Mathematics Department

University *of* Pennsylvania

Final Exam, Math 240: Calculus III

May 3, 2007

No calculators books and notes can be used, other than a two-sided handwritten A4 sheet of paper.

- Name:
- Penn Id (last 4 of the middle 8 digits):
- Instructor:

◯ Dr. Rimmer ◯ Dr. Temkin ◯ Dr. Katzarkov

The duration of the exam is 2 hours. There are seven free response questions which worth 10 points and eight multiple choice questions which worth 5 points. Thus, the total number of points is 110, and each grade above 100 will be cut to 100. Show your work in the space provided after each question and **circle your answer** in the multiple choice questions. No part credit is given in the multiple choice part of this exam, but you must show your work: blind guessing will not be credited. Part credit may be given for free response part, so be sure to show all details of your solution. Good luck!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

total

1. (10 points) Let A and B be real $n \times n$ matrix. Decide whether each of the following statements is true or false, you do not need to justify your answer.

(i) (3 points) Always $A^T B^T = (AB)^T$.

(ii) (3 points) If A and B are invertible then always $B^{-1}A^{-1} = (AB)^{-1}$.

(iii) (4 points) If A is diagonalizable then it has n distinct eigenvalues.

(i)	a) true	b) false
(ii)	a) true	b) false
(iii)	a) true	b) false

2. (10 points) Show that if a real 3×3 matrix A satisfies $A^T = -A$, then its rank is smaller than 3.

3. (10 points) Given a matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

find its diagonalization, i.e. find an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

P= ____________________

D= ____________________

4. (10 points) Decide whether each of the following statements is true or false, you do not need to justify your answer.

(i) (3 points) If y_1 and y_2 are two solutions of the differential equation $e^x y'' + \sin(x)y' + \cos(x)y = 0$, then $y_1 - y_2$ is also a solution.

(ii) (3 points) If line integrals $\int_C Pdx + Qdy$ are independent of path in $\mathbf{R}^2$, then $Pdx + Qdy$ is an exact differential, i.e. $Pdx + Qdy = d\Phi$ for some function Φ on $\mathbf{R}^2$.

(iii) (4 points) If X is an open ball in $\mathbf{R}^3$, a vector field $\mathbf{F}$ on X has continuous partial derivatives and $\text{curl}(\mathbf{F}) = 0$, then $\mathbf{F}$ is potential, i.e. $\mathbf{F} = \text{grad}(\Phi)$ for some function $\Phi(x, y, z)$ on X.

(i)	a) true	b) false
(ii)	a) true	b) false
(iii)	a) true	b) false

5. (10 points) Solve the differential equation $y'' - 2xy = 0$ by a power series expansion about $x = 0$. You are done when you have written out the recurrence relation for the coefficients and the first three **non-zero** terms of two linearly independent solutions y_1 and y_2.

coefficient recurrence:____________________

y_1=____________________________________

y_2=____________________________________

6. (10 points) Find the solution of the system of linear first order differential equations $\mathbf{Y}' = A\mathbf{Y}$ that satisfies the initial condition $\mathbf{Y}(0) = \mathbf{a}_0$, where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \mathbf{a}_0 = \begin{pmatrix} 16 \\ -1 \end{pmatrix}$$

y_1=________________________

y_2=________________________

7. (10 points) Use **Laplace transform** to solve the initial value problem $y' + 3y = \sin t$, $y(0) = 1$.

y=______________________________

8. (5 points) Solve the system of linear equations

$$\begin{cases} 3x + y + z = -3 \\ 2x + 3z = 2 \\ -2x - 3y + z = 3 \end{cases}$$

Then x equals to

a) 1; b) 0; c) 3; d) -1;
e) -3; f) 4; g) -2; h) 2.

