

1. Let A be a 5×5 matrix with $\text{rank}(A) = 3$. For each of the following statements, indicate whether it must be true for all such A .
- 0 is an eigenvalue of A .
 - The first four rows of A are linearly dependent.
 - A is diagonalizable.
 - $\text{rank}(A^2) \geq 1$.
 - A is defective.
-

2. Let V be the vector space over \mathbb{R} consisting of all polynomials $p(x, y) \in \mathbb{R}[x, y]$ in two variables x and y with real coefficients whose total degree is at most 2. (If $p(x, y) \in V$ is not the zero polynomial, then the $\deg(p(x, y))$ is 0, 1 or 2.)

- Find a basis of V and determine the dimension of V .
- Let $T : V \rightarrow V$ be the linear transformation from V to itself, which sends every element $p(x, y) \in V$ to

$$T(p(x, y)) := y \frac{dp}{dx} + x \frac{dp}{dy}.$$

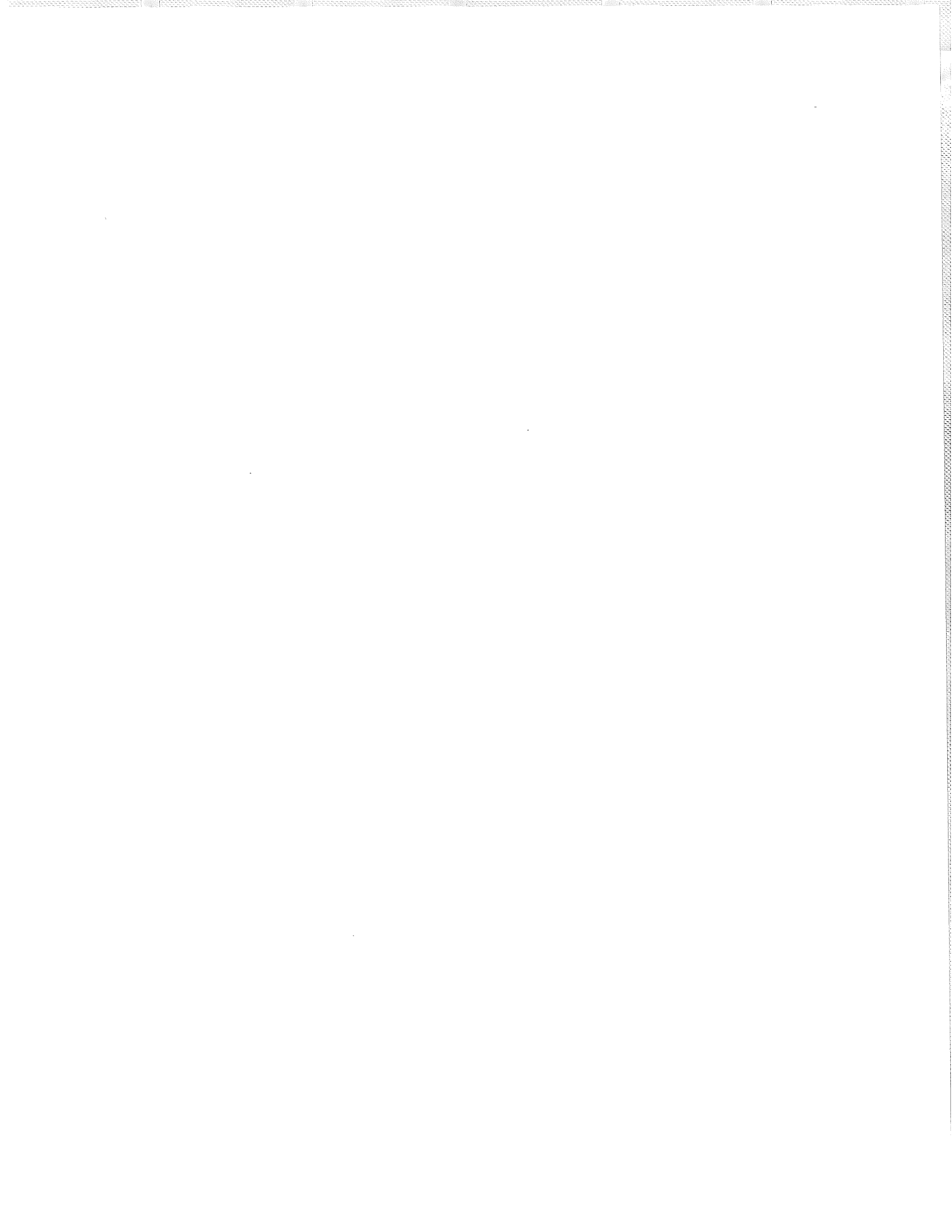
Find the matrix representation of this linear transformation T with respect to the basis you picked in (a).

