Math 240: Some More Challenging Linear Algebra Problems

Although problems are categorized by topics, this should not be taken very seriously since many problems fit equally well in several different topics.
Note that for lack of time some of the material used here might not be covered in Math 240.

Basic Definitions

1. Which of the following sets are linear spaces?
   a) $\{X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$
   b) The set of solutions x of $Ax = 0$, where A is an $m \times n$ matrix.
   c) The set of 2×2 matrices A with $\det(A) = 0$.
   d) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\,dx = 0$.
   e) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$.

2. Which of the following sets of vectors are bases for $\mathbb{R}^2$?
   a). $\{(0,\ 1),\ (1,\ 1)\}$
   b). $\{(1,\ 0),\ (0,\ 1),\ (1,\ 1)\}$
   c). $\{(1,\ 0),\ (-1,\ 0\}$
   d). $\{(1,\ 1),\ (1,\ -1)\}$
   e). $\{((1,\ 1),\ (2,\ 2)\}$
   f). $\{(1,\ 2)\}$

3. For which real numbers x do the vectors: $(x,1,1,1)$, $(1,x,1,1)$, $(1,1,x,1)$, $(1,1,1,x)$ *not* form a basis of $\mathbb{R}^4$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

4. If A is a 5×5 matrix with $\det A = -1$, compute $\det(-2A)$.

5. Let A be an $n \times n$ matrix of real or complex numbers. Which of the following statements are *equivalent* to: "the matrix A is invertible"?
   a) The columns of A are linearly independent.
   b) The columns of A span $\mathbb{R}^n$.
   c) The rows of A are linearly independent.
   d) The kernel of A is 0.
   e) The only solution of the homogeneous equations $Ax = 0$ is $x = 0$.
   f) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is 1-1.
   g) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is onto.
   h) The rank of A is n.

i) The adjoint, A^*, is invertible.

j) $\det A \neq 0$.

Linear Equations

6. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $AX = Y$.

   c) If $n > k$ the nullspace of A has dimension greater than zero.

   d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

   e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

7. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

   a) A is 1-to-1 (hence $n \leq k$).

   b) $\dim \ker(A) = 0$.

   c) A has a *left inverse* B, so $BA = I$.

   d) The columns of A are linearly independent.

8. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

   a) A is onto (hence $n \geq k$).

   b) $\dim \operatorname{im}(A) = k$.

   c) A has a *right inverse* B, so $AB = I$.

   d) The columns of A span $\mathbb{R}^k$.

9. Let A be a 4×4 matrix with determinant 7. Give a proof or counterexample for each of the following.

   a) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

   b) some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

   c) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution.

   d) For all vectors $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

10. Let A and B be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

a) $BA = 0$

b) Either $A = 0$ or $B = 0$ (or both).

c) If $\det A = -3$, then $B = 0$.

d) If B is invertible then $A = 0$.

e) There is a vector $V \neq 0$ such that $BAV = 0$.

11. Consider the system of equations

$$\begin{aligned} x + y - \ \ z &= a \\ x - y + 2z &= b. \end{aligned}$$

a) Find the general solution of the homogeneous equation.

b) A particular solution of the inhomogeneous equations when $a = 1$ and $b = 2$ is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of the inhomogeneous equations.

c) Find some particular solution of the inhomogeneous equations when $a = -1$ and $b = -2$.

d) Find some particular solution of the inhomogeneous equations when $a = 3$ and $b = 6$.

[Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

12. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

a) Find the general solution $\mathbf{Z}$ of the homogeneous equation $A\mathbf{Z} = 0$.

b) Find some solution of $A\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c) Find the general solution of the equation in part b).

d) Find some solution of $A\mathbf{X} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and of $A\mathbf{X} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

e) Find some solution of $A\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

f) Find some solution of $A\mathbf{X} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. [Note: $\binom{7}{2} = \binom{1}{2} + 2\binom{3}{0}$].

[Remark: After you have done parts a), b) and e), it is possible immediately to write the solutions to the remaining parts.]

13. Consider the system of equations

$$\begin{aligned} x+y-\ z &= a \\ x-y+2z &= b \\ 3x+y\quad &= c \end{aligned}$$

a) Find the general solution of the homogeneous equation.

b) If $a=1$, $b=2$, and $c=4$, then a particular solution of the inhomogeneous equations is $x=1$, $y=1$, $z=1$. Find the most general solution of these inhomogeneous equations.

c) If $a=1$, $b=2$, and $c=3$, show these equations have *no* solution.

d) If $a=0$, $b=0$, $c=1$, show the equations have *no* solution. [Note: $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\2\\4\end{smallmatrix}\right) - \left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$].

e) Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$. Compute $\det A$.

