MATH 241 FINAL EXAM MAY 10, 2011 (CLEE, GUPTA, SHATZ)

The examination consists of 15 problems each worth 10 points; answer all of them. The first 10 are multiple choice, NO PARTIAL CREDIT given, but NO PENALTY FOR GUESSING. To answer, **CIRCLE THE ENTIRE STATEMENT YOU DEEM CORRECT** in the problem concerned. No work is required to be shown for these 10 problems, use your bluebooks for computations and scratch work. The next 5 are long answer questions, PARTIAL CREDIT GIVEN–YOU MUST SHOW YOUR WORK. SHOW THE WORK FOR THESE IN THE SPACE PROVIDED ON THE EXAM SHEETS. DO NOT USE BLUE BOOKS FOR THESE 5 QUESTIONS; DO NOT HAND IN YOUR BLUE BOOKS. No books, tables, notes, calculators, computers, phones or electronic equipment allowed; one 8 and 1/2 inch by 11 inch paper allowed, **handwritten on both sides, in your own handwriting**; no substitutions of this aid allowed.

NB. In what follows, we write u_x for $\frac{\partial u}{\partial x}$ and u_{xx} for $\frac{\partial^2 u}{\partial x^2}$, etc. in those problems concerning PDE.

YOUR NAME (print please):

YOUR PENN ID NUMBER:

YOUR SIGNATURE:

INSTRUCTOR'S NAME (CIRCLE ONE): CLEE GUPTA SHATZ

SCORE:

Do not write below–for grading purposes only.

I	VI	XI
II	VII	XII
III	VIII	XIII
IV	IX	XIV
V	X	XV

TOTAL:

I) If γ is the ellipse $x^2 + \frac{y^2}{4} = 1$, traced counterclockwise, evaluate $\oint_\gamma \frac{e^z}{z^2(z-2)}\, dz$.

a) $\frac{5\pi i}{2}$

b) $\frac{-3\pi i}{2}$

c) 0

d) $2\pi i$

e) $-4\pi i$

II) If $H(x) = -1$ for $-\pi \leq x < 0$ and $H(x) = 1$ for $0 \leq x < \pi$ and

if we extend H to be 2π periodic, then when we expand $H(x)$ in a complex Fourier Series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, we find the sum $c_{-2} + c_{-1} + c_o$ equals

a) $2\pi i$

b) $3\pi i$

c) $\frac{i}{\pi}$

d) $\frac{2i}{\pi}$

e) -1

III) The complex number $(i\pi)^i$ may have many values. One of its values is:

a) $e^{\frac{-\pi}{2}}(cos(log\pi) + i\ \ sin(log\pi))$

b) $log\pi + i\ \ log\pi$

c) $e^{\pi}(cos(log\pi) + i\ \ sin(log\pi))$

d) $cos(log\pi) + i\ \ sin(log\pi)$

e) $e^{-\pi}(cos(e^{\pi}) + i\ \ sin(e^{\pi}))$.

IV) For the Sturm-Liouville Problem

$$y'' + \lambda y = 0, \;\; y(0) = y(\pi) = 0,$$

the eigenvalues are:

a) $\lambda = k^2, \;\; k = 1, 2, 3, \cdots$

b) $\lambda = \frac{(2k+1)^2}{4}, \;\; k = 1, 2, 3, \cdots$

c) $\lambda = k, \;\; k = 1, 2, 3, \cdots$

d) $\lambda = \frac{k^2}{2}, \;\; k = 1, 2, 3, \cdots$

e) $\lambda = (2k+1)^2, \;\; k = 1, 2, 3, \cdots.$

V) Consider the function $f(x) = |x|$ for $-\pi < x \leq \pi$. We extend f to be 2π periodic and compute its Fourier Series

$$\frac{a_o}{2} + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx).$$

Then the sum $a_o + a_1 + b_1 + a_2 + b_2 + a_3 + b_3$ is:

a) $\frac{\pi^2}{2}$

b) $\frac{3\pi^2-10}{3\pi}$

c) $\frac{\pi^2-8}{\pi}$

d) $\frac{7\pi^2-22}{7\pi}$

e) $\frac{9\pi^2-40}{9\pi}$.

