University of Pennsylvania
Math 241 Fall 2008 **NAME_____________________________**
Final Exam

Student ID #_______________

Circle the name of your professor

NICOARA **RIMMER**

Work all problems in the space provided. You may use the back of each sheet for additional space. Please indicate when you do so.

There are 5 multiple choice questions each worth 10 points graded for partial credit on a scale of 0-4-7-10.

There are 5 free-response questions each worth 15 points.

Please write legibly so that the proper credit may be given. You must show all work, an unsupported answer will receive little or no credit.

Do not write below this line. The table is for grading purposes only.

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total Pts.	

Multiple Choice: Circle the correct answer choice. These questions will be graded for partial credit so be careful to show all work legibly. A correct answer with no work will receive little or no credit.

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{6}$ D) $\frac{\pi}{2}$

E) $\frac{\pi}{4}$ F) $\frac{4\pi}{3}$ G) 0 H) none of these

2. $\displaystyle P.V.\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+13}$

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{6}$ D) $\frac{\pi}{2}$

E) $\frac{\pi}{4}$ F) $\frac{4\pi}{3}$ G) 0 H) none of these

3. $\displaystyle\int_{C_1} \frac{e^{z^2}}{(z-i)^4}dz = \alpha$ $\qquad$ $\displaystyle\int_{C_2} \frac{e^{i\pi z}}{2z^2-5z+2}dz = \beta$

$C_1 : |z| = 2$ $\qquad$ $C_2 : |z| = 1$

Find $\alpha+\beta$

A) $\frac{\pi}{3}\left(2-\frac{1}{e}\right)$ B) $\frac{\pi}{3}\left(2-\frac{2}{e}\right)$ C) $\frac{2\pi}{3}\left(2-\frac{1}{e}\right)$ D) $\frac{2\pi}{3}\left(1-\frac{2}{e}\right)$

E) $\frac{4\pi}{3}\left(1-\frac{1}{e}\right)$ F) $\frac{\pi}{3}\left(4-\frac{1}{e}\right)$ G) 0 H) none of these

4. The solution $u(x,t)$ to a certain wave equation is known to be given by

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_{2n}\cos\left(\frac{2n\pi a}{L}t\right)+B_{2n}\sin\left(\frac{2n\pi a}{L}t\right)\right)\sin\left(\frac{2n\pi}{L}x\right).$$

$u(x,0)$ can be any of the following **EXCEPT** for

A) $u(x,0)=3\sin\left(\frac{2\pi}{L}x\right)+\sin\left(\frac{6\pi}{L}x\right)$

B) $u(x,0)=2\sin\left(\frac{2\pi}{L}x\right)-5\sin\left(\frac{4\pi}{L}x\right)$

C) $u(x,0)=7\sin\left(\frac{2\pi}{L}x\right)+5\sin\left(\frac{8\pi}{L}x\right)$

D) $u(x,0)=4\sin\left(\frac{2\pi}{L}x\right)-6\sin\left(\frac{7\pi}{L}x\right)$

E) $u(x,0)=3\sin\left(\frac{4\pi}{L}x\right)+5\sin\left(\frac{8\pi}{L}x\right)$

F) $u(x,0)=\sin\left(\frac{6\pi}{L}x\right)-3\sin\left(\frac{10\pi}{L}x\right)$

G) $u(x,0)=5\sin\left(\frac{2\pi}{L}x\right)+3\sin\left(\frac{4\pi}{L}x\right)-7\sin\left(\frac{6\pi}{L}x\right)$

H) $u(x,0)=4\sin\left(\frac{4\pi}{L}x\right)-3\sin\left(\frac{8\pi}{L}x\right)$

5. Consider the following statements:

(I) The heat equation $ku_{xx}=u_t$ on a rod of length L with boundary conditions $u(0,t)=u(L,t)=0$ only has the trivial solution $u(x,t)=0$.

(II) The wave equation $a^2u_{xx}=u_{tt}$ on a string of length L with boundary conditions $u(0,t)=u(L,t)=0$ only has the trivial solution $u(x,t)=0$.