[Remark: After you have done parts a), and c), it is possible immediately to write the solutions to the remaining parts with no additional computation.]

14. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A = ?$

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

Linear Maps

15. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

e) Find the inverse of each of these maps.

16. a) Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

b) Find a 3×3 matrix A mapping $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates the x_1 x_3 plane by 60 degrees and leaves the x_2 axis fixed.

17. Find a real 2×2 matrix A (other than $A = I$) such that $A^5 = I$.

18. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

a) If L is invertible, then L^{-1} is also invertible.

b) If $LV = 5V$ for all vectors V, then $L^{-1}W = (1/5)W$ for all vectors W.

c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

e) The zero map ($0\mathbf{V} = 0$ for all vectors $\mathbf{V}$) is invertible.

f) The identity map ($I\mathbf{V} = \mathbf{V}$ for all vectors $\mathbf{V}$) is invertible.

g) If L is invertible, then $L^{-1}0 = 0$.

h) If $L\mathbf{V} = 0$ for some non-zero vector $\mathbf{V}$, then L is not invertible.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

19. Let L, M, and P be linear maps from the (two dimensional) plane to the plane given in terms of the standard $\mathbf{i}, \mathbf{j}$ basis vectors by:

$L\mathbf{i} = \mathbf{j}, \quad L\mathbf{j} = -\mathbf{i}$ (rotation by 90 degrees counterclockwise)

$M\mathbf{i} = -\mathbf{i}, \quad M\mathbf{j} = \mathbf{j}$ (reflection across the vertical axis)

$NV = -V$ (reflection across the origin)

a) Draw pictures describing the actions of the maps L, M, and N and the compositions: LM, ML, LN, NL, MN, and NM.

b) Which pairs of these maps commute?

c) Which of the following identities are correct—and why?

1) $L^2 = N$	2) $N^2 = I$	3) $L^4 = I$	4) $L^5 = L$
5) $M^2 = I$	6) $M^3 = M$	7) $MNM = N$	8) $NMN = L$

d) Find matrices representing each of the linear maps L, M, and N.

20. Give a proof or counterexample the following. In each case your answers should be brief.

a) Suppose that u, v and w are vectors in a vector space V and $T : V \to W$ is a linear map. If u, v and w are linearly dependent, is it true that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent? Why?

b) If $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map, is it possible that the nullspace of T is one dimensional?

21. Let V be a vector space and $\ell : V \to \mathbb{R}$ be a linear map. If $z \in V$ is not in the nullspace of ℓ, show that every $x \in V$ can be decomposed uniquely as $x = v + cz$, where v is in the nullspace of ℓ and c is a scalar. [Moral: The nullspace of a linear functional has codimension one.]

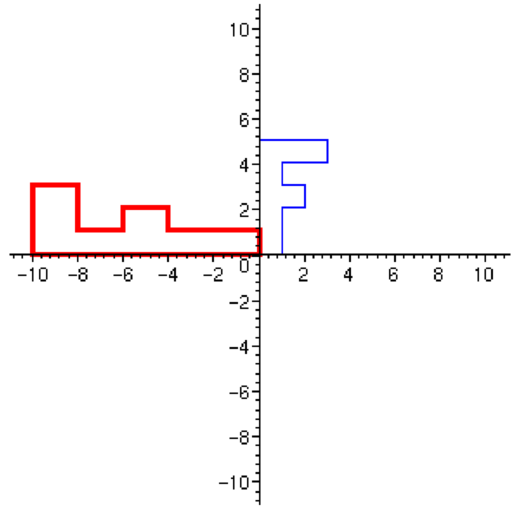
22. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$. Show that BA can *not* be invertible.

23. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Show that the area$(Q') = |ad - bc|$. [More generally, the area of any region is magnified by $|ad - bc|$, which is called the *determinant* of A.]

24. a). Find a linear map of the plane, $A : \mathbb{R}^2 \to \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



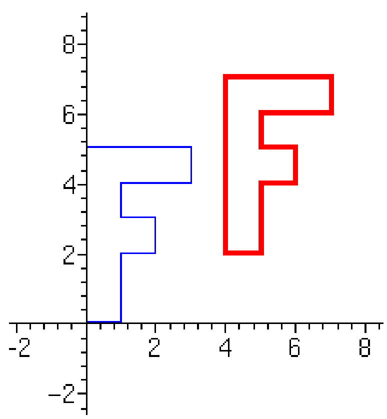
b). Find a linear map of the plane that inverts this map, that is, it maps the larger **F** to the smaller.