VI) Consider the heat equation $u_t = u_{xx}$ with boundary values $u(0,t) = u(1,t) = 0$ and with initial condition $u(x,0) = sin(6\pi x)$. Compute $u(\frac{1}{4}, \frac{1}{4})$:

a) $-exp(-5\pi^2)$

b) $exp(-7\pi^2)$

c) $-exp(-9\pi^2)$

d) $exp(-3\pi^2)$

e) $-exp(-\pi^2)$

VII) For the PDE: $u_x + 3u_y = 0$, we are interested only in solutions of the form $u(x,y) = X(x)Y(y)$. For such a solution, suppose we know $u(0,1) = e$ and $u(1,0) = e^2$. Compute $u(1,1)$:

a) $exp(\frac{1}{4})$

b) $exp(\frac{3}{4})$

c) $exp(\frac{5}{4})$

d) $exp(\frac{7}{4})$

e) $exp(\frac{9}{4})$

VIII) A certain function $u(x,y)$ is defined on the square whose vertices are (0,0), (1,0), (0,1) and (1,1). The function u satisfies the two PDE's: $u_{xx} = 0$ and $u_{yy} = 0$ and has values at the corners of the square given by

$$u(0,0) = 0, \;\; u(1,0) = 3, \;\; u(0,1) = -2, \;\; u(1,1) = 1.$$

For this function u, the value $u(\frac{1}{2}, \frac{1}{2})$ is:

a) 1

b) -2

c) 0

d) -3/2

e) 1/2

IX) We expand the function $f(z) = \frac{e^z}{(z-1)^2}$ in a Laurent Series valid in the annulus $1 < |(z-1)| < 2$. This series has the form

$$\sum_{m=1}^{\infty} b_m \frac{1}{(z-1)^m} + \sum_{k=o}^{\infty} a_k (z-1)^k.$$

Then the value of the product $b_2 b_1 a_o a_1 a_2$ is:

a) e^5

b) $e^5/288$

c) $e^5/120$

d) $e^5/84$

e) $e^5/316$.

X) For the function $f(z) = \frac{2z-1}{z(z-5)^2}$, find its residue at $z = 5$.

a) 0

b) i/2

c) 2/5

d) $\pi/4$

e) 1/25

THE NEXT 5 PROBLEMS ARE LONG ANSWER–WORK IS REQUIRED TO BE SHOWN.

XI) Compute the integral $\int_{-\infty}^{\infty} \frac{sin^2(x)}{1+x^2}\, dx$.

XII) If $u(x,y)$ is a harmonic function (that is, it satisfies Laplace's DE: $\Delta u = 0$) defined on the whole plane, then it is known that there is another harmonic function, call it $v(x,y)$, so that the complex function

$$f(z) = f(x,y) = u(x,y) + i\ \ v(x,y)$$

is an entire function (i.e., is holomorphic (= analytic) on the entire complex plane). Write $g(z)$ for the function

$$g(z) = e^{-f(z)}.$$

a) If $u(x,y)$ is always ≥ 0, show, with an *explicit* upper bound, that $|g(z)|$ is bounded.

b) Again assume $u(x,y) \geq 0$ for all x, y. Use a) to find all such (everywhere non-negative harmonic) functions—be careful, clear and explicit in your reasoning.

XIII) Consider the wave equation $u_{tt} = 4u_{xx}$ on the interval $[0, \pi]$ with boundary values $u(0,t) = u(\pi, t) = 0$. If the initial conditions are

$$u(x,0) = \frac{1}{3} sin(3x) + \frac{1}{4} sin(4x)$$
$$\mathrm{u}_t(x,0) = \frac{1}{5} sin(5x) + \frac{1}{6} sin(6x),$$

find (explicitly) the solution, $u(x,t)$, of the problem.

XIV) A certain rod (insulated along its length) has length 10 and is kept at 0^o at its ends. Physical units are adjusted so that its temperature, $u(x,t)$, satisfies the heat equation $u_t = 2u_{xx}$. Suppose that initially the temperature of the rod is $u(x,0) = sin(\frac{\pi x}{2})$. Find an explicit formula for the temperature, $u(x,t)$, of the rod at any time t. In particular, at what time will the temperature at the middle of the rod be $(1/2)^o$?

XV) Suppose $f(x) = |x| - \pi$ on $[-\pi, \pi]$ and is made 2π periodic. We compute the Fourier Series for $f(x)$:

$$\frac{a_o}{2} + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx).$$

Find the sum of the series

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k.$$

Be explicit with your reasoning—a simple numerical answer is not sufficient.

END OF THE EXAM