Math 240 FINAL EXAM May 9, 2008

Professor Popa

Circle one: Professor Storm

Professor Ziller

Name: Solutions

Penn Id#: ______

Signature: ______

TA: ______

Recitation Day and Time: ______

You need to show your work, even for multiple choice problems. A correct answer with no work will get 0 points. The only exception are True/False problems, where no work needs to be shown. Each problem is worth 10 points.

You are NOT allowed to use a calculator. The extra double sided sheet of paper needs to be hand written in your own hand writing (no copies allowed).

(Do not fill these in; they are for grading purposes only.)

1)	9)
2)	10)
3)	11)
4)	12)
5)	13)
6)	14)
7)	15)
8)	

Total

1. The matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

has $K = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ as an eigenvector. What is the corresponding eigenvalue?

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvalue is 0.

[redacted]

2. The matrix

$$A = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 9$ (with multiplicity 2). An eigenvector corresponding to λ_1 is $v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and an eigenvector corresponding to λ_2 is $v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

First I'm looking for a vector $\vec{w}$ which is $\perp$ to $\vec{v}_1$ & $\vec{v}_2$ and is an e-vector with e-value 9.

$$\begin{pmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \quad w_1 + 2w_2 - 2w_3 = 0$$

to be an e-vec. for e-value 9.

Now add the orthogonality conditions to get the system:

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & -2 \\ -2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -4/5 \end{pmatrix}$$

$-2 + \frac{8}{5}$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & -4/5 \end{pmatrix} \rightsquigarrow$$

I can choose $\vec{w} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$.

Now normalize $\vec{v}_1, \vec{v}_2,$ & $\vec{w}$ to have length 1.

$$P = \begin{pmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{-2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$$

3. Compute the determinant

$$\begin{vmatrix} 1 & 2 & 1 & 4 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{vmatrix}$$

Answer:

(a) 2 (b) 4 (c) -4 (d) 8 (e) -2 (f) -8

[redacted]

$$-4 \cdot \begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -4 \cdot (2 \cdot (1-2)) = 8.$$

[redacted]

4. For each of the following statements, determine whether they are true or false. All the matrices below are assumed to have real entries. No work needs to be shown for this problem.

(a) Any orthogonal 3x3 matrix has an eigenvalue equal to 1 or -1 . True● False○

(b) There is a symmetric matrix having i and $-i$ as eigenvalues. True○ False●

(c) If A is any 2x2 matrix of rank 1, then the system True○ False●

$AX = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has infinitely many solutions.

(d) If A is any 3x2 matrix of rank 2 and B a 2x3 matrix True● False○
such that $AB = 0$, then $B = 0$.

(e) If A is a 2x2 matrix with $A^2 = I$ (the identity matrix) True○ False●
then $A = \pm I$.

[redacted]

5. Consider the vector field

$$\mathbf{F} = \left(\frac{-z^2}{5} - z + \pi y e^{\sin x}\cos x\right)\mathbf{i} + \left(\pi e^{\sin x} - x\right)\mathbf{j} - \frac{2xz}{5}\mathbf{k}$$

and the curve C given by

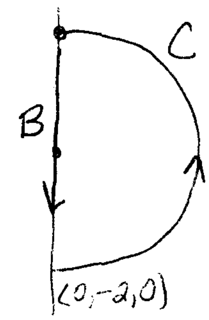
$$(2\cos t, 2\sin t, 0)$$

for $-\pi/2 \le t \le \pi/2$. Evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

(a) $2\pi\sqrt{2}$ (b) 0 (c) 4π (d) $-\pi$ (e) -2π (f) 2π (g) none of the above

$$\nabla\times F = \hat{i}\left(0-0\right) - \hat{j}\left(-\frac{2z}{5}+\frac{2z}{5}+1\right) + \hat{k}\left(-1\right) = -\hat{j}-\hat{k}.$$