(III) The Laplace equation $u_{xx}+u_{yy}=0$ on a square of side length 1 with boundary conditions $u(0,y)=u(1,y)=u(x,0)=u(x,1)=0$ only has the trivial solution $u(x,y)=0$.

Out of these three statements, the following are true:

A) I only
B) I and II only
C) II only
D) I and III only
E) III only
F) II and III only
G) I, II, and III
H) none of these

Free Response:

6. Each of the following are real numbers. Use only the principal value $\left(\operatorname{Arg} z\right)$ in each case. Find the real number for each.

$a)\ \cos\left(2i\ln\left(\frac{1-i}{1+i}\right)\right)$ $b)\ \arcsin\left(\left[\frac{\sqrt{3}+i}{\sqrt{3}-i}\right]^{12}\right)$ $c)\ \left(i\right)^{\ln i}$

7. Let $f\left(z\right)=\frac{-3}{\left(z-2\right)\left(z+1\right)}$

Find the Laurent series valid

$a)$ in the annulus $1<|z|<2$

$b)$ in the region $|z|>2$

$c)$ in the region $0<|z+1|<3$

8. Find the Fourier series of $f\left(x\right)=\sin^4 x$ and justify your answer.
(Hint: You might not have to integrate after all...)

9. The Fourier-Legendre series of a function $f(x)$ defined on the interval $(-1,1)$ is given by

$f(x)=\sum_{n=0}^{\infty} c_n P_n(x)$, where $c_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$.

The Legendre polynomials are given by $P_0=1, P_1=x, P_2=\frac{1}{2}(3x^2-1), P_3=\frac{1}{2}(5x^3-3x),\ldots$

For all n, if n is odd, then $P_n(x)$ is an odd function and if n is even, then $P_n(x)$ is an even function.

a) Assume $f(x)$ is an odd function on $(-1,1)$. Show that its Fourier-Legendre series is given by $f(x)=\sum_{n=0}^{\infty} c_{2n+1}P_{2n+1}(x)$, where $c_{2n+1}=(4n+3)\int_{0}^{1} f(x)P_{2n+1}(x)\,dx.$

b) The expression in part a) can be used to expand a function $f(x)$ defined only on $(0,1)$. If $f(x)=1$ on $(0,1)$, what are the first two non-zero coefficients in its Fourier-Legendre expansion?

c) The Fourier-Legendre expansion for $f(x)=1$ on $(0,1)$ converges on all of $(-1,1)$. What function does it represent on $(-1,1)$?

10. a) Compute the Fourier Integral of

$$f(x)=\begin{cases} e^x & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

b) To what values does the Fourier integral of $f(x)$ converge at -1 and 1? Justify your answer.

Answers:

1. A
2. B
3. D
4. D
5. E
6. a) -1 b) $\frac{\pi}{2}$ c) $e^{\frac{-\pi^2}{4}}$
7. a) $\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{z^{n+1}}+\frac{z^n}{2^{n+1}}\right]$ b) $\sum_{n=0}^{\infty}\frac{(-1)^n-2^n}{z^{n+1}}$ c) $\sum_{n=0}^{\infty}\frac{(z+1)^{n-1}}{3^n}$
8. $\frac{1}{8}\cos(4x)-\frac{1}{2}\cos(2x)+\frac{3}{8}$
9. a) Proof b) $c_1=\frac{3}{2}, c_3=\frac{-7}{8}$ c) $f(x)=\begin{cases}-1 & \text{if } -1<x<0\\ 1 & \text{if } \ 0<x<1\end{cases}$
10. a) $A(\alpha)=\dfrac{\frac{\alpha(e^2+1)}{e}\sin\alpha+\frac{e^2-1}{e}\cos\alpha}{\alpha^2+1}$ $B(\alpha)=\dfrac{\frac{e^2+1}{e}\sin\alpha+\frac{\alpha(1-e^2)}{e}\cos\alpha}{\alpha^2+1}$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\left(A(\alpha)\cos(\alpha x)+B(\alpha)\sin(\alpha x)\right)d\alpha$$

b) at $x=-1$: $\dfrac{1}{2e}$ at $x=1$: $\dfrac{e}{2}$