Final exam, Math 240: Calculus III

April 29, 2005

No books, calculators or papers may be used, other than
a hand-written note card at most $5'' \times 7''$ in size.

For this web version, answers are at the end of the exam.

This examination consists of eight (8) long-answer questions and four (4) multiple-choice questions. Each problem is worth ten points. Partial credits will be given only for long-answer questions, when a substantial part of a problem has been worked out. Merely displaying some formulas is not sufficient ground for receiving partial credits.

- Your name, printed:
- Your Penn ID (last 4 of the middle 8 digits):
- Your signature:
- Your lecture section (circle one):

Chai Caldararu

1	2	3	4	5	6	7	8	9-12	Total

Part I. Long-answer Questions.

1. Compute $\det(A^3)$, where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}.$$

2. Let C be the oriented curve

$$C = \{(x, y) \,:\, 4x^2 + 9y^2 = 36,\ x \geq 0,\ y \geq 0\}$$

from $(3, 0)$ to $(0, 2)$. Compute the line integral

$$\int_C (x + 1)\, dy + y\, dx\,.$$

3. Let D be the cube

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, 0 \leq x, y, z \leq 1 \right\},$$

and let $S = \partial D$ be the boundary surface of D, oriented by the unit normal vector field $\vec{n}$ on S pointing away from D. Compute the oriented surface integral

$$\iint_S (x^2\, \vec{i} + xyz\, \vec{j} + z^3\, \vec{k}) \cdot \vec{n}\, dS\,.$$

4. Let S be the surface

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1,\ z \geq 0 \right\},$$

the upper half of the unit sphere centered at the origin, oriented by the unit normal vector field $\vec{n} = x\, \vec{i} + y\, \vec{j} + z\, \vec{k}$ on S. Compute the surface integral

$$\iint_S (x\, \vec{i} - y\, \vec{j} + z\, \vec{k}) \cdot \vec{n}\, dS\,.$$

5. Let C be the boundary of the rectangle with vertices $(3, 2)$, $(-5, 2)$, $(-5, -7)$ and $(3, -7)$, oriented counter-clockwise. Compute the line integral

$$\oint_C \frac{y\, dx - x\, dy}{x^2 + y^2}\,.$$

6. Let $y(t)$ be a function which satisfies the differential equation

$$y''(t) + (1+t)y'(t) + (1+t+t^2)y(t) = 0$$

$y(0) = 1$, $y'(0) = 0$. Determine the values of $y''(0)$ and $y'''(0)$.

7. Suppose that a vector-valued function $\vec{x}(t)$ satisfies the following system of ordinary differential equations

$$\vec{x}'(t) - A \cdot \vec{x}(t) = \vec{0}, \quad \text{where } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

and $\vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Determine the function $\vec{x}(t)$ explicitly.

8. Find one ("particular") solution of the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} + y & = t \\ \dfrac{dy}{dt} - 2x & = 0 \end{cases}.$$

In other words, find a pair of real-valued functions $(x(t), y(t))$ satisfying the above system of equations. (There are many such solutions.)

[Hint: Try to replace the above system by a single differential equation, then try to find a particular solution of that equation.]

PART II BEGINS ON THE NEXT PAGE

Part II. Multiple Choice Questions. Please **circle** your answer.

9. Consider the following matrices

$$A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \; A_3 = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}, \; A_4 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Which ones can be diagonalized over the real numbers? (In other words, there exists an invertible matrix P with coefficients in real numbers such that $P^{-1} \cdot A_i \cdot P$ is a diagonal matrix.)

A. A_1 and A_3 only B. A_2, A_3 and A_4 only C. A_3 and A_4 only

D. A_2 and A_4 only E. A_1, A_3 and A_4 only F. A_1 and A_4 only

G. A_1, A_2, A_3 and A_4

10. Let A be a symmetric 4×4 matrix with real entries. Consider the following statements.

I. A must have four distinct eigenvalues.

II. There exists an invertible matrix C with real entries such that $C \cdot A \cdot C^{-1}$ is a diagonal matrix.

III. The four roots of the characteristic polynomial of A are all real numbers.

IV. A^2 is a symmetric matrix.

Which ones among the above statements are true?

A. I, II, III only. B. II, III, IV only. C. I, III, IV only.

D. III and IV only. E. II and III only. F. II and IV only.

G. I, II, III, IV are all true. H. None of the above.

11. Suppose that a function $x(t)$ satisfies the differential equation

$$t^2\,\frac{d^2x}{dt} - 2t\frac{dx}{dt} + 2x(t) = 0\,,$$

and $x(1) = 3$, $\frac{dx}{dt}(1) = 1$. What is the value of $x(2)$?

A. 0 B. -1 C. 3 D. -4 E. 2 F. 1 G. 5 H. None of the above.

12. Suppose that a function $y(t)$ satisfies the differential equation

$$y''(t) + 2y'(t) + y(t) = e^{-2t}\,,$$

and $y(0) = y'(0) = 0$. What is the value of the Laplace transform $\mathcal{L}\{y(t)\}(s)$ of $y(t)$ at $s = 1$?

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{24}$ D. $\frac{1}{4}$ E. $\frac{1}{3}$ F. $\frac{1}{36}$ G. None of the above.

Answers:

1. 1728
2. 2
3. $\frac{5}{2}$
4. $\frac{2\pi}{3}$
5. $-2\,\pi$
6. $y''(0) = -1,\ y'''(0) = 0$ 7. $\vec{x}(t) = \begin{bmatrix} -e^t + 2te^t \\ 2e^t \end{bmatrix}$
8. $x = -\frac{3}{2}t - 4$, $y = t + \frac{3}{2}$ [Take the derivative of the first equation; then use the second].
9. Only A_3 and A_4 can be diagonalized by *real* matrices.
10. Only II, III, and IV are true.
11. $x(t) = 5t - 2t^2,\ x(2) = 2$
12.