$\vec{n} = \hat{k}.$

$(\nabla\times F)\cdot\hat{k} = -1.$

$B(t) = \cancel{(a,0,t)}\ (0,-t,0)$ for $-2 \le t \le 2.$

$$\int_C F\cdot dr + \int_B F\cdot dr = \iint_{\substack{\text{half}\\ \text{disk}}} (-1)\cdot dA = \frac{\pi\cdot 2^2}{2}\cdot(-1) = -2\pi.$$

$$\int_B F\cdot dr = \int (\pi y\,\hat{i} + \pi\,\hat{j})\cdot\left(\frac{dy}{dt}\,dt\right)\hat{j} = -\pi\int_{-2}^{2} dt = -4\pi.$$

$$\Rightarrow \int_C F\cdot dr = 2\pi.$$

[redacted]

[redacted]

6. In the following true/false problems, **F** is any vector field in 3-dimensions and f is any function in 3 variables. (You may assume **F** and f have continuous derivatives.) You do not need to show any work. For each problem, state whether the given identity is true or false.

(*a*) $\operatorname{div}(\nabla f) = 0$	True○	False●
(*b*) $\operatorname{curl}(\nabla f) = \mathbf{0}$	True●	False○
(*c*) $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$	True●	False○
(*d*) $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \mathbf{0}$	True○	False●
(*e*) $\nabla(\operatorname{div}\mathbf{F}) = \mathbf{0}$	True○	False●

(a) $\operatorname{div}\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(b) $\operatorname{curl}\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = 0.$

(c) $\operatorname{div}\left(\hat{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \hat{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)$

$= 0$

(d) $\operatorname{curl}(\operatorname{curl}(x^2\hat{j})) = \operatorname{curl}(2x\,\hat{z}) = -2\hat{j}.$

(e) $\nabla(\operatorname{div}(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})) = \nabla(2x + 2y + 2z) = 2(\hat{i} + \hat{j} + \hat{k})$

[redacted]

7. Define the function [redacted]

$$f(x,y,z) = e^{(\sin x \cdot \cos y)} \cdot \left(z + \frac{\pi}{2}\right).$$

Let C be the curve

$$\left(t\ \cos^2(2t),\ t\ \sin(t),\ t\right)$$

for $0 \leq t \leq \pi/2$. Compute the integral

$$\int_C \frac{\partial f}{\partial x}\,dx + \int_C \frac{\partial f}{\partial y}\,dy + \int_C \frac{\partial f}{\partial z}\,dz.$$

$$C(0) = (0,0,0) \qquad C\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right).$$

So

$$\left(e^{1\cdot 0} \cdot \pi\right) - e^{0\cdot 1} \cdot \frac{\pi}{2} = \frac{\pi}{2}$$

[redacted]

8. Let S be the closed surface in 3-space formed by the cone

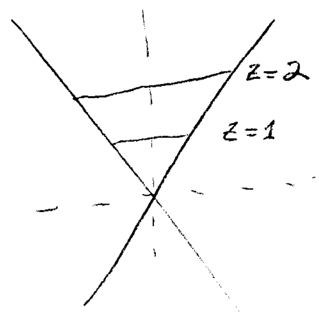
$$x^2 + y^2 - z^2 = 0, \quad 1 \le z \le 2,$$

the disk $x^2 + y^2 \le 4$ in the plane $z = 2$, and the disk $x^2 + y^2 \le 1$ in the plane $z = 1$. Define the vector field

$$\mathbf{F}(x, y, z) = xy^2\mathbf{i} + x^2y\mathbf{j} + \sin x\,\mathbf{k}$$

and let $\mathbf{n}$ be the outward pointing unit normal vector to S. Compute the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$



$(\nabla \cdot F) = y^2 + x^2 = r^2$ in cylindrical coords

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint (\vec{\nabla} \cdot \vec{F})\, dV$$

$$= \int_1^2 \int_0^z \int_0^{2\pi} r^2 \cdot r \, d\theta\, dr\, dz$$

$$= 2\pi \int_1^2 \frac{z^4}{4} = \frac{\pi}{2} \cdot \left[\frac{z^5}{5}\right]_1^2 = \frac{\pi}{2 \cdot 5} \cdot [32 - 1] = \frac{31\pi}{10}$$

[redacted]

9. Find the solution of the differential equation $y' - y = y^2$ with $y(0) = \frac{1}{3}$.

$$\frac{y'}{y+y^2} = 1 \qquad \int \frac{dy}{y+y^2} = \int dx$$

For this problem I present two solutions. This, the first, is by separation of variables.

