NAME:

PROFESSOR: (A) Donagi ; (B) Ghrist ; (C) Krieger

240 SPRING 2009: Calculus

FINAL EXAM

INSTRUCTIONS:

1. WRITE YOUR NAME at the top and indicate your professor's name.
2. As you solve problems on the exam, FILL IN COMPLETELY the letter(s) of your solution(s) ON THIS PAGE. You must mark your answer clearly to obtain credit.
3. Show your work, if possible. It might not give you partial credit, but it doesn't hurt to be clear.
4. There is to be no cheating. No kinda-sorta-cheating. No anything that even comes close. If in doubt, don't do it, for your honor (and perhaps more) is at stake. Play fair.
5. There are to be no calculators, no cell phones, and no books during this exam. You are permitted one hand-written sheet of paper, double-sided, as an aide. Use a pencil, an eraser, your sheet, and your head: but not in that order.
6. You have 120 minutes to complete the exam. Do the exam quickly, but read carefully! You gain nothing by doing a different problem than what is asked.
7. If you have a question, re-read carefully. If that still does not help, please raise your hand and someone will help you directly.

ANSWERS:

PROBLEM 1	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 2	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 3	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 4	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 5	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 6	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 7	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 8	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 9	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 10	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 11	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 12	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 13	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 14	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)
PROBLEM 15	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)

PROBLEM 1: The portion of the plane $z = 10 + 2x + 3y$ over the disc $x^2 + y^2 \leq 1$ has area equal to: (choose one)

(A) $\pi/\sqrt{14}$
(B) $\sqrt{14}$
(C) $10 + \sqrt{14}$
(D) $\sqrt{14\pi}$
(E) $\pi\sqrt{14}$
(F) $4\pi\sqrt{13}$
(G) $4\pi/\sqrt{14}$
(H) 14π

PROBLEM 2: What is $\det(A)$, where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 8 & 64 \end{bmatrix}$$

(choose one)

(A) 0
(B) 1
(C) 4
(D) 8
(E) 16
(F) 48
(G) 64
(H) None of the above.

PROBLEM 3: The function $y = y(x)$ satisfies $x^2y'' + 7xy' + 8y = 0$, with $y(1) = 0$ and $y'(1) = 32$. What is $y(2)$? (choose one)

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 16
(G) 64
(H) None of the above.

PROBLEM 4: The flux of the field $\vec{F} = x\hat{\imath} + y\hat{\jmath} - z\hat{k}$ across the cylindrical surface $\{x^2 + z^2 = 1\}$, $0 \leq y \leq 3$ [sides only! end caps not included!], with outward pointing normal, equals: (choose one)

(A) -3π
(B) 0
(C) π
(D) 2π
(E) 3π
(F) 4π
(G) 6π
(H) 9π

PROBLEM 5: The general solution to the differential equation

$$x''' - 4x'' + 3x' = 8e^{-t}$$

is (choose one)

(A) $x(t) = c_1 e^{3t} + c_2 e^{t} + c_3 + e^{-t}$
(B) $x(t) = c_1 + c_2 e^{-t} + c_3 e^{-3t} + te^{-t}$
(C) $x(t) = c_1 e^{3t} + c_2 e^{t} + c_3 - e^{-t}$
(D) $x(t) = c_1 t + c_2 e^{t} + c_3 e^{3t} - e^{-t}$
(E) $x(t) = c_1 e^{3t} + c_2 e^{t} + c_3 e^{-t}$
(F) $x(t) = c_1 e^{-3t} + c_2 te^{-t} + c_3 - e^{-t}$
(G) $x(t) = c_1 + c_2 e^{-t} + c_3 e^{3t} + e^{t}$
(H) $x(t) = c_1 e^{3t} + c_2 e^{t} + c_3 t + e^{-t}$

PROBLEM 6: Let S denote the sphere of radius r in centered at the origin. If the flux of a vector field $\vec{F}$ across S (with the outward pointing normal) is $8\pi r^3/3$, which of the following could be $\vec{F}$? (circle one)

