Instructors D. Krashen and P. Storm — December 12, 2007

Math 241
Final examination

Instructions. Answer the following problems carefully and completely. Show all your work. Do not use a calculator. You may use both sides of one $8\frac{1}{2} \times 11$ sheet of paper for handwritten notes you wrote yourself. Please turn in your sheet of notes with your exam. There are 100 points possible. Good luck!

Name ________________________

Instructors's name ________________________

TA's name and time ________________________

1.	(2)	________
2.	(14)	________
3.	(6)	________
4.	(2)	________
5.	(3)	________
6.	(8)	________
7.	(8)	________
8.	(5)	________
9.	(5)	________
10.	(10)	________
11.	(11)	________
12.	(6)	________
13.	(6)	________
14.	(14)	________
Total	(100)	________

Here are some integrals you can use:

$$\int_0^\infty xe^{-x} \sin(cx)\, dx = \frac{2c}{(1+c^2)^2}$$
$$\int_0^\infty xe^{-x} \cos(cx)\, dx = \frac{1-c^2}{(1+c^2)^2}$$

1. Write whether the following statement is true or false. (You do not need to show any work.) The product of an odd function f with an odd function g is an odd function.

2. Use a Fourier transform, a sine transform, or a cosine transform to find the displacement $u(x,t)$, for $x > 0$ and $t > 0$, of a semi-infinite string if

$$u(0,t) = 0, \quad u(x,0) = xe^{-x}, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

You may assume the constant a^2 of the wave equation is equal to 1. Your final answer may contain an integral.

Scratch paper

3. Find *any two* independent solutions $u(x, y)$ to the following PDE:

$$\frac{\partial^2 u}{\partial x \partial y} = u$$

Neither of your solutions can be the zero function.

4. Find a and b real numbers such that

$$\frac{10 - 5i}{6 + 2i} = a + ib.$$

5. Let

$$
\begin{aligned}
z_1 &= 2\cos(\pi/8) + 2i\sin(\pi/8) \\
z_2 &= 4\cos(3\pi/8) + 4i\sin(3\pi/8)
\end{aligned}
$$

Find a and b real numbers such that

$$\frac{z_1}{z_2} = a + ib.$$

6. Show the complex function $f(z) = \overline{z}$ is not analytic at $z = 0$.

7. Find all points z in $\mathbb{C}$ satisfying the equation

$$\sin z = 2.$$

Write the solutions in the form $a + ib$ for a and b real numbers.

8. Compute the contour integral

$$\oint_C \frac{z}{z^2 - \pi^2}\, dz$$

where C is the circle $|z| = 3$.

9. Determine the pole(s) of $5 - 6/z^2$. Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

10. Determine the pole(s) of

$$\frac{1}{1 - e^z}.$$

Find the order(s) of the pole(s). Compute the residue(s) at the pole(s).

Scratch paper

11. Compute the integral

$$\int_0^\pi \frac{1}{5+4\cos\theta}\,d\theta$$

Scratch paper

12. Let C be the curve in the complex plane parametrized by $C(t) = \cos(t) + i\sin(t)$, for $0 \leq t \leq \pi$. (Note the π!) Compute the value of the contour integral

$$\int_C \frac{dz}{z^2}$$

13. Consider the function

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x \le 2 \end{cases}$$

defined on the interval $[0, 2]$. Let

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)$$

be a sine series for $f(x)$. Using the same values for B_n, for all x in the real line define a function

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right).$$

Find $g(-5/2)$ and $g(-5)$.

14. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ for a function $u(x, y)$ with $0 \leq x \leq 2$, $0 \leq y \leq 1$ and boundary conditions:

$$u(0, y) = 0, \quad \frac{\partial u}{\partial x}(2, y) = 0, \quad u(x, 0) = 0,$$

$$u(x, 1) = 3 \sin\left(\frac{\pi x}{4}\right) - 2 \sin\left(\frac{5\pi x}{4}\right).$$

Scratch paper

More scratch paper