 \end{vmatrix}$$

(a) (5 points) Separate variables and determine all product solutions of this problem.

(b) (5 points) Find the solution for which the initial temperature is $u(x,0) = 3\sin(\pi x) + \sin(3\pi x)$.

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(3) 10 POINTS Let $f(x)$ be a piecewise smooth function. Denote by $g(x)$ the Fourier series of the function $f(x)$ on the interval $[-\pi, \pi]$, that is,

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx =: g(x).$$

Decide whether the following statements are true or false. To obtain full credit, you must justify your answers.

(i) (2.5 points) If f is an odd function, then $b_n = 0$ for all $n = 1, 2, \cdots$. T / F

(ii) (2.5 points) If f is continuous, then g must be continuous. T / F

(iii) (2.5 points) If f is bounded, then g must be bounded. T / F

(iv) (2.5 points) For any given function f, the Fourier coefficient b_n can be computed by $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$. T / F

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(4) 10 POINTS Solve the damped wave equation

$$u_{tt} = u_{xx} - u_t, \qquad 0 < x < 1,\ t > 0,$$

with initial conditions

$$u(x,0) = \sin(2\pi x)$$
$$u_t(x,0) = 0$$

and boundary conditions

$$u(0,t) = 0$$
$$u(1,t) = 0.$$

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(5) 10 POINTS Consider the eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + (\lambda - x^4)\phi = 0 \\ \dfrac{d\phi}{dx}(0) = \dfrac{d\phi}{dx}(1) = 0. \end{cases}$$

Answer the following questions.

(i) (2 points) Is the above eigenvalue problem a *regular* Sturm-Liouville eigenvalue problem?

(ii) (5 points) Show that all eigenvalues λ are non-negative.

(iii) (3 points) Is $\lambda = 0$ an eigenvalue?

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(6) 10 POINTS The displacement function $u(r, \theta, t)$ describing the vibration of a circular membrane $D:\ 0 < r < 2,\ -\pi < \theta < \pi$ satisfies the boundary value problem

$$\left|\begin{array}{c} u_{tt} = 9\nabla^2 u \text{ on } D \\ u(2, \theta, t) = 0 \\ u(r, \theta, 0) = 0 \\ u_t(r, \theta, 0) = -J_0\left(\frac{z_6}{2} r\right), \end{array}\right.$$

where $J_0(z)$ is the 0-th Bessel function of the first kind with roots $z_1,\ z_2,\ \ldots,\ z_n,\ \ldots$.

(a) (7 points) Compute $u(r, \theta, t)$.

(b) (3 points) Find the value $u\left(2, \frac{\pi}{6}, 1\right)$.

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(7) 10 POINTS Let $u := u(x,t)$ satisfy the following initial and boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + x + 2e^{-\frac{\pi^2}{4}t}\sin\frac{\pi x}{2} \qquad \text{for } 0 < x < 1,\ t > 0 \\ u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(1,t) &= t \\ u(x,0) &= 5\sin\frac{3\pi x}{2}. \end{aligned}$$

Answer the following questions.

(ii) (8 Points) Find the solution u to the above initial and boundary value problem.

(iii) (2 Points) Prove or disprove

$$\lim_{t\to\infty} u(1,t) = 0.$$

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(8) 10 POINTS Let $u(x,t)$ be the solution of

$$\left| \begin{aligned} u_t &= 2u_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= \sin(3\pi x). \end{aligned} \right.$$

(a) (3 points) Use Euler's formula to rewrite the initial data in terms of exponentials.

(b) (7 points) Use a Fourier transform in x to find $u(x,t)$.

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