Math 240 Final Exam

Name: Answer Key

Instructor: ______________________

TA's Name: ______________________

Recitation Day/Time: ______________________

I pledge that I have abided by the guidelines of the Penn Honor Code on this exam.

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(*signature*)

Directions:

1. Show all of your work on this test paper and **clearly indicate your answers**. In order to get full or partial credit, you must show work and **justify your answers**!
2. Calculator use is not permitted on this test. You may consult one handwritten 8 1/2" x 11" sheet of paper filled out on both sides.
3. Check that your test has 12 pages and 10 problems.
4. Each problem is worth 10 points, for a total of 100 points.
5. Make sure you sign the honor pledge.
6. Good luck!

For official use only. Do not write in the boxes below.

Problem	Score (out of)	Problem	Score (out of)
1	(10)	**6**	(10)
2	(10)	**7**	(10)
3	(10)	**8**	(10)
4	(10)	**9**	(10)
5	(10)	**10**	(10)

Total Score: __________(100)

1. Decide if the given statements are true or false, and **justify your answers**.

(i) **T / F :** A 3×3 matrix need not have any real eigenvalues.

False - since complex eigenvalues come in conjugate pairs, every 3 x 3 matrix must have 1 or 3 real eigenvalues.

(ii) **T / F :** The surface S parametrized by $x = s^3,\ y = s\cos t,\ z = s\sin t$, where $s \geq 0$ and $0 \leq t \leq 2\pi$, is smooth.

False - since X_s = (3s^2, cos t, sin t) and X_t=(0, -s sin t, s cos t), the normal vector X_s x X_t = (s, -3s^3 cos t, -3s^3 sin t) is zero when s = 0.

(iii) **T / F :** If an $n \times n$ matrix A is invertible, then it is diagonalizable.

False - e.g., the matrix {{1,0},{1,1}} is invertible but not diagonalizable.

(iv) **T / F :** Given vector functions $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$, if the Wronskian $W[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n](t)$ is zero at every point in an interval I, the functions are dependent on I.

False - this only holds if the vector functions form a set of solutions to a differential equation.

(v) **T / F :** Let $a_0(x)$, $a_1(x)$, and $f(x)$ be continuous on an interval I. The initial value problem

$$\begin{cases} y'' + a_0(x)y' + a_1(x)y = f(x) \\ y(0) = y_0, y'(0) = v_0 \end{cases}$$

may have no solution on I.

False - the Existence and Uniqueness Theorem guarantees that the IVP has a unique solution on I.

2. Let R be the region in $\mathbb{R}^3$ enclosed by $z = x^2 + y^2$ and $z = 9$, and let S be the boundary of R oriented with outward normals. Compute the flux integral $\iint_S \vec{F} \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = \langle y - 2, x^3 z, z^2 \rangle.$$

Answer: 486pi (Divergence Theorem)

3. Carefully determine whether or not the set $\{3, x-3, 5x+e^{-x}\}$ forms a basis for the space of solutions of the differential equation $y''' + y'' = 0$ on the interval $[0, 1]$.

Answer (circle one): **Yes** / **No** (Must check that the functions are all solutions and that they are linearly independent.)

4. Evaluate $\iint_S \text{curl}\vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle z^2, -3xy, x^3y^3 \rangle$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$, oriented upward.

Answer: 0 (Stokes' Theorem)

5. For what values of k does the system

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

have (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions? **Justify your answers.**

Answers:
(i) e.g., k=0
(ii) k = -2
(iii) k = 1

6. Determine the general (real-valued) solution to the system $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}.$$

Answer: y = c_1e^{5t}(-1,2) + c_2e^{5t}[t(-1,2) + (-1/2,0)]

7. Solve the following initial value problem:

$$\begin{cases} y'' + 7y' + 10y = e^{-2x} \\ y(0) = 0, y'(0) = 0 \end{cases}$$

Answer: y = 1/9e^{-5x} - 1/9e^{-2x} + 1/3xe^{-2x}

8. Determine whether or not each of the following sets S is a subspace of the given real vector space V. For each set that is a subspace, **write down a basis** for the subspace.

(i) $V = M_3(\mathbb{R})$, $S = \{A \in V : \operatorname{rank}(A) = 3\}$.

Not a subspace - e.g., does not contain the zero matrix.

(ii) $V = M_2(\mathbb{R})$, $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V : a + d = 0 \right\}$.

Subspace with basis vectors {{1,0},{0,-1}}, {{0, 1},{0,0}}, {{0,0},{1,0}}.

9. Determine the general (real-valued) solution to the system $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Answer: ________________________________

y= c_1e^{3t}(1,0,0) + c_2e^{-2t}(0,-1/2(cost + sint),cost) + c_3e^{-2t}(0,-1/2cost - sint,-sint)

10. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $L(\vec{e}_1) = \vec{e}_2$, $L(\vec{e}_2) = 2\vec{e}_1 + \vec{e}_2$, and $L(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \vec{e}_3$. Find a nonzero vector $\vec{v}$ such that $L(\vec{v}) = k\vec{v}$ for some real number k.

Answer: e.g., (2,2,1)