9. (5 points) Find $\det(A^{-1}BA^T)$, where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 4 & 0 \\ 6 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 8 & 0 & 4 \\ 7 & 6 & 5 \end{pmatrix}$$

a) 120; b) 144; c) 24; d) -120;
e) -144; f) 96; g) -24; h) -96.

10. (5 points) Let C be the arc of the parabola $x = t, y = 2 - t - t^2$ given by $-2 \le t \le 1$, and $\mathbf{F} = 2xe^{x^2-1}\cos(y)\mathbf{i} - e^{x^2-1}\sin(y)\mathbf{j}$ be a vector field. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

a) $\frac{e+e^{-1}}{2}$; b) $\frac{e^2+e^{-2}}{2}$; c) $e^3 - e^{-1}$; d) $\frac{e-e^{-1}}{2}$;
e) 1; f) 0; g) $1 - e^3$; h) $e^3 - 1$.

11. (5 points) Let C denote the circumference $(x-2)^2+(y-2)^2=1$ traversed counterclockwise. Evaluate the line integral

$$\oint_C (x^6+3y)dx+(2x-e^{y^2})dy$$

a) 0; b) e^4; c) $-\pi$; d) -2π;
e) 2π; f) π; g) $-e^4$; h) $\pi-e^4$.

12. (5 points) Let S be the portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ lying above the xy-plane. We orient S by a unit upward normal $\mathbf{n}$. Given a vector field $\mathbf{F} = y\mathbf{i} + \sin(z^2)\mathbf{j} + \cos(x^2)\mathbf{k}$, evaluate the surface integral

$$\iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS$$

a) $\sin(\pi^2)$; b) π; c) $-\pi$; d) -2π;
e) $\cos(\pi^2) - \sin(\pi^2)$; f) 2π; g) $-\sin(\pi^2)$; h) 0.

13. (5 points) Let S be the sphere $x^2 + y^2 + z^2 = 4$ oriented by the outward unit normal $\mathbf{n} = \frac{1}{2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ and

$$\mathbf{F} = (xy + x)\mathbf{i} + (y - y^2)\mathbf{j} + (yz + z)\mathbf{k}$$

be a vector field. Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

a) -32π; b) 32π; c) -8π; d) 16π;
e) $\frac{16\pi}{3}$; f) 8π; g) -16π; h) 0.

14. (5 points) Solve the initial value problem $2y'' + 2y' + y = x + 2$, $y(0) = 1$, $y'(0) = 0$. Then $y(\pi)$ equals to

a) $\pi + 2$; b) $\pi + e^{-\frac{\pi}{2}}$; c) $e^{-\pi}$; d) $\pi + \frac{e^{-\pi}}{2}$;
e) $-e^{-\frac{\pi}{2}}$; f) $\pi - e^{-\pi}$; g) $2e^{-\pi}$; h) $\pi - e^{-\frac{\pi}{2}}$.

15. (5 points) A linear differential equation

$$3x^2y'' + (x - x^2)y' + (x - 1)y = 0$$

has solutions $y_1 = x^{r_1}(1 + c_1x + c_2x^2 + \dots)$ and $y_2 = x^{r_2}(1 + d_1x + d_2x^2 + \dots)$ about $x = 0$. Find r_1 and r_2.

a) $r_1 = 0, r_2 = 1$; b) $r_1 = -1, r_2 = 3$;

c) $r_1 = \frac{1-\sqrt{13}}{6}, r_2 = \frac{1+\sqrt{13}}{6}$; d) $r_1 = 0, r_2 = \frac{1}{3}$;

e) $r_1 = -\frac{1}{3}, r_2 = 1$; f) $r_1 = \frac{3-\sqrt{21}}{2}, r_2 = \frac{3+\sqrt{21}}{2}$;

g) $r_1 = \frac{1-\sqrt{13}}{2}, r_2 = \frac{1+\sqrt{13}}{2}$; h) $r_1 = \frac{3-\sqrt{21}}{6}, r_2 = \frac{3+\sqrt{21}}{6}$.