Mathematics 241, Final Exam, Friday, December 13, 2002, 1:30-3:30PM

This is a closed book exam. You may use one 5x7 card on which you have written on both sides. No other books, papers, calculators or other materials may be used during this exam. For the multiple choice questions, a correct answer scores +2 points, no answer scores 0 points, and an incorrect answer scores −1 point. Questions 9, 10 and 11 require you to provide a written response. Write your solutions to these questions in the exam blue book. Turn in the answer sheet with your multiple choice selections for problems 1 through 8 as well as the exam blue books with solutions to problems 9, 10 and 11.

Important: Do not ask the proctors any questions about what the exam questions mean or about what is wanted, etc. If you think that a question is ambiguous then choose an interpretation that seems reasonable to you, state clearly what that interpretation is, and then answer the question as you interpret it. Of course if we think that your interpretation is not reasonable then you will get no credit.

Some formulas that may be useful

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{1}$$

$$E = mc^2 \tag{2}$$

$$\mathcal{L}\{\mathrm{t}\} = \frac{1}{s^2} \tag{3}$$

$$\mathcal{L}\{\mathrm{y}'(\mathrm{t})\} = sY(s) - y(0) \tag{4}$$

$$\mathcal{L}\{\mathrm{y}''(\mathrm{t})\} = s^2Y(s) - sy(0) - y'(0) \tag{5}$$

$$\int_A^\infty te^{-xt}dt = e^{-Ax}\frac{(1+Ax)}{x^2} \tag{6}$$

$$\text{Area} = \pi R^2 \tag{7}$$

$$\int_m^{m+1} x\cos(n\pi x)dx = \frac{1}{n^2\pi^2}(-1)^{mn}((-1)^n - 1)) \tag{8}$$

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 0 & \text{if } n \neq -1 \text{ or } z_0 \text{ is not inside } \mathcal{C}; \\ 2\pi i & \text{if } n = -1 \text{ and } z_0 \text{ is inside } \mathcal{C}. \end{cases} \tag{9}$$

$$\mathcal{L}\{\mathrm{e}^{-\mathrm{at}}\} = \frac{1}{s+a} \tag{10}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad ; \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{11}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad ; \quad \cosh z = \frac{e^{z} + e^{-z}}{2} \tag{12}$$

$$\int \cos^2(nx)dx = \frac{x}{2} + \frac{\sin(2nx)}{4n} + C \quad ; \quad \int \sin^2(nx)dx = \frac{x}{2} - \frac{\sin(2nx)}{4n} + C; \tag{13}$$

$$\sin A\cos B = (\sin(A+B) + \sin(A-B))/2 \tag{14}$$

ANSWER SHEET

Print your name:

Sign your name:

Circle the name of your Professor: Haglund Wilf

Circle just one choice below for each of the 8 multiple choice questions. Keep this page covered during the exam except when you are recording your choices. Hand in this sheet when you leave, as well as the exam blue book with your answers to questions 9,10,11, and keep the questions themselves. PRINT YOUR NAME ALSO ON THE COVER OF THE EXAM BLUEBOOK.

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)

1. The value of

$$\oint_C \frac{2z^2+1}{z^2-z}dz,$$

where C is the circle $|z| = 2$ traversed counterclockwise, is

   (a) πi
   (b) $2\pi i$
   (c) $3\pi i$
   (d) $4\pi i$
   (e) 0

2. The inverse Laplace transform of $1/(s^2 - 1)$ is

   (a) e^{2t}
   (b) $e^{2t} - 1$
   (c) $e^{2t} + 1$
   (d) $\cosh t$
   (e) $\sinh t$

3. The Laplace transform of $u(t - \pi)2t$ (where $u(t)$ is the Heaviside function, or unit step function) is

   (a) $2\pi s(s + 1)e^{-\pi s}$
   (b) $2((\pi/s)^2 + 1/s)e^{-\pi s}$
   (c) $2(\pi/s + 1/s^2)e^{-\pi s}$
   (d) $2(2\pi + 1/s)e^{-\pi s}$
   (e) $2(\pi/s - 2/s^3)e^{-\pi s}$