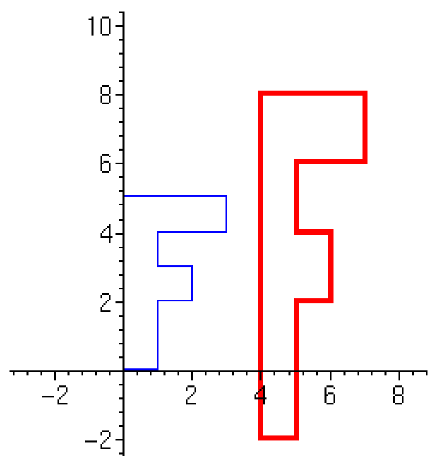
25. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

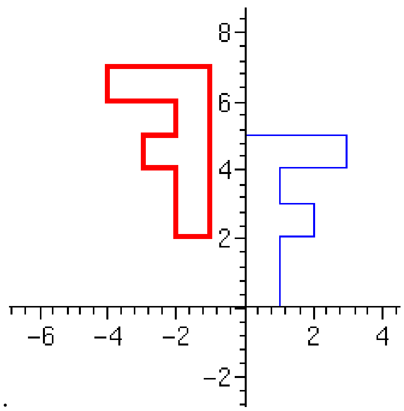
$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

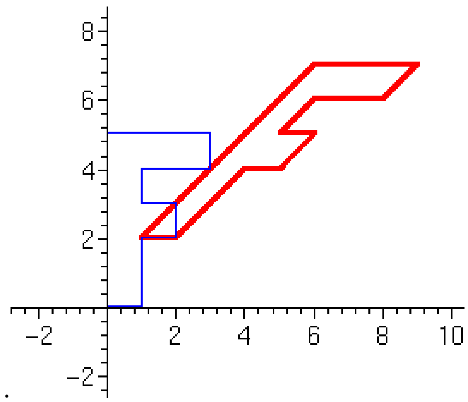
Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].

a). 

b). 



c).



d).

26. Find all linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane $\{\,(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\,\}$.

27. Let V be an n-dimensional vector space and $T : V \to V$ a linear transformation such that the image and kernel of T are identical.

    a) Prove that n is even.

    b) Give an example of such a linear transformation T.

28. Let $V \subset \mathbb{R}^{11}$ be a linear subspace of dimension 4 and consider the family $\mathcal{A}$ of all linear maps $L : \mathbb{R}^{11} -> \mathbb{R}^9$ each of whose nullspace contain V.

    Show that $\mathcal{A}$ is a linear space and compute its dimension.

29. Let L be a 2×2 matrix. For each of the following give a proof or counterexample.

    a) If $L^2 = 0$ then $L = 0$.

    b) If $L^2 = L$ then either $L = 0$ or $L = I$.

    c) If $L^2 = I$ then either $L = I$ or $L = -I$.

30. Let $\mathcal{S} \subset \mathbb{R}^3$ be the subspace spanned by the two vectors $v_1 = (1, -1, 0)$ and $v_2 = (1, -1, 1)$ and let $\mathcal{T}$ be the orthogonal complement of $\mathcal{S}$.

    a) Find an orthogonal basis for $\mathcal{S}$ and use it to find the 3×3 matrix P that projects vectors orthogonally into $\mathcal{S}$.

    b) Find an orthogonal basis for $\mathcal{T}$ and use it to find the 3×3 matrix Q that projects vectors orthogonally into $\mathcal{T}$.

    c) Verify that $P = I - Q$. How could you seen this in advance?

31. a) Let $\mathbf{v} := (\alpha, \beta, \gamma)$ and $\mathbf{x} := (x, y, z)$ be any vectors in $\mathbb{R}^3$. Viewed as column vectors, find a 3×3 matrix $A_{\mathbf{v}}$ so that the *cross product* $\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x}$.

ANSWER:

$$\mathbf{v} \times \mathbf{x} = A_{\mathbf{v}}\mathbf{x} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where the anti-symmetric matrix $A_{\mathbf{v}}$ is defined by the above formula.

b) From this, one has $\mathbf{v} \times (\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}(\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}^2\mathbf{x}$ (why?). Combined with the cross product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{w}$, show that

$$A_{\mathbf{v}}^2\mathbf{x} = \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{v} - \|\mathbf{v}\|^2\mathbf{x}.$$

c) If $\mathbf{n} = (a, b, c)$ is a unit vector, use this formula to show that (perhaps surprisingly) the orthogonal projection of $\mathbf{x}$ into the plane perpendicular to $\mathbf{n}$ is given by

$$\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n} = -A_{\mathbf{n}}^2\mathbf{x} = -\begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}$$

(See Problem 83, parts a-b)).

