MATH 241 MAKE-UP FINAL JANUARY 12, 2012 (LU, SHATZ)

The examination consists of 15 problems, the first ten are worth 10 points each, the second five are worth 20 points each. Answer all of them. The first 10 are multiple choice, NO PARTIAL CREDIT given, but NO PENALTY FOR GUESSING. To answer, **CIRCLE THE ENTIRE STATEMENT YOU DEEM CORRECT** in the problem concerned. No work is required to be shown for these 10 problems, use your bluebooks for computations and scratch work. The next 5 are long answer questions, PARTIAL CREDIT GIVEN–YOU MUST SHOW YOUR WORK. SHOW THE WORK FOR THESE IN THE SPACE PROVIDED ON THE EXAM SHEETS. DO NOT USE BLUE BOOKS FOR THESE 5 QUESTIONS; DO NOT HAND IN YOUR BLUE BOOKS. No books, tables, notes, calculators, computers, phones or electronic equipment allowed; one 8 and 1/2 inch by 11 inch paper allowed, **handwritten on both sides, in your own handwriting**; no substitutions of this aid allowed.

NB. In what follows, we write u_x for $\frac{\partial u}{\partial x}$ and u_{xx} for $\frac{\partial^2 u}{\partial x^2}$, etc. in those problems concerning PDE.

YOUR NAME (print please):

YOUR PENN ID NUMBER:

YOUR SIGNATURE:

INSTRUCTOR'S NAME (CIRCLE ONE): LU SHATZ

SCORE:
Do not write below–for grading purposes only.

I	VI	XI
II	VII	XII
III	VIII	XIII
IV	IX	XIV
V	X	XV

TOTAL:

I) If γ is the ellipse $\frac{x^2}{9} + \frac{y^2}{2} = 1$, traced counterclockwise, evaluate $\oint_\gamma \frac{cos^2(\pi iz)}{z^2-iz+2}\,dz$.

a) $\pi/3$

b) $-2\pi/3$

c) 0

d) -4π

e) $4\pi/3$

II) If $g(x) = 3$ for $-\pi \leq x < 0$ and $g(x) = 0$ for $0 \leq x < \pi$ and if

we extend g to be 2π periodic, then when we expand $g(x)$ in a complex Fourier Series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, we find the limit

$$\lim_{R\to\infty} \sum_{k=-R}^{R} c_k$$

equals

a) $\frac{2}{\pi}$

b) 0

c) 2π

d) $1/2$

e) $3/2$

III) The complex function $f(z) = zcot^2 z$ is defined for $z \neq 0$ near $z = 0$.

a) $z = 0$ is a pole of order 2 with residue 0

b) $z = 0$ is a removable singularity with residue 0

c) $z = 0$ is an essential singularity with residue 1/6

d) $z = 0$ is a pole of order 3 with residue 1/6

e) $z = 0$ is a pole of order 1 with residue 1.

IV) For the Sturm-Liouville Problem

$$y'' + \lambda y = 0, \;\; y'(0) = y(2) = 0,$$

the eigenvalues are:

a) $\lambda = \frac{k\pi}{4}, k$ odd

b) $\lambda = \frac{k\pi}{8}, k$ odd

c) $\lambda = \frac{k^2\pi^2}{4}, k$ odd

d) $\lambda = \frac{k^2\pi^2}{16}, k$ odd

e) $\lambda = k^2\pi^2, k$ odd.

V) Consider the function $f(x) = |x|$ for $-\pi < x \leq \pi$. We extend f to be 2π periodic and compute its Fourier Series

$$\frac{a_o}{2} + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx).$$

Then the sum of the infinite series $\sum_{k=1}^{\infty} a_k^2 b_k^2$ is:

a) 1

b) $\frac{-\pi}{2}$

c) 0

d) $\frac{-\pi}{4}$

e) 2π.

