Math 241 Final Exam
Spring 2008

____
Name

1. Which of the following families of functions are orthogonal on the indicated sets? Justify your answers.

   (a) $\{\sin(n\pi x) : n = 1, 2, \ldots\}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

   (b) $\{e^{2ni\pi x} : n = 1, 2, 3, \ldots\}$ on the interval $[0, \frac{3}{4}]$. Remember: for complex valued functions we use the inner product defined by

$$\langle f, g\rangle = \int_0^{\frac{3}{4}} f(x)\overline{g(x)}dx. \tag{1}$$

2. Which of the following orthogonal families are complete on the indicated sets?

   (a) $\{\sin(2n\pi x) : n = 1, 2, \ldots\}$ on the interval $[0, \frac{1}{2}]$.

   (b) $\{\sin(2n\pi x)\sin(m\pi x) : n = 1, 2, 3, \ldots; m = 1, 2, 3 \ldots\}$ on the unit square $[0, 1] \times [0, 1]$.

3. Find a solution to the stationary heat equation

$$\nabla^2 u = 0$$

in a cylinder described as the the region in three-space $\mathbb{R}^3$ delimited by the surfaces

$$z = 0, \qquad z = 1, \qquad x^2 + y^2 = 4$$

assuming that:

- the lower base and the lateral surface are in thermal contact with a thermostat at temperature $u = 0$

- the upper base is in thermal contact with a thermostat at a temperature f depending on the radial coordinate r according to the law $f(r) = a(2-r)$, $0 \le r \le 2$.

(Write the integrals to compute the coefficients in the final series expansions without attempting to evaluate them).

4. Draw pictures of the following subsets of the complex plane (give as much detail and be as accurate as possible):

   (a) $\{z : z = \bar{z}\}$

   (b) $\{z : \operatorname{Im} z > (\operatorname{Re} z)^2\}$

5. For each of the following sums say whether it is absolutely convergent or not, and give a justification for your answer:

   (a) $\sum_{n=1}^{\infty} \left(\frac{2}{2+i}\right)^n$

   (b) $\sum_{n=1}^{\infty} \frac{n-i}{n+i}$

6. Give the radius of convergence of the following power series, justify your answer.

   (a) $\sum_{n=1}^{\infty} n^4 z^n$

   (b) $\sum_{n=1}^{\infty} \frac{z^n}{(n!)^2}$

7. Evaluate the following contour integral

$$\oint_{|z|=4} \frac{dz}{z^2-4} - \oint_{|z+2|=1} \frac{dz}{z^2-4} \tag{2}$$

8. Evaluate the following contour integral

$$\oint_{|z|=1} (2z^2 + 3\bar{z})dz. \tag{3}$$

9. For the following function, locate and classify all the singularities as a pole of some finite order, essential singularity, or removeable singularity. Compute the residues at the poles of finite order:

$$f(z) = \frac{e^{i\frac{z}{z+1}}}{z(z-2)^2}. \tag{4}$$

10. For the following functions give the radius of convergence of the Taylor series centered at the indicated point

    (a) $f(z) = \frac{e^z - 1}{z}$ centered as $z_0 = 1$.

    (b) $f(z) = \frac{(z+1)(z-2)}{(z-2i)^2}$ centered at $z_0 = 2$.

11. For the function

$$f(z) = \frac{1}{z(z-4)},$$

find the Laurent expansions valid in

   (a) $0 < |z| < 4$

   (b) $4 < |z|$

   (c) $0 < |z-4| < 4$

   (d) $4 < |z-4|$

Extra Credit: Suppose that $f(z)$ is an analytic function in the disk $\{z : |z-1| < 2\}$, and that $f^{[j]}(1) = 0$, for $j = 0, 1, 2, \ldots$ What can we conclude about $f(z)$ in this disk?