# Makeup Final Exam, Math 240: Calculus III

September, 2011

No books, papers or may be used, other than a hand-written note sheet at most $8.5'' \times 11''$ in size. All electronic devices must be switched off during the exam.

This examination consists of nine (9) long answer questions and one (1) true/false question. Partial credits will be given only when a substantial part of a long answer question has been worked out. Merely displaying some formulas is not sufficient ground for receiving partial credits.

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- Your name, printed:

- Your Penn ID (last 4 of the middle 8 digits):

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
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1. For what values of $k$ is the following matrix singular?

$$\begin{pmatrix} 2 & -3 & 1 \\ 2 & k & 0 \\ -k & -6 & 4 \end{pmatrix}$$

ANS. $k =$ __________.

2. Consider the following differential equation

$$\left[x(1-x)\frac{d^2}{dx^2} + (3-4x)\frac{d}{dx} - 2\right] u(x) = 0\,.$$

(a) Find a real number $\mu$ such that there exists a solution of the form

$$u(x) = x^{\mu} \cdot \left(1 + \sum_{n\geq 1} a_n x^n\right)\,.$$

ANS. $\mu =$__________.

(b) For the value of $\mu$ you found in (a), determine the coefficients $a_1, a_2, a_3$.

ANS. $a_1 =$__________, $a_2 =$__________, $a_3 =$__________.

3. Find a solution of the differential equation

$$\frac{d}{dt}\,\vec{x}(t) = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \vec{x}(t), \qquad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

which satisfies

$$\lim_{t\to\infty} \vec{x}(t) = 0 \quad \text{and} \quad x_2(0) = 1\,.$$

ANS. $\vec{x}(t) =$ ____________________

4. Let $A$ be the $4 \times 4$ matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

(a) (4 pts) Compute the matrix $A^2$ and find all eigenvalues of $A$

ANS. $A^2 =$

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The eigenvalues of $A$ are ____________________.

(b) (6 pts) Is $A$ diagonalizable? If so, find an invertible matrix $C$ such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix. If not, explain why such a matrix $C$ does not exist.

5. Let $D = \{(x,y,z) \in \mathbb{R}^3 \mid 4 \le x^2+y^2+z^2 \le 9\}$, the solid region between the sphere of radius 3 and the sphere of radius 2, both centered at the origin. Let $S$ be the boundary of $D$, consisting of the sphere $S_3$ of radius 3 and the sphere $S_2$ of radius 2, both centered at the origin. Orient $S$ by the unit normal vector field on $S$ such that

$$\vec{N}(x,y,z) = \begin{cases} \frac{1}{3}(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}) & \text{if } (x,y,z) \in S_3 \\ -\frac{1}{2}(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}) & \text{if } (x,y,z) \in S_2 \end{cases}$$

Compute the surface integral

$$\iint_S x\,\vec{i} \cdot \vec{N}\,\mathrm{d}A\,,$$

i.e. the flux of the vector field $\vec{F}(x,y,z) = x\,\vec{i}$ through the boundary $S$ of the solid $D$.

ANS. The surface integral is ______________.

6. Let $A$ and $B$ be two $5 \times 5$ matrices such that $AB = B^3$ and 4 is an eigenvalue of $B$, find one eigenvalue of $A$.

Ans. One eigenvalue of $A$ is __________.

7. Find all values of $r$ such that the following differential equation

$$2x^2y'' + 4x^2y' + 3y = 0$$

has a solution of the form $y = x^r \cdot (1 + \sum_{n=1}^{\infty} c_n(x)^n)$.

ANS. $r =$ __________.

8. Find the general solution to the following system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

ANS. $\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} =$

9. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1$ be the unit sphere in $\mathbb{R}^3$ centered about the origin. Orient $S$ by the unit normal vector field $\vec{N} := x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ on $S$. Compute the oriented surface integral

$$\iint_S \operatorname{curl}\left(\frac{x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}}{\sqrt{x^2 + 4y^2 + 9z^2}}\right) \cdot \vec{N}\,\mathrm{d}A$$

ANS. This integral, also written as $\displaystyle\iint_S \operatorname{curl}\left(\frac{x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}}{\sqrt{x^2 + 4y^2 + 9z^2}}\right) \cdot \vec{N}\mathrm{d}S$, is __________.

10. True/False questions. For each of the following statements, decide whether it is true or false and CIRCLE YOUR ANSWER.

(a) Let $x_1(t)$ and $x_2(t)$ be two solutions of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6\,x(t) = t^3 + \cos(t)\,.$$

Then $\lim_{t\to\infty}(x_1(t) - x_2(t)) = 0$.
Answer. TRUE FALSE

(b) Let $A$ be a $4 \times 4$ matrix whose characteristic polynomial $f(x) := \det(x \cdot \mathrm{I}_4 - A)$ is equal to $(x-1)^2(x-2)(x-3)$. Then there does not exist an invertible $4 \times 4$ matrix $C$ such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix.
Answer. TRUE FALSE

(c) Suppose that $C$ is a smooth closed curve on $U = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1\}$, oriented counterclockwise, and $P(x,y)$ and $Q(x,y)$ are two continuously differentiable functions on $U = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1\}$. Suppose moreover that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $U$, then

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = 0\,.$$

Answer. TRUE FALSE

(d) Let $\vec{F}(x,y,z)$ and $\vec{G}(x,y,z)$ are two continuously differentiable vector fields on $\mathbb{R}^3$ such that $\vec{F}(x,y,z) = \vec{G}(x,y,z)$ for all $(x,y,z)$ on the sphere $\{(x,y,z) \mid x^2+y^2+z^2 = 100\}$. Let $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 100\}$. Then

$$\iiint_B \operatorname{div}(\vec{F})\,dx\,dy\,dz = \iint_B \operatorname{div}(\vec{G})\,dx\,dy\,dz\,.$$

Answer. TRUE FALSE

(e) Suppose $A$ and $B$ are two *non-zero* $3 \times 3$ matrices such that $A \cdot B = 0 \cdot \mathrm{I}_3$. Then both $\det(A) = \det(B) = 0$.
Answer. TRUE FALSE