- Find the eigenvalues of this matrix.
-

3. Let $A = \begin{pmatrix} 2 & 0 & k \\ 0 & 3 & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is a parameter. For which values of the parameter k is the matrix A diagonalizable?
-

4. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ Find A^{11} .
-





5. (a) Find the general solution to the system of differential equations:

$$x' = 2x + 3y, \quad y' = x + 4y$$

- (b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions

$$x(0) = 0, \quad y(0) = 2.$$

6. Solve the initial-value problem

$$y'' + 3y' = 6t, \quad y(0) = 0, \quad y'(0) = 0$$

for $y(t)$.

7. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 4e^{-2t} \ln t, \quad t > 0$$

8. Let $B := \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. Find the general solution of the system of first order linear differential equations

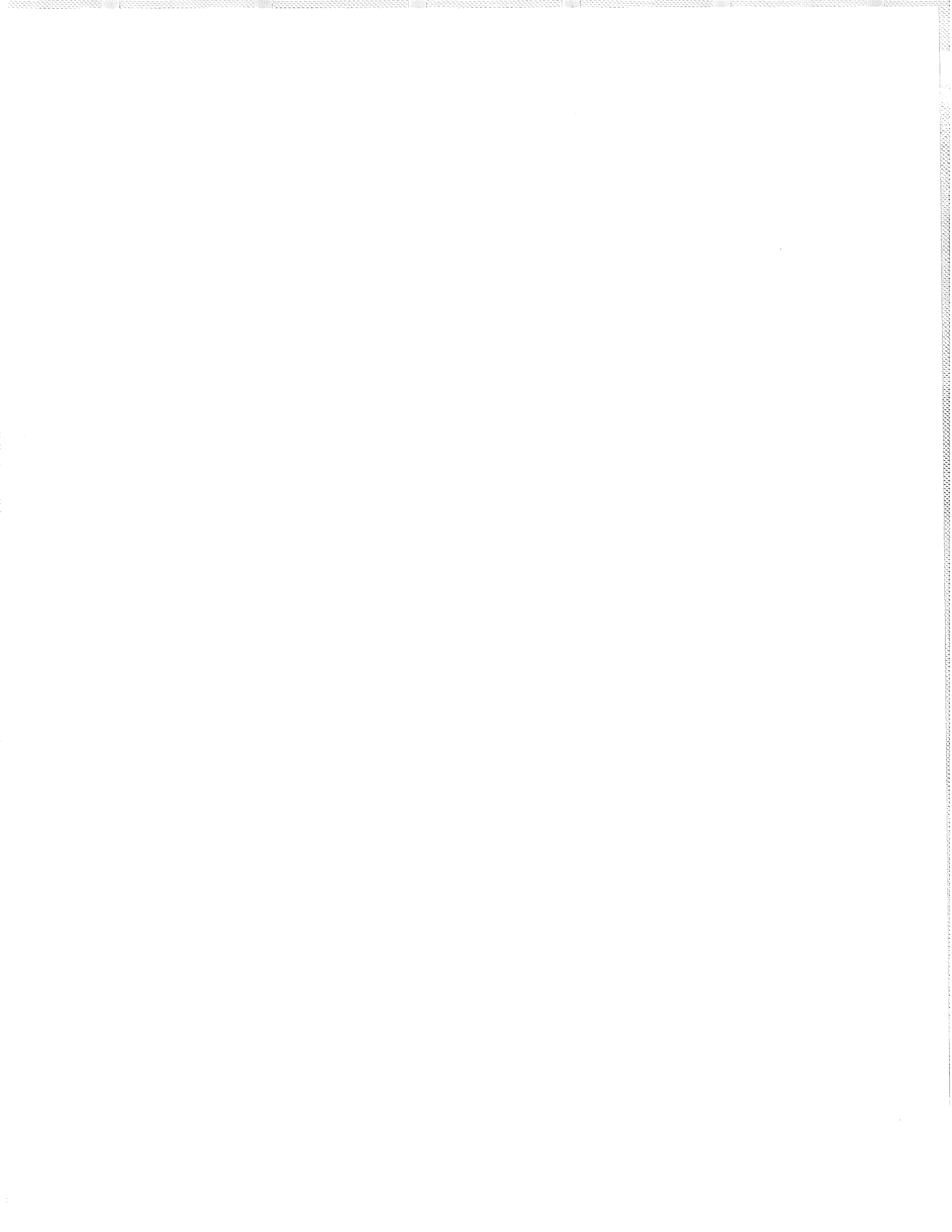
$$\frac{d}{dt}\mathbf{x}(t) = B \cdot \mathbf{x}(t),$$

