Math 241 Makeup Final Answers

1) True False

1. True.
2. False.
3. True.
4. False.
5. False.
6. False.
7. False.
8. True.

2) Short Answer

1. 12.
2. $2\pi i$.
3. $e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.
4. Essential.
5. 0. The function is analytic.
6. 7. All the Fourier coefficients are zero except for 7, the coefficient on $\cos 2x$.
7. $\sqrt{2}$.
8. 3.
9. -4.

3)

1. The integrand has simple poles at $z = i, -i$. Both poles are inside the contour C.

$$\mathrm{Res}(\frac{z^2-4}{z^2+1}, i) = \frac{i^2-4}{2i} = \frac{-5}{2i} = \frac{5}{2}i.$$

$$\mathrm{Res}(\frac{z^2-4}{z^2+1}, -i) = \frac{(-i)^2-4}{-2i} = \frac{-5}{-2i} = -\frac{5}{2}i.$$

So

Answer:
$$\int_C \frac{z^2-4}{z^2+1}\,dz = 2\pi i(\frac{5}{2}i - \frac{5}{2}i) = 0$$

2. The integrand has a pole of order 2 at 2. The pole is inside the contour C.

$$\mathrm{Res}(\frac{e^z}{(z-2)^2}, 2) = \lim_{z\to 2}\frac{d}{dz}[(z-2)^2\frac{e^z}{(z-2)^2}] = \lim_{z\to 2} e^z = e^2.$$

So

Answer:
$$\int_C \frac{e^z}{(z-2)^2}\,dz = 2\pi i(e^2) = 2e^2\pi i.$$

3. The integrand, which we think of as $\frac{\sin z}{z\cos z}$, has simple poles at 0 and all odd multiples of $\frac{\pi}{2}$. The poles inside C are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{5\pi}{2}$. Note that $z = 0$ is a removable singularity and hence its residue is 0.

$$\mathrm{Res}(\frac{\sin z}{z\cos z}, \frac{\pi}{2}) = \frac{1}{0-(\frac{\pi}{2})} = -\frac{2}{\pi}$$

$$\mathrm{Res}(\frac{\sin z}{z\cos z}, \frac{3\pi}{2}) = \frac{-1}{0-(\frac{3\pi}{2})(-1)} = -\frac{2}{3\pi}$$

Similarly $\mathrm{Res}(\frac{\sin z}{z\cos z}, \frac{3\pi}{2}) = -\frac{2}{5\pi}$. So

Answer:
$$\int_C \frac{\tan z}{z}\,dz = -2\pi i(\frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi}) = -2\pi i\frac{30+10+6}{15\pi} = -\frac{92}{15}i.$$

4)

1. $f(z)$ has three different Laurent expansions about $z = 1$. One good in $|z-1| < 1$ one good in $1 < |z-1| < 4$ and one good in $|z-1| > 4$.

2. Using partial fractions you find

$$f(z) = -\frac{\frac{1}{5}}{z+3} + \frac{\frac{1}{5}}{z-2}$$

Recall $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ good for $|w| < 1$. So

$$\frac{1}{4+(z-1)} = \frac{1}{4(1+\frac{z-1}{4})} = \frac{1}{4}\sum_{n=0}^{\infty}(-\frac{1}{4})^n(z-1)^n = -\sum_{n=0}^{\infty}(-\frac{1}{4})^{n+1}(z-1)^n$$

Similarly $\frac{1}{1-w} = \sum_{n=0}^{\infty} -(\frac{1}{w})^{n+1}$ for $|w| > 1$. So

$$\frac{1}{-1+(z-1)} = \frac{1}{-(1-(z-1))} = -\sum_{n=0}^{\infty} -(z-1)^{-(n+1)} = \sum_{n=0}^{\infty}(z-1)^{-(n+1)}$$

Thus

Answer:

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{5}(-\frac{1}{4})^{n+1}(z-1)^n + \sum_{n=0}^{\infty}\frac{1}{5}(z-1)^{-(n+1)}$$