$$\int \left(\frac{1}{y} - \frac{1}{1+y}\right) dy = \int dx$$

$$\log \frac{y}{1+y} = \text{[redacted]}\, x + C_1$$

$$\frac{y}{1+y} = C_2 e^x$$

$$y = C_2 e^x + y\, C_2 e^x$$

$$y = \frac{C_2 e^x}{1 - C_2 e^x} \qquad y(0) = \frac{C_2}{1-C_2} = \frac{1}{3}.$$

$$3C_2 = 1 - C_2 \Rightarrow C_2 = 1/4.$$

$$y(x) = \frac{e^x}{4 - e^x}.$$

$$\frac{(4-e^x)e^x + e^{2x}}{(4-e^x)^2} = \frac{4e^x}{(4-e^x)^2} \checkmark.$$

[redacted]

9. Find the solution of the differential equation $y' - y = y^2$ with $y(0) = \frac{1}{3}$.

This solution is using the equation of Bernoulli.

Substitute $y = u^{-1}$.

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{-1}{y^2} \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2 \frac{du}{dx}$$

So the equation $y' - y - y^2 = 0$ becomes

$$-y^2 \frac{du}{dx} - y - y^2 = 0 \Rightarrow \frac{du}{dx} + u + 1 = 0.$$

integrating factor is $e^{\int dx} = e^x \Rightarrow e^x u'(x) + e^x u(x) = -e^x$

$$\Rightarrow \frac{d}{dx}\left(e^x u(x)\right) = -e^x$$

$$\Rightarrow e^x u(x) = -e^x + C \Rightarrow u(x) = \frac{-e^x + C}{e^x}.$$

$$y(x) = \frac{1}{u(x)} = \frac{e^x}{C - e^x} \quad \& \quad y(0) = \frac{1}{3} = \frac{1}{C-1}.$$

$$\Rightarrow C = 4 \Rightarrow y(x) = \frac{e^x}{4 - e^x}.$$

[redacted]

10. Let y be the solution of $y'' = e^{-3t} - y'$ that passes through the origin and has a horizontal tangent line there. Then $\lim_{t\to\infty} y(t)$ is equal to:

Answer:

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) -2 (e) $-\frac{1}{4}$

$$y''(x) + y'(x) = e^{-3t}.$$

$$y_h'(x) = C e^{-x} \rightsquigarrow y_h(x) = + C_1 e^{-x} + C_2.$$

$$9ke^{-3t} - 3ke^{-3t} \qquad 6k = 1$$

$$y(x) = C_1 e^{-x} + C_2 + \frac{1}{6} e^{-3t}$$

$$y(0) = C_1 + C_2 + \frac{1}{6} = 0 \Rightarrow C_1 + C_2 = -\frac{1}{6}$$

$$y'(0) = -C_1 - \frac{1}{2} = 0 \Rightarrow C_1 = -\frac{1}{2} \qquad C_2 = \text{[redacted]} \ \frac{1}{3}.$$

$$y(x) = -\frac{1}{2} e^{-x} + \frac{1}{3} + \frac{1}{6} e^{-3t}.$$

$$\lim_{x\to\infty} y(x) = \frac{1}{3}.$$

[redacted]

11:19

11. Let $y(t) = (y_1(t), y_2(t))$ be any non-zero solution of the system of differential equations

$$y_1' = y_1 + 2y_2$$
$$y_2' = 3y_1 + 2y_2$$

such that $\lim_{t\to\infty} y(t) = 0$. Then $\frac{y_1(1)}{y_2(1)}$ is equal to

Answer:

(a) -1 (b) 1 (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$ (e) $\frac{3}{2}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1-x & 2 \\ 3 & 2-x \end{pmatrix} \rightsquigarrow (x-1)(x-2)-6 = x^2-3x-4.$$

$$\frac{3 \pm \sqrt{9+16}}{2} = 4, -1.$$

The sol'n corresponding to e-value 4 will go to ∞ as $t \to \infty$, so we can ignore it.