(A) $\vec{F} = 2rx\,\hat{\imath} + 2ry\,\hat{\jmath} + 2rz\,\hat{k}$
(B) $\vec{F} = x\,\hat{\imath} + 2y\,\hat{\jmath} + z\,\hat{k}$
(C) $\vec{F} = x\,\hat{\imath} + z\,\hat{\jmath} + y\,\hat{k}$
(D) $\vec{F} = x\,\hat{\imath} - xz\,\hat{\jmath} + z\,\hat{k}$
(E) $\vec{F} = \hat{\imath} + \hat{\jmath} + \hat{k}$
(F) $\vec{F} = 2x\,\hat{\imath} + 2y\,\hat{\jmath} + 2z\,\hat{k}$
(G) $\vec{F} = z\,\hat{\imath} + x\,\hat{\jmath} + y\,\hat{k}$
(H) $\vec{F} = 2\,\hat{\imath} + 2\,\hat{\jmath} + 2\,\hat{k}$

PROBLEM 7: Find $y(1)$, if $x(t)$, $y(t)$, and $z(t)$ are solutions of the system:

$$\begin{aligned} x' &= x + 3y \\ y' &= x - y \\ z' &= 3z \end{aligned}$$

satisfying the initial conditions $x(0) = y(0) = z(0) = 2$.

(A) 0
(B) -2
(C) 2
(D) $e^2 - e^{-2}$
(E) $e^2 + e^{-2}$
(F) $3e^2 - e^{-2}$
(G) $3e^2 + e^{-2}$
(H) $e^2 - 3e^{-2}$

PROBLEM 8: The eigenvalues of a 4-by-4 matrix A are $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = -1$ and $\lambda_4 = 0$. Fill in the statements below which are definitely TRUE. (*i.e.*, "maybe-yes-maybe-no" or "not-enough-information" means it's FALSE) (choose all that are true)

(A) $A\mathbf{v} = \mathbf{v}$ for some vector $\mathbf{v}$.
(B) A is invertible.
(C) The determinant of A is $2 + 2 - 1 + 0 = 3$.
(D) A is diagonalizable.
(E) The rank of A is 4.
(F) The equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution for $\mathbf{x}$.
(G) A has a zero on its diagonal.
(H) There are at least two linearly independent eigenvectors of A with eigenvalue 2.

PROBLEM 9: The work done by the vector field

$$\vec{F} = (1+y)\hat{\imath} + (x-1-y^2)\hat{\jmath}$$

along the curve $x = t$, $y = \sin(t)$, as t goes from 0 to π is: (choose one)

(A) 0
(B) $\pi/2$
(C) $-\pi/2$
(D) π
(E) $-\pi$
(F) -4π
(G) 4π
(H) $4-\pi$

PROBLEM 10: The change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u \\ v - u^2 \end{pmatrix}$$

transforms the integral

$$\int_0^2 \int_{u^2}^{u^2+2} 4u \sin(v - u^2) e^{2uv - 2u^3} dv\, du$$

into which integral with respect to (x, y) coordinates? (choose one)

(A) $\int_0^2 \int_0^1 4x \sin(y) e^{xy} dx\, dy$
(B) $\int_0^4 \int_0^2 4x \sin(y) e^{xy} dy\, dx$
(C) $\int_0^2 \int_0^4 2x \sin(y) e^{xy} dx\, dy$
(D) $\int_0^2 \int_0^2 2x \sin(y) e^{xy} dx\, dy$
(E) $\int_0^2 \int_0^4 2x \sin(y) e^{xy} dy\, dx$
(F) $\int_0^4 \int_0^2 x \sin(y) e^{xy} dy\, dx$
(G) $\int_0^2 \int_0^2 x \sin(y) e^{xy} dx\, dy$
(H) $\int_0^2 \int_0^1 x \sin(y) e^{xy} dx\, dy$

PROBLEM 11: The curl of $\vec{F}$ equals $\nabla \times \vec{F} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$. What is the work done by $\vec{F}$ along the oriented square path from $(1,-1,3)$ to $(2,-1,3)$ to $(2,0,3)$ to $(1,0,3)$ and back to $(1,-1,3)$? (Choose one)

