Math 240 Fall 2002 Final Exam

1. Solve the initial value problem

$$2xy\ dy + (y^2 - 6x^2)\ dx = 0, \qquad y(-2) = 0.$$

2. Solve the initial value problem

$$y'' + 4y' + 8y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

3. Find the general solution of the differential equation

$$y'' - 2y' + y = 2e^t.$$

4. Find an equation of the plane through the origin parallel to the line $x = 1 - t$, $y = 3t$, $z = 2 + t$ and orthogonal to the plane $x + y + z = 5$.

5. Let V be the vector space of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbf{R}^5$ such that

$$x_1 + x_3 = 0$$
$$x_1 - x_2 + 2x_3 - x_4 = 0.$$

What is the dimension of V?

6. Let A be the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

Find the inverse of the matrix A.

7. A certain 4×4 matrix A has eigenvalues 0, 1, and 2. Suppose also that

$$A \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad A \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

   (a) Determine whether there exist a diagonal matrix D and an invertible matrix P such that $A = P^{-1}DP$.

   (b) If your answer to (a) is positive, find the sum of all entries of D. *Hint: this does not require finding D or P.*

8. Solve the initial value problem

$$xy' = y \ln x, \qquad y(e) = e.$$

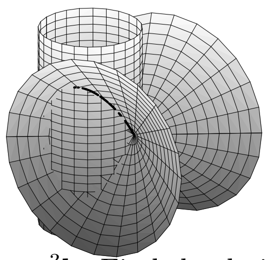
9. Find the solution to the system

$$X' = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} X$$

that also satisfies

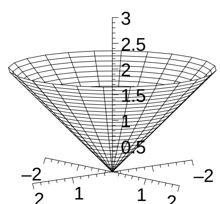
$$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

10. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = z^2\mathbf{i} + (\sin y)\mathbf{j} + 2xz\mathbf{k}$, and C is the curve with endpoints $A = (0,0,0)$ and $B = (1,1,1)$ which is obtained by intersecting the surfaces $x^2 - 2y^2 + z^2 = 0$ and $(x-1)^2 + y^2 = 1$, and oriented from A to B.



11. Let $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$. Find the derivative of the scalar function $\operatorname{div} \mathbf{F}$ at the point $P = (1,2,3)$ in the direction of the vector $(1,1,-1)$.

12. Let S be the part of the cone $x^2 + y^2 = z^2$ between $z = 0$ and $z = 2$ planes. Evaluate $\int_S z \, dA$.



13. Does there exist an orthogonal matrix A such that

$$A \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \; ?$$

Justify your answer.

14. Determine whether it is true that for every solution $X(t)$ to the system

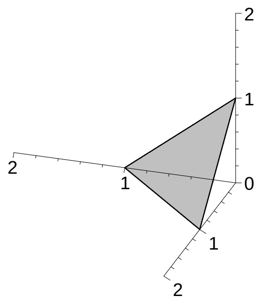
$$X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X$$

we have

$$\lim_{t \to +\infty} X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

15. Find the volume of the parallelepiped with sides given by the vectors $(1,-1,2)$, $(2,1,3)$, and $(-1,-4,1)$.

16. Let $\mathbf{F} = (y^2+z^2)\mathbf{i}+(z^2+x^2)\mathbf{j}+(x^2+y^2)\mathbf{k}$. Find the circulation $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where C is the boundary of the triangle cut from the plane $x+y+z=1$ by the first octant, counterclockwise as viewed from above.



17. Evaluate $\oint_C (3y\ dx + 2x\ dy)$, where C is the boundary of the region $0 \le x \le \pi$, $0 \le y \le \sin x$, traversed counterclockwise.

18. Find the outward flux of the vector field $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$ through the boundary of the region inside the solid cylinder $x^2+y^2 \le 4$ between the plane $z=0$ and the paraboloid $z=x^2+y^2$.