5)

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 (3x-4)\sin\frac{n\pi}{3}x\,dx \\ &= \frac{2}{3}\left(-\frac{3}{n\pi}(3x-4)\cos\frac{n\pi}{3}x + \frac{9}{n^2\pi^2}3\sin\frac{n\pi}{3}x\right)\Big|_0^3 \\ &= \frac{2}{3}(-\frac{15}{n\pi}(-1)^n + 0 + \frac{-12}{n\pi} - 0) = -\frac{2}{3n\pi}(12+(-1)^n 15) \end{aligned}$$

So the Fourier sine expansion is

Answer:

$$\sum_{n=1}^{\infty} -\frac{2}{3n\pi}(12+(-1)^n 15)\sin\frac{n\pi}{3}x$$

6) 1. Make the substitution $z = e^{ix}$ so $dx = \frac{1}{iz}dz$ and $\cos x = \frac{1}{2}(z + \frac{1}{x})$. Thus if C is the unit circle our integral becomes

$$\int_C \frac{1}{2 - \frac{1}{2}(z + \frac{1}{z})} \frac{1}{iz} dz = 2i \int_C \frac{1}{z^2 - 4z + 1} dz.$$

The integrand has simple poles at $z = 2 \pm \sqrt{3}$, but only $2 - \sqrt{3}$ is inside the contour C. Thus

$$\int_C \frac{1}{z^2 - 4z + 1} dz = 2\pi i(\text{Res}(\frac{1}{z^2 - 4z + 1}, 2 - \sqrt{3}) = 2\pi i \frac{1}{2(2 - \sqrt{3}) - 4} = -\frac{\pi i}{\sqrt{3}}.$$

Thus

Answer:

$$2i(-\frac{\pi i}{\sqrt{3}}) = \frac{2\pi}{\sqrt{3}}.$$

2. For this type of integral we compute $\int_{-\infty}^{\infty} \frac{x^2 e^{(i\pi x)}}{(x^2+1)(x^2+2)} dx$ and take the real part. So

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2 e^{(i\pi x)}}{(x^2 + 1)(x^2 + 2)} dx = \lim_{R\to\infty} \int_{I_R} \frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)} dz$$
$$= \lim_{R\to\infty} \left(\int_{C_R} \frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)} dz + \int_{S_R} \frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)} dz \right).$$

Where $I_R = \{-R \leq x \leq R\}, S_R$ is the circle of radius R about the origin and C_R is their union. The only singularities inside C_R are two poles at i and $\sqrt{2}i$. so

$$\int_{C_R} \frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)} dz = 2\pi i(\text{Res}(\frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)}, i) + \text{Res}(\frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)}, \sqrt{2}i))$$
$$= 2\pi i(\frac{-e^{-\pi}}{2i} + \frac{-2e^{-\pi\sqrt{2}}}{-2\sqrt{2}i}) = -\frac{\pi(2e^{-\pi\sqrt{2}} + \sqrt{2}e^{-\pi})}{\sqrt{2}}$$

Also

$$\lim_{R\to\infty} \int_{S_R} \frac{z^2 e^{(i\pi z)}}{(z^2 + 1)(z^2 + 2)} dz = 0$$

by a theorem from class. Thus

Answer:

$$-\frac{\pi(2e^{-\pi\sqrt{2}} + \sqrt{2}e^{-\pi})}{\sqrt{2}}$$

7) Take the Laplace transform of both sides to get

$$\mathcal{L}\{y'' - 5y' + 4y = 0\} = 0$$

so

$$s^2\hat{y} - sy(0) - y'(0) - 5s\hat{y} + 5y(0) + 4\hat{y} = 0$$
$$(s^2 - 5s + 4)\hat{y} = 2$$
$$\hat{y} = \frac{2}{s^2 - 5s + 4} = \frac{2}{3}\frac{1}{s-4} - \frac{2}{3}\frac{1}{s-1}.$$

Thus

$$y(t) = \mathcal{L}^{-1}\{\hat{y}\} = \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s-4}\} - \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s-1}\}.$$

So

Answer:

$$y(t) = \frac{2}{3}e^{4t} - \frac{2}{3}e^{t}.$$

8) 1. Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function. The real part of f is $u(x,y)$.