Algebra of Matrices

32. Every real upper triangular $n \times n$ matrix (a_{ij}) with $a_{ii} = 1$, $i = 1, 2, \ldots, n$ is invertible. Proof or counterexample.

33. Let $L\colon V \to V$ be a linear map on a vector space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \ge 1$.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \ge j$. Does your proof require that V is finite dimensional?

c) Let A be an $n \times n$ matrix. If $A^j = 0$ for some integer j, show that $A^n = 0$.

34. Let $L\colon V \to V$ be a linear map on a vector space V and $z \in V$ a vector with the property that $L^{k-1}z \ne 0$ but $L^k z = 0$. Show that z, Lz, $\ldots L^{k-1}z$ are linearly independent.

35. Let A, B, and C be any $n \times n$ matrices.

a) Show that $\operatorname{trace}(ABC) = \operatorname{trace}(CAB) = \operatorname{trace}(BCA)$.

b) $\operatorname{trace}(ABC) \stackrel{?}{=} \operatorname{trace}(BAC)$. Proof or counterexample.

36. Let A be an $n \times n$ matrix. If $AB = BA$ for *all* invertible matrices B, show that $A = cI$ for some scalar c.

37. There are no square matrices A, B with the property that $AB - BA = I$. Proof or counterexample.

Eigenvalues and Eigenvectors

38. Let A be an invertible matrix. If $\mathbf{V}$ is an eigenvector of A, show it is also an eigenvector of both A^2 and A^{-2}. What are the corresponding eigenvalues?

39. Let the $n \times n$ matrix A have an eigenvalue λ with corresponding eigenvector v.

    True or False

    a) $-v$ is an eigenvalue of $-A$ with eigenvalue $-\lambda$.
    b) If v is also an eigenvector of the $n \times n$ matrix B with eigenvalue μ, then $\lambda\mu$ is an eigenvalue of AB.
    c) Let c be a scalar.. Then $(\lambda + c)^2$ is an eigenvalue of $A^2 + 2cA + c^2 I$.
    d) Let μ be an eigenvalue of the $n \times n$ matrix$B$, Then $\lambda + \mu$ is an eigenvalue of $A + B$.
    e) Let c be a scalar. Then $c\lambda$ is an eigenvalue of cA.

40. Let A and B be $n \times n$ complex matrices that commute: $AB = BA$. If λ is an eigenvalue of A, let $\mathcal{V}_\lambda$ be the subspace of all eigenvectors having this eigenvalue.

    a) Show there is an vector $v \in \mathcal{V}_\lambda$ that is also an eigenvector of B, possibly with a different eigenvalue.
    b) Give an example showing that some vectors in $\mathcal{V}_\lambda$ may not be an eigenvectors of B.
    c) If all the eigenvalues of A are distinct (so each has algebraic multiplicity one), show that there is a basis in which both A and B are diagonal. Also, give an example showing this may be false if some eigenvalue of A has multiplicity greater than one.

41. Let A be a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding linearly independent eigenvectors V_1, V_2, V_3 which we can therefore use as a basis.

    a) If $X = aV_1 + bV_2 + cV_3$, compute AX, A^2X, and $A^{35}X$ in terms of λ_1, λ_2, λ_3, V_1, V_2, V_3, a, b and c (only).
    b) If $\lambda_1 = 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$, compute $\lim_{k\to\infty} A^k X$. Explain your reasoning clearly.

42. Let Z be a complex square matrix whose self-adjoint part is positive definite, so $Z+Z^*$ is positive definite.

 a) Show that the eigenvalues of Z have positive real part.

 b) Is the converse true? Proof or counterexample.

43. Let A be a square matrix with the property that the sum of the elements in each of its columns is 1. Show that $\lambda = 1$ is an eigenvalue of A. [These matrices arise in the study of Markov chains.]

44. Given any real monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, find an $n \times n$ real matrix with this as its characteristic polynomial. [This is related to writing an n^{th} order linear ordinary differential equation as a system of first order linear equations.]

45. Compute the value of the determinant of the 3×3 complex matrix X, provided that $\operatorname{tr}(X) = 1$, $\operatorname{tr}(X^2) = -3$, $\operatorname{tr}(X^3) = 4$. [Here $\operatorname{tr}(A)$ denotes the the *trace*, that is, the sum of the diagonal entries of the matrix A.]

46. Let $A := \begin{pmatrix} 4 & 4 & 4 \\ -2 & -3 & -6 \\ 1 & 3 & 6 \end{pmatrix}$. Compute

 a) the characteristic polynomial,

 b) the eigenvalues,

 c) *one* of the corresponding eigenvectