Math 241 Final Exam
Fall 2012

Name:

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Recitation Number and Day/Time:

Please *turn off and put away all electronic devices*. You may use both sides of a 8.5″ × 11″ sheet of paper for handwritten notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**. Please **clearly mark** your final answer. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Your signature

Question Number	Points Possible	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	

Question Number	Points Possible	Your Score
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

A Partial Table of Integrals

$$\int_0^x u\cos nu\ du = \frac{\cos nx + nx\sin nx - 1}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x u\sin nu\ du = \frac{\sin nx - nx\cos nx}{n^2} \qquad \text{for any real } n \neq 0$$

$$\int_0^x e^{mu}\cos nu\ du = \frac{e^{mx}(m\cos nx + n\sin nx) - m}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x e^{mu}\sin nu\ du = \frac{e^{mx}(-n\cos nx + m\sin nx) + n}{m^2+n^2} \qquad \text{for any real } n, m$$

$$\int_0^x \sin nu\cos mu\ du = \frac{m\sin nx\sin mx + n\cos nx\cos mx - n}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \cos nu\cos mu\ du = \frac{m\cos nx\sin mx - n\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

$$\int_0^x \sin nu\sin mu\ du = \frac{n\cos nx\sin mx - m\sin nx\cos mx}{m^2-n^2} \qquad \text{for any real numbers } m \neq n$$

Formulas Involving Bessel Functions

- Bessel's equation: $r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cJ_n(\alpha r)$.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

  $J_0(0) = 1$, $J_n(0) = 0$ if $n > 0$. z_{nm} is the mth positive zero of $J_n(x)$.

- Orthogonality relations:

$$\text{If } m \neq k \text{ then } \int_0^1 xJ_n(z_{nm}x)J_n(z_{nk}x)\,dx = 0 \quad \text{and} \quad \int_0^1 x(J_n(z_{nm}x))^2\,dx = \frac{1}{2}J_{n+1}(z_{nm})^2.$$

- Recursion and differentiation formulas:

$$\frac{d}{dx}(x^nJ_n(x)) = x^nJ_{n-1}(x) \quad \text{or} \quad \int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + C \quad \text{for } n \geq 1 \tag{1}$$

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x) \quad \text{for} \quad n \geq 0 \tag{2}$$

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x) \tag{3}$$

$$J_n'(x) - \frac{n}{x}J_n(x) = -J_{n+1}(x) \tag{4}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{5}$$

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{6}$$

- Modified Bessel's equation: $r^2R'' + rR' - (\alpha^2r^2 + n^2)R = 0$ – The only solutions of this which are bounded at $r = 0$ are $R(r) = cI_n(\alpha r)$.

$$I_n(x) = i^{-n}J_n(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Formulas Involving Associated Legendre and Spherical Bessel Functions

- Associated Legendre Functions: $\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right)+\left(\mu-\frac{m^2}{\sin\phi}\right)g=0$. Using the substitution $x=\cos\phi$, this equation becomes $\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right)+\left(\mu-\frac{m^2}{1-x^2}\right)g=0$. This equation has bounded solutions only when $\mu=n(n+1)$ and $0\le m\le n$. The solution $P_n^m(x)$ is called an associated Legendre function of the first kind.

- Associated Legendre Function Identities:

$$P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \text{ and } P_n^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \text{ when } 1\le m\le n$$

- Orthogonality of Associated Legendre Functions: If n and k are both greater than or equal to m,

$$\text{If } n\neq k \text{ then } \int_{-1}^{1}P_n^m(x)P_k^m(x)dx=0 \text{ and } \int_{-1}^{1}\left(P_n^m(x)\right)^2dx=\frac{2(n+m)!}{(2n+1)(n-m)!}.$$

- Spherical Bessel Functions: $(\rho^2 f')'+(\alpha^2\rho^2-n(n+1))f=0$. If we define the spherical Bessel function $j_n(\rho)=\rho^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\rho)$, then only solution of this ODE bounded at $\rho=0$ is $j_n(\alpha\rho)$.

- Spherical Bessel Function Identity:

$$j_n(x)=x^2\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right).$$

- Spherical Bessel Function Orthogonality: Let z_{nm} be the m-th positive zero of j_n.

$$\text{If } m\neq k \text{ then } \int_0^1 x^2 j_n(z_{nm}x)j_n(z_{nk}x)dx=0 \quad\text{and}\quad \int_0^1 x^2(j_n(z_{nm}x))^2dx=\frac{1}{2}(j_{n+1}(z_{nm}))^2.$$

One-Dimensional Fourier Transform

$$\mathcal{F}[u](\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x)e^{i\omega x}dx,\qquad \mathcal{F}^{-1}[U](x)=\int_{-\infty}^{\infty}U(\omega)e^{-i\omega x}d\omega$$

