Final Exam Fall 2003 Math 240

Instructions: This is a closed book exam. No calculators are allowed. You are allowed two sides of a 5" by 7" index card of (handwritten) notes. Write your name and Penn ID # on the answer sheet on the last page of this exam. Do all of your work on the corresponding page of the exam. Questions 1 through 11 are multiple choice, and questions 12 through 15 are free response. For both the multiple choice and free response record your answers on the answer sheet.

If you need more paper, raise your hand with your exam in it and we will give a blank booklet. We will not answer any questions during the exam about the problems. If you are sure a problem is wrong or has a typo indicate that on your answer sheet and move on to the next problem. At the end of the exam, turn in your entire exam and any extra scratch paper you have written on. Each problem is worth 10 points apart from the last one which is worth 20 points. A partial credit of 5 for the multiple choice problems is given if you make a minor mistake in an otherwise correct solution. If you circle the correct answer but give a wrong or incomplete solution in the answer booklet, you will not receive any credit for this problem.

1. Evaluate

$$\int_C y\,dx + z\,dy + x\,dz,$$

where C is the line segment starting at $(0,0,0)$ and ending at $(6,8,5)$. The answer is

(*a*) 55 (*b*) 56 (*c*) 57 (*d*) 58

(*e*) 59 (*f*) 60 (*g*) 61 (*h*) 62

2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (2x + e^{-y})\mathbf{i} + (4y - xe^{-y})\mathbf{j}$ along the curve C given by $y = x^4$, $0 \leq x \leq 1$. The answer is

(a) 3 (b) $3 + e^{-1}$ (c) $3 + 2e^{-1}$ (d) 4

(e) $4 + e^{-1}$ (f) $4 + 2e^{-1}$ (g) 0 (h) 1

3. Find the surface area of that portion of the paraboloid $z = 4 - x^2 - y^2$ that is above the plane $z = 1$. The answer is

(a) $\frac{\pi}{6}(13^{3/2} - 1)$ (b) $\frac{\pi}{4}(13^{3/2} - 1)$ (c) $\frac{\pi}{6}(17^{3/2} - 1)$ (d) $\frac{\pi}{4}(17^{3/2} - 1)$

(e) $\frac{\pi}{6}(11^{3/2} - 1)$ (f) $\frac{\pi}{4}(11^{3/2} - 1)$ (g) 2π (h) $\pi/2$

4. Let $\mathbf{F} = y^3\mathrm{i} + x^3\mathrm{j} + z^3\mathbf{k}$. Find

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outward normal and S is the surface bounded below by $z = 0$, above by $z = 4 - x^2 - y^2$, and on the sides by $x^2 + y^2 = 3$. The answer is

(a) $\dfrac{249\pi}{4}$ (b) $\dfrac{250\pi}{4}$ (c) $\dfrac{251\pi}{4}$ (d) $\dfrac{252\pi}{4}$

(e) $\dfrac{253\pi}{4}$ (f) $\dfrac{254\pi}{4}$ (g) $\dfrac{255\pi}{4}$ (h) $\dfrac{256\pi}{4}$

5. Find the determinant and the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 6 \end{pmatrix}.$$

The pair $\det A$, $\operatorname{rank} A$ in that order is

(a) 1, 0	(b) 1, 1	(c) 1, 2	(d) 1, 3
(e) 2, 0	(f) 2, 1	(g) 2, 2	(h) 2, 3

6. Let $f(t)$ be the function that is periodic with period 2 on the positive half-axis, and for $0 \leq t < 2$ is defined as follows

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & 1 \leq t < 2 \end{cases}$$

Let $F(s)$ denote its Laplace transform. Then $F(1)$ equals

$(a)\ \dfrac{1}{1-e^{-2}}$ $(b)\ \dfrac{1+e}{1-e^{-2}}$ $(c)\ 0$ $(d)\ \dfrac{1-e^{-1}}{1-e^{-2}}$

$(e)\ \dfrac{1}{1-e^{-1}}$ $(f)\ \dfrac{1+e}{1-e^{-1}}$ $(g)\ \dfrac{1-e}{1-e^{-1}}$ $(h)\ 1$

7. Let $y(x)$ be the solution to the equation

$$(-8x + 4y)\,dx + (4x - 2y)\,dy = 0.$$

If $y(0) = 0$, what is $y(1)$?