4. The coefficient of $z + 2i$ in the Laurent series expansion of

$$\frac{z^4}{(z+2i)^2}$$

with center $-2i$ is

   (a) 1
   (b) $z - 2i$
   (c) $-i$
   (d) $-3 + 2i$
   (e) $-8i$

5. Let $\mathrm{Ln}(z)$ denote the principal value of the natural logarithm $\ln(z)$. Then the imaginary part of $\mathrm{Ln}(\sqrt{3}+i)$ is

   (a) $\pi/6$
   (b) $\sqrt{10}$
   (c) $1/\sqrt{3}$
   (d) 1
   (e) $1/\sqrt{2}$

6. A vibrating string, stretched between $x = 0$ and $x = \pi$, satisfies the following special case of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

   If the ends are fixed, there is zero initial velocity and initial deflection $0.02 \sin x$, then the coefficient of $\sin x$ in the solution for the deflection $u(x,t)$ (obtained by separating variables) is

   (a) 0
   (b) 0.02
   (c) $0.02 \cos t$
   (d) $0.02 e^{-\pi t}$
   (e) None of the above

7. This problem deals with eigenfunctions for the periodic Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(\pi) = y(-\pi), \qquad y'(\pi) = y'(-\pi).$$

   Let A, B, C, D, E be the following sets of possible candidates for orthonormal eigenfunctions for the problem.

$$A = \{\frac{\cos kx}{\sqrt{\pi}}\}, \quad k = 1, 2, \ldots \qquad B = \{\frac{\sin kx}{\sqrt{\pi}}\}, \quad k = 1, 2, \ldots$$
$$C = \{\frac{\cos kx}{\sqrt{2\pi}}\}, \quad k = 1, 2, \ldots \qquad D = \{\frac{\sin kx}{\sqrt{2\pi}}\}, \quad k = 1, 2, \ldots$$
$$E = \{\frac{1}{\sqrt{2\pi}}\}$$

   Which of the following statements about the Sturm-Liouville problem are true ?

   (a) The union[1] of sets A, B and E form a set of orthonormal eigenfunctions.
   (b) The union of sets A and E form a set of orthonormal eigenfunctions, but the union of sets A, B and E do not.
   (c) Set B forms a set of orthonormal eigenfunctions, but the union of sets A, B and E do not.
   (d) The union of sets C, D and E form a set of orthonormal eigenfunctions.
   (e) The union of sets C and E form a set of orthonormal eigenfunctions, but the union of sets C, D and E do not.

[1] The union of a family of sets is the set whose members are all of the things that belong to any set in the family

8. The radius of convergence of the Maclaurin series for $1/(1-3z^5)$ is

   (a) 0

   (b) 1/3

   (c) ∞

   (d) 1

   (e) None of the above

9. (a) (1 point) Find $u(x, y)$ and $v(x, y)$, the real and imaginary parts of $\sin(x + iy)$.

   (b) (1 point) Write a short Maple program that will do the following. First, define a function whose name is `laplacian`, which, given some expression f that depends on x and y, will produce the Laplacian of f, that is, it will produce the 2nd partial of f with respect to x plus the 2nd partial of f with respect to y (that should all be done in one Maple line). Then use your `laplacian` function to print the Laplacian of the functions $u(x, y)$ and $v(x, y)$ that you found in part (a) above. Altogether your program should be three or four lines long.

10. List the three solutions to the equation

$$z^3 = 2i$$

   Write your answers in polar form, using the principal value of the argument. Circle your final answers. Show your work. Plot the three solutions as points in the complex plane, showing clearly in your sketch the absolute value and arguments of each of them. (Each fully correct answer is worth 1 point, each wrong answer is worth 0 points).

11. Let I be the integral

$$I = \int_0^{\pi} \frac{d\theta}{3 + \cos\theta}.$$

   (a) Express I as a contour integral. Show your work. Circle your answer. (1 point for a fully correct answer, 0 points for an incorrect answer).

   (b) Use the residue theorem to express the contour integral from part (a) as a finite sum of residues. Circle your answer. (1 point for a fully correct answer, 0 points for an incorrect answer).

   (c) By calculating the residues from part (b), obtain the value of I. Show your work. Circle your final answer. (1 point for a fully correct answer, 0 points for an incorrect answer).