Table of Fourier Transform Pairs

Fourier Transform Pairs ($\alpha>0$)		Fourier Transform Pairs ($\beta>0$)	
$u(x)=\mathcal{F}^{-1}[U]$	$U(\omega)=\mathcal{F}[u]$	$u(x)=\mathcal{F}^{-1}[U]$	$U(\omega)=\mathcal{F}[u]$
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\beta}}e^{-\frac{x^2}{4\beta}}$	$e^{-\beta\omega^2}$
$e^{-\alpha\|x\|}$	$\frac{1}{2\pi}\frac{2\alpha}{x^2+\alpha^2}$	$\frac{2\beta}{x^2+\beta^2}$	$e^{-\beta\|\omega\|}$
$u(x)=\begin{cases}0 & \|x\|>\alpha\\ 1 & \|x\|<\alpha\end{cases}$	$\frac{1}{\pi}\frac{\sin\alpha\omega}{\omega}$	$2\frac{\sin\beta x}{x}$	$U(\omega)=\begin{cases}0 & \|\omega\|>\beta\\ 1 & \|\omega\|<\beta\end{cases}$
$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	$e^{-i\omega_0 x}$	$\delta(\omega-\omega_0)$
$\frac{\partial u}{\partial t}$	$\frac{\partial U}{\partial t}$	$\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 U}{\partial t^2}$
$\frac{\partial u}{\partial x}$	$-i\omega U$	$\frac{\partial^2 u}{\partial x^2}$	$(-i\omega)^2U$
xu	$-i\frac{\partial U}{\partial\omega}$	x^2u	$(-i)^2\frac{\partial^2 U}{\partial\omega^2}$
$u(x-x_0)$	$e^{i\omega x_0}U$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}f(s)g(x-s)ds$	FG

1. Suppose the concentration $u(x,t)$ of a chemical satisfies Fick's law: $\phi = -k\frac{\partial u}{\partial x}$. Assume there is no source to generate chemicals in the region $0 \leq x \leq 1$ and that cross-section area $A(x) = 1$. Answer the following questions under the assumption that initial concentration is $u(x,0) = x(1-x)$ and the flow of chemical at both ends is specified by the boundary conditions $-k\frac{\partial u}{\partial x}(0,t) = a$ and $-k\frac{\partial u}{\partial x}(1,t) = b$.

   (a) Determine the rate of change of the total amount of chemical in the region.

   (b) Determine the total amount of chemical in the region as a function of time.

   (c) Under what condition is there an equilibrium distribution of the chemical and what is it?

2. Solve the Laplace equation on the half disk $0 \le \theta \le \pi$, $0 \le r \le 1$, subject to the boundary conditions $u(1, \theta) = \theta + 2$ and $\frac{\partial u}{\partial \theta}(r, 0) = \frac{\partial u}{\partial \theta}(r, \pi) = 0$.

3. Consider

$$x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

(a) Determine the b_n.

(b) For which values of x is $x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$?

(c) Use your answer from (a) to compute the Fourier cosine series for x^2.

4. Consider the displacement u of a string with source term $Q = -\alpha u$ (where α is positive constant), given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \alpha u.$$

(a) What is the physical effect of the source term Q on the string?

(b) Using separation of variables, solve the equation subject to the constraints

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = f(x).$$

(c) In the case of part (b), what are the (circular) frequencies of vibration of the string?

5. Express the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + 4\frac{d\phi}{dx} = -\lambda\frac{x^2}{1+3x^2}\phi \text{ on } [1,2] \text{ with } \phi(1) = \phi(2) = 0$$

in the standard Sturm-Liouville form. Explicitly verify that the Sturm-Liouville problem is regular and give the orthogonality relationship satisfied by the eigenfunctions.

6. Write the general solution of Laplace's equation on the unit ball in $\mathbb{R}^3$ as an infinite series. Verify that $u(x, y, z) = (x^2 + y^2)z - \frac{2}{3}z^3$ satisfies the Laplace equation on the unit ball in $\mathbb{R}^3$, then express this solution u as an infinite series, i.e., determine exactly what each coefficient in your infinite series must equal so that the sum of the series is u. (Hint: $P_3^0(\cos\phi) = 1$ at the north pole of the unit sphere.)

7. Let $u(x,t)$ be the solution of the initial/boundary value problem for the heat equation:

$$u_t = u_{xx} \quad \text{for } 0 < x < 1,\ t > 0$$

$$u(x,0) = 0, \quad u(0,t) = 0, \quad u(1,t) = 1.$$

(a) What is $\lim_{t\to\infty} u(x,t)$?

(b) Find $u(x,t)$.

(c) Use the equilibrium solution and one term from the Fourier series of $u(x,t)$ to estimate how long from $t = 0$ it will take for $u(\frac{1}{2},t)$ to attain half its equilibrium value (in other words, estimate the value of T such that

$$u\left(\tfrac{1}{2},t\right) < \frac{1}{2}\lim_{t\to\infty} u\left(\tfrac{1}{2},t\right) \quad \text{for } t < T$$

and

$$u\left(\tfrac{1}{2},t\right) > \frac{1}{2}\lim_{t\to\infty} u\left(\tfrac{1}{2},t\right) \quad \text{for } t > T.$$

(Bonus points for explaining why this is a really good estimate.)

8. Consider the following forced wave equation on the interval $[0, L]$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \sin \alpha t.$$

Suppose that boundary conditions are prescribed so that one end is fixed at zero and the other end is free. For what values of α is resonance possible? How does the situation change if both ends are free? Note that you do not need to write down the solution of this PDE to answer this question.

9. Let

$$f(x) = \begin{cases} \cos x & \text{for } -10\pi \le x \le 10\pi \\ 0 & \text{for } |x| > 10\pi \end{cases}$$

(a) Compute the Fourier transform of f. (Hint: You may wish to use Euler's formula to rewrite cosine in terms of complex exponentials.)

(b) Solve initial-value problem for the wave equation

$$u_{tt} = 100\pi^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = 0$$

with $f(x)$ as given at the beginning of this problem.

(c) Draw a reasonably accurate graph of $u(x,3)$.

10. Solve the initial-value problem

$$u_t = 7u_{xx}\,, \qquad u(x,0) = e^{-x^2}$$

for $u(x,t)$, $t > 0$ and $-\infty < x < \infty$. Fully simplify your answer.