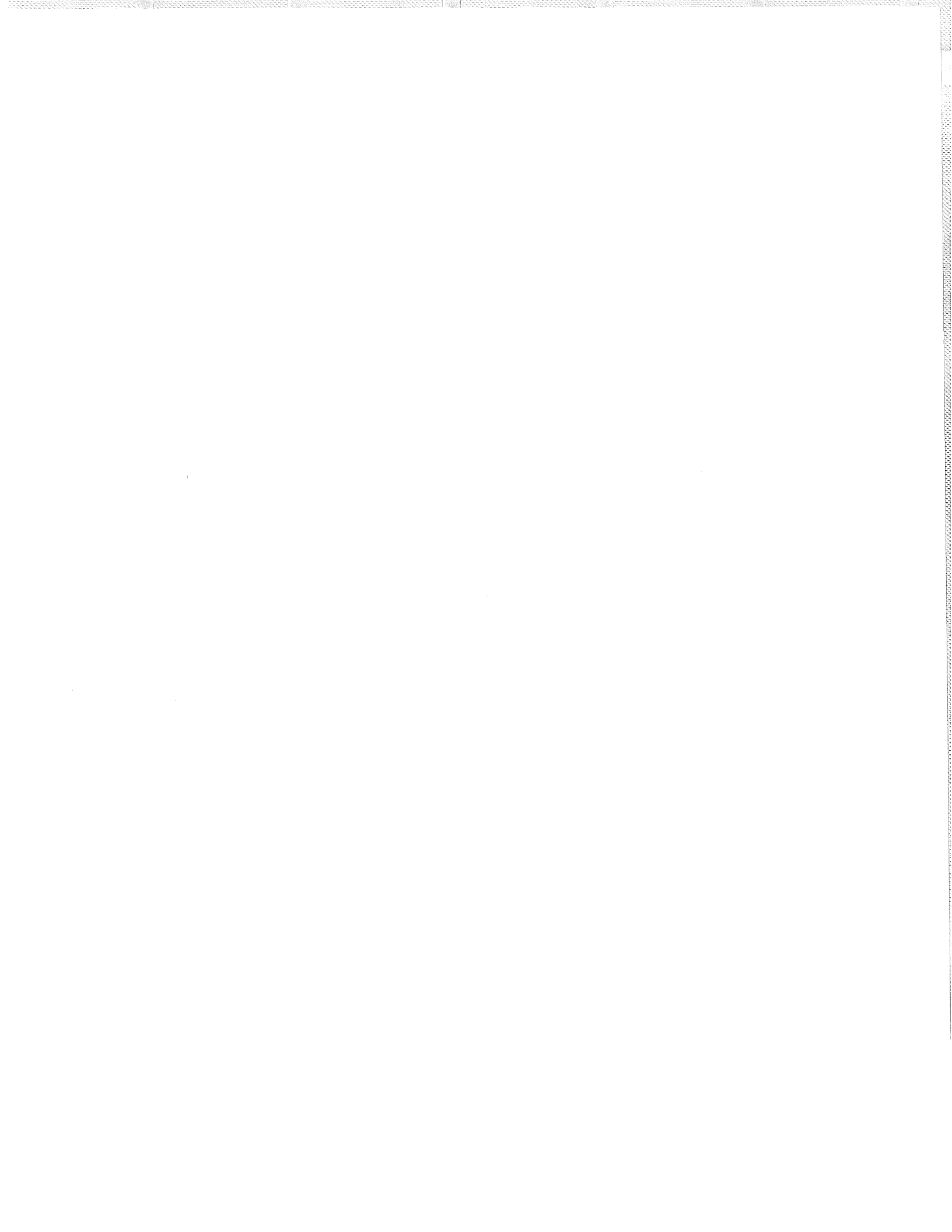
where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

9. Find a (particular) solution $y_p(x)$ of the differential equation

$$\left(\frac{d}{dx} + 1\right)^3 \left(\frac{d}{dx} - 1\right) y(x) = -240x^2e^{-x} + 120e^{-x}.$$





10. Let $A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -9 & 11 \end{pmatrix}$. Compute the matrix exponential e^{tA} .

11. Consider the following differential equation with a real parameter k

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + 8ky = \cos(2t) \quad (1)$$

Determine all values of the parameter $k \in \mathbb{R}$ such that every solution $y(t)$ of the differential equation (1) is bounded as $t \rightarrow \infty$ (i.e. there exists a constant C depending on the solution $y(t)$ such that $|y(t)| \leq C$ for all $t \geq 0$).

12. Find all equilibrium points of the differential equation

$$\frac{dx}{dt} = y(t), \quad \frac{dy}{dt} = -4\sin(x(t)) - \frac{y(t)}{25}$$

and determine/classify each of the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc.