We need only the e-vec for -1:

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightsquigarrow e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[redacted]

12. Find the solution of $xy'' + y' = -\frac{y}{x}$ with $y(1) = 0$, $y'(1) = 2$. Do not use a power series approach.

Cauchy-Euler eqn.:

$$x^2y'' + xy' + y = 0.$$

Guess the sol'n X^m.

$$m(m-1) + m + 1 = m^2 + 1 = 0$$

$$X^{\pm i} = e^{\pm i \log x} = \cos(\log x) \pm i \sin(\log x)$$

$$\rightsquigarrow \quad y(x) = c_1 \cos(\log x) + c_2 \sin(\log x).$$

$$y(1) = c_1 = 0$$

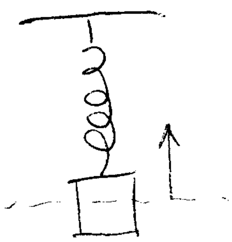
$$y'(1) = c_2 \cos(\log 1) \cdot \frac{1}{1} = c_2 = 2.$$

$$y(x) = 2 \sin(\log x).$$

[redacted]

[redacted]

13. A spring satisfies the differential equation $x'' + 16x = 0$. It is released one meter above its equilibrium position with a downward velocity of 3 meters per second. What is its highest position above the equilibrium position?



$x(0) = 1$

$x'(0) = -3.$

$x(t) = c_1 \cos(4t) + c_2 \sin(4t)$

$x(0) = c_1 = 1$

$x'(0) = 4c_2 = -3$

$x(t) = \cos(4t) - \frac{3}{4}\sin(4t).$

Amplitude $\sqrt{1 + \frac{9}{16}} = \sqrt{\frac{16+9}{16}} = \frac{5}{4}$

5/4 meters

[redacted]

14. Find an inhomogenous linear second order differential equation with constant coefficients having $y_p = \frac{1}{2}x^2 - x$ as a particular solution, and $y_1 = 3$ and $y_2 = e^{2x}$ as solutions of the associated homogeneous differential equation.

3 and e^{2x} are sol'ns. So the characteristic eqn is

$(\lambda - 0)(\lambda - 2) = \lambda^2 - 2\lambda$.

$\Rightarrow$ the homogeneous eqn. is $y''(x) - 2y'(x) = 0$.

Then $\frac{d^2}{dx^2}\left(\frac{1}{2}x^2 - x\right) - 2\frac{d}{dx}\left(\frac{1}{2}x^2 - x\right) = 1 - 2(x-1) = 3 - 2x$.

So the answer is

$$y''(x) - 2y'(x) = 3 - 2x$$

15. Consider the differential equation $2xy'' + y' + y = 0$.

a) Show that the equation has two *linearly independent* series solutions. You do not have to compute the coefficients of the series.

b) Find the coefficient a_2 in a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $y(0) = 1$, $y'(0) = -1$.

There is a soln of the form $\sum_{n=0} a_n x^{n+r}$. The eqn. becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\cdot 2 a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0. \quad (*)$$

The lowest order term is x^{r-1}. It's coefficients are

$$r\cdot(r-1)\cdot 2\cdot a_0 + r\cdot a_0 = 0$$

$$\Rightarrow 2(r^2 - r) + r = 2r^2 - r = 0 \Rightarrow r = 0, \tfrac{1}{2}.$$

Therefore there is a soln of the form $\sum_{n=0}^{\infty} a_n x^n$ (for $r=0$) and a soln of the form $\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$. They are linearly independent.

b) Looking for the a_2 coefficient of the soln $\sum_{n=0}^{\infty} a_n x^n$.

$y(0) = a_0 = 1.$ $\quad y'(0) = a_1 = -1.$

Plugging $r=0$ into the above eqn $(*)$ yields

$$\sum_{n=0}^{\infty} n\cdot(n-1)\cdot 2 a_n x^{n-1} + \sum_{n=0}^{\infty} n a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} (n+1)\cdot n\cdot 2 a_{n+1} x^n + \sum_{n=-1}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

For $n > -1$ the coeff. is $\big((n+1)\cdot n\cdot 2 + (n+1)\big) a_{n+1} + a_n = 0.$

For $n=1$ this is $(4+2) a_2 + -1 = 0$

$$\boxed{\Rightarrow a_2 = 1/6.}$$