(*a*) -5 (*b*) -4 (*c*) -3 (*d*) -2

(*e*) -1 (*f*) 0 (*g*) 1 (*h*) 2

8. Find div $\mathbf{F}$ and curl $\mathbf{F}$ for $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. The pair div $\mathbf{F}$, curl $\mathbf{F}$ in that order is

(a) $x + y + z,\ x\mathbf{i} + 2\mathbf{j}$

(b) $xz + yz + xy,\ x\mathbf{i} + 2\mathbf{j}$

(c) $z\mathbf{i} + z\mathbf{j},\ x\mathbf{i} + 2\mathbf{j}$

(d) $0,\ x\mathbf{i} + 2\mathbf{j}$

(e) $x + y + z,\ (x - y)\mathbf{i} + (x - y)\mathbf{j}$

(f) $2z,\ (x - y)\mathbf{i} + (x - y)\mathbf{j}$

(g) $0,\ (x - y)\mathbf{i} + (x - y)\mathbf{j}$

(h) $z,\ (x - y)\mathbf{i} + (x - y)\mathbf{j}$

9. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (2z+x)\mathbf{i}+(y-z)\mathbf{j}+(x+y)\mathbf{k}$ and C is the boundary of the triangle that is the part of the plane $x+y+z=1$ that lies in the first octant oriented upward. The answer is

(a) 1	(b) $\frac{3}{2}$	(c) 2	(d) $\frac{5}{2}$
(e) 3	(f) $\frac{7}{2}$	(g) 4	(h) $\frac{9}{2}$

10. Let $\mathcal{U}(t)$ denote the Heaviside function. If

$$F(s) = e^{-s}\frac{s}{s^2+9}$$

then $f(0)$ equals

(a) $\cos(3t)$ (b) $\sin(3t)$ (c) $\mathcal{U}(t-1)\sin(3(t-1))$ (d) $\mathcal{U}(t-1)\cos(3(t-1))$

(e) $\cos(t)$ (f) $\sin(t)$ (g) $\mathcal{U}(t)\sin(3t)$ (h) $\mathcal{U}(t)\cos(3t)$

11. Find the Laplace transform of

$$t\int_0^t \sin\tau\, d\tau.$$

The answer is

$(a)\ \dfrac{s^2+s+1}{s^2(s^2+1)}$ $(b)\ \dfrac{2s^2+s+1}{s^2(s^2+1)}$ $(c)\ \dfrac{3s^2+s+1}{s^2(s^2+1)}$ $(d)\ \dfrac{4s^2+s+1}{s^2(s^2+1)}$

$(e)\ \dfrac{s^2+1}{s^2(s^2+1)^2}$ $(f)\ \dfrac{2s^2+1}{s^2(s^2+1)^2}$ $(g)\ \dfrac{3s^2+1}{s^2(s^2+1)^2}$ $(h)\ \dfrac{4s^2+1}{s^2(s^2+1)^2}$

12. (free response) The motion of a weight on a spring in a liquid in presence of a driving force is described by

$$x'' - 10x' + 25x = 30t + 3$$

where $x(t)$ is the position of the weight at time t. Using either method of undetermined coefficients or variation of parameters (but **not** the Laplace transform) find the general solution $x(t)$. No credit will be given for a solution that uses the Laplace transform.

13. (free response) Let

$$A = \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}.$$

Use the method of eigenvalues and eigenvectors to find the general solution to the system

$$X' = AX$$

where $X(t)$ is the column vector of the unknowns $x_1(t)$ and $x_2(t)$. (Do not solve this problem using the Laplace Transform).

14. (free response) Find out if the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

is diagonalizable. If yes, find a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

15. (free response) (worth 20 points) Use the Laplace transform to solve the following system of diffferential equations.

$$2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1$$

$$\frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 0.$$

with initial conditions $x(0) = 0$, $y(0) = 0$.