VI) Consider the heat equation $u_t = u_{xx}$ with boundary values $u(0,t) = u(\pi, t) = 0$ and with initial condition $u(x, 0) = sin(3x)$. Then $u(\frac{\pi}{6}, \frac{1}{2})$ is:

a) $exp(-9/2)$

b) $exp(-7/2)$

c) $exp(-5/2)$

d) $exp(-3/2)$

e) $exp(-1/2)$

VII) Consider Laplace's PDE: $u_{xx} + u_{yy} = 0$, with three different sets of boundary conditions:

$$(BC_1): \quad u(0,y) = 0, u(1,y) = y, u(x,0) = 0, u(x,1) = -2x,$$
$$(BC_2): \quad u(0,y) = 0, u(1,y) = -y, u(x,0) = u(x,1) = 0,$$
$$(BC_3): \quad u(0,y) = u(1,y) = 0, u(x,0) = 0, u(x,1) = -x.$$

Write u_2 for a solution with (BC_2) and u_3 for a solution with (BC_3). Which combination below satisfies the DE and (BC_1)?

a) $2u_3 - u_2$

b) $2u_2 - u_3$

c) $u_2 + u_3$

d) $u_3 - u_2$

e) $2u_2 + u_3$

VIII) Consider three curves (all traced counter clockwise):

$$\gamma_1 : |z| = 2, \gamma_2 : |z - i| = 2, \gamma_3 : |z - \pi| = 2.$$

Define I_k as $\oint_{\gamma_k} z^3 exp(\frac{-1}{z^2})\, dz$, with $k = 1, 2, 3$. Then:

a) $I_1 = I_2 \neq I_3$

b) $I_2 = I_3 \neq I_1$

c) $I_1 \neq I_2 \neq I_3 \neq I_1$

d) $I_1 = I_3 \neq I_2$

e) $I_1 = I_2 = I_3$.

IX) We expand the function $f(x) = 0$ for $-\pi < x < 0$ and $f(x) = 1$ for $0 < x < \pi$ in a real Fourier Series: $\frac{a_o}{2} + \sum_{k=1}^{\infty} a_k coskx + b_k sinkx$. Then the value of the series $a_o + \sum_{k=1}^{\infty} a_k^2 b_k$ equals:

a) 3π

b) 2π

c) 2

d) π

e) 1.

X) Consider the Sturm-Liouville Problem on $[1, 2]$ given by

$x^2 y'' + (\lambda - x^3)y = 0$ and certain boundary conditions. The eigenfunctions, $f_j(x)$, satisfy an orthogonality relation

$$\int_1^2 f_j(x) f_k(x) w(x)\, dx = 0,\ j \neq k.$$

Then $w(x)$ equals:

a) x

b) 1

c) $1/x$

d) $\frac{1}{x^2}$

e) $\frac{1}{x^3}$

THE NEXT 5 PROBLEMS ARE LONG ANSWER–WORK IS REQUIRED

TO BE SHOWN ON THE EXAM SHEETS–NOT IN YOUR BLUE BOOKS.

XI) Compute the integral $\int_0^\infty \frac{cos6x}{x^2+9}\,dx$.

XII) Let $u(x, y)$ be a harmonic function (that is, it satisfies Laplace's DE: $\Delta u = u_{xx} + u_{yy} = 0$) in the infinite strip: $0 < x < \pi,\ \ y \leq 0$ and say it satisfies the partial boundary conditions: $u(0, y) = u(\pi, y) = 0$ and $u(x, y)$ is bounded as $y \to -\infty$.

a) Find the general solution with just these partial boundary conditions.

b) Now assume the last boundary condition $u(x, 0) = 5sin14x$. What is the explicit solution with all boundary conditions satisfied?

c) What is the value of $u(x, y)$ along the midline when $x = \pi/2$?

XIII) Consider the modified wave equation $u_{tt} = u_{xx} + cosx$ on the interval $[0, \pi]$. It has a special solution $u(x,t) = cosx$.
a) Solve this DE with the boundary conditions: $u(0,t) = 1, u(\pi,t) = -1$.

b) If now initial conditions are given: $u(x,0) = cosx$ and $u_t(x,0) = 1$, explicitly find the solution, $u(x,t)$, of the problem.

XIV) Given $f(x) = x$ on $[0, \pi]$, we extend it to be an even function $(f(-x) = f(x))$ on $[-\pi, \pi]$.

a) Compute its Fourier Series on $[-\pi, \pi]$.

b) What is the sum of the series $1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \frac{1}{15^2} + \cdots$?

XV) Compute the complex Fourier Series for $cosx$ on the interval $[-1, 1]$, where we extend $cosx$ to be periodic with period 2.

END OF THE EXAM (EXTRA SHEET OF PAPER FOLLOWS)