(A) -6
(B) -3
(C) -2
(D) -1
(E) 1
(F) 2
(G) 3
(H) 6

PROBLEM 12: Solving the equation $y'' + 4y = 0$ via a series solution $y = \sum_{k=0}^{\infty} a_k x^k$ yields which recurrence relation? (choose one)

(A) $a_{k+1} = \dfrac{-4a_k}{(k+1)(k+2)}$

(B) $a_{k+2} = \dfrac{4a_k}{k(k+1)}$

(C) $a_{k+1} = \dfrac{a_k}{4k(k+1)}$

(D) $a_{k+2} = \dfrac{-4a_k}{(k+1)(k+2)}$

(E) $a_{k+2} = \dfrac{4a_k}{(k+1)(k+2)}$

(F) $a_{k+1} = \dfrac{-a_k}{4(k+1)(k+2)}$

(G) $a_{k+2} = \dfrac{4a_k}{(k-1)(k-2)}$

(H) $a_{k+2} = \dfrac{a_k}{(k+1)(k+2)}$

PROBLEM 13: Let $\mathbf{x}(t)$ be the solution to the equation

$$\mathbf{x}' = \begin{bmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \mathbf{x} \quad ; \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For which values of a, b, and c is the solution $\mathbf{x}(t)$ bounded (*e.g.*, all components of $\mathbf{x}(t)$ are less than a billion) for $0 \le t < \infty$? (choose the best answer)

(A) Any $a, b, c \le 0$.
(B) Any $a, b, c < 0$.
(C) Any c and a; Any $b \le 0$.
(D) Any c and a; Any $b < 0$.
(E) Any c and b; Any $a \le 0$.
(F) Any c and b; Any $a < 0$.
(G) Any c; Any $a, b < 0$.
(H) Any c; Any $a, b \le 0$ with $a \ne b$.

PROBLEM 14: Mark (*i.e.*, fill in the letters for) each of the following statements which is TRUE: if there is any exception, then the answer is FALSE. (choose all that are true)

(A) The planar vector field $\vec{F} = (-2xy)\hat{\imath} + (y^2 - x^2)\hat{\jmath}$ is conservative (*i.e.*, is the gradient of a scalar field).
(B) The matrix exponential e^A is defined as $I + A + A^2/2 + A^3/3 + \cdots + A^n/n + \cdots$
(C) $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 3 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 1 & -2 & 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{b} = \begin{pmatrix} 4 \\ 1 \\ -3 \\ -2 \end{pmatrix} \quad ; \quad \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(D) An undamped, unforced spring-mass system with mass $m > 0$ and spring constant $k > 0$ has as its solution a sum of $\sin$ and $\cos$ waves with a constant amplitude: the amplitude neither increases nor decreases with time.
(E) Let S be the boundary surface of a solid region in $\mathbb{R}^3$, and $\vec{F}$ a continuously differentiable vector field. Then we have $\int\int_S \text{curl}\vec{F} \cdot \hat{n}\, dS = 0$, where $\hat{n}$ is the unit normal vector to S with $inward$ orientation.
(F) An airplane flies along a smooth curve in $\mathbb{R}^3$. At the point along the curve where the plane is coldest, the gradient of the temperature function is perpendicular to the curve.
(G) If A is a 2-by-2 matrix, then $\det(3A) = 3\det(A)$.
(H) A square matrix with repeated eigenvalues is not diagonalizable.

PROBLEM 15: Consider the differential equation

$$x^2(1-x^2)y'' + y' + (1+x)y = 0$$

Which of the following is true? (choose one)

(A) $x = 0$ is an irregular singular point and $x = -1$, $x = +1$ are regular singular points.
(B) $x = +1$ is an irregular singular point and $x = -1$, $x = 0$ are regular singular points.
(C) $x = 0$ is a regular singular point and $x = -1$, $x = +1$ are irregular singular points.
(D) $x = -1$ is a regular singular point and $x = 0$, $x = +1$ are irregular singular points.
(E) $x = 0$, $x = -1$ and $x = +1$ are regular singular points.
(F) $x = 0$, $x = -1$ and $x = +1$ are irregular singular points.
(G) $x = 0$ is an irregular singular point.
(H) There are no singular points.