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{xy} - v_{xy} = 0,$$

where the second equality follows from the Cauchy Riemann equations.
2. The function $u(x,y) = xy - 1$ is clearly harmonic. If $v(x,y)$ is its harmonic conjugate then $v_y = u_x = y$ and $v_x = -u_y = -x$. Integrating the first equation with respect to y gives

$$v(x,y) = \frac{1}{2}y^2 + g(x).$$

Thus $v_x = g'(x)$ but this must be $-x$ so $g(x) = -\frac{1}{2}x^2 + c$. So

Answer:

$$v(x,y) = \frac{1}{2}(y^2 - x^2 + c)$$

9) The equation becomes

$$X''T - 2XT'' + 7XT' + XY = 0.$$

Dividing by XT and rearranging we get

$$\frac{X''}{X} = \frac{1}{T}(2T'' - 7T') - 1.$$

Since each side is a function of a different variable they must both be constant. Let the constant be k. So we get equations for X and T:

$$X'' - kX = 0$$

$$2T'' - 7T' - (k+1)T = 0$$

The boundary conditions for u imply $X(0) = 0$ and $X(7) = 0$. As we have seen many times this leads to solutions

$$X_n(x) = \sin(\frac{n\pi}{7}x).$$

Plugging into the equation for T give

$$T_n(x) = a_n e^{\lambda_n t} + b_n e^{\lambda'_n t}$$

where $\lambda_n = \frac{7+\sqrt{49+8((1-\frac{n\pi}{7})^2)}}{4}$ and $\lambda_n = \frac{7-\sqrt{49+8((1-\frac{n\pi}{7})^2)}}{4}$. Thus the solution to the PDE that can be written as the product to $X(x)$ and $T(t)$ are

Answer:
$$u(x,t) = \sin(\frac{n\pi}{7}x)(a_n e^{\lambda_n t} + b_n e^{\lambda'_n t})$$

10) If $\lambda = 0$ then the solutions to $y'' + \lambda y = 0$ are $y(x) = Ax + B$. We now want $y'(0) = 0$ so $B = y'(0) = 0$. Moreover we want $y'(3) = 0$ which tells us nothing new. Thus when $\lambda = 0$ we get the solution $y = 1$ with eigenvalue $\lambda = 0$.

Now if $\lambda < 0$, so we can write it $\lambda = -\nu^2$ then our solutions to the ODE are $y(x) = Ae^{\nu x} + Be^{-\nu x}$. Plugging in the boundary conditions we see

$$\nu A + \nu B = 0 \quad \text{and} \quad A\nu e^{3\nu} - B\nu e^{-3\nu} = 0.$$

One may easily check that the only A and B satisfying this are $A = 0 = B$. Thus we only get the trivial solution again.

Now if $\lambda > 0$, so we can write it $\lambda = \nu^2$ then out solutions to the ODE are $y(x) = A\cos\nu x + B\sin\nu x$. Plugging in the boundary conditions we see $B\nu = 0$ so $B = 0$ and $A\nu\sin 3\nu = 0$. To get non trivial solutions we need $3\nu = n\pi$. To get all eigenfunctions we just need to consider non-negative integers n. (Note: when $n = 0$ we get the eigen function $y = 1$ with eigenvalue $\lambda = 0$.) Thus we have

Answer: Eigenvalues $\lambda_n = \left(\frac{n\pi}{3}\right)^2$ with eigenfunctions $\cos \frac{n\pi}{3}x$, where n runs through all non-negative integers.

11) We are looking for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$. So $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$. Thus the equation is

$$y'' + 2y' + y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n]x^n = 0.$$

So

Answer:

$$a_{n+2} = \frac{-a_n - (n+1)a_{n+1}}{(n+2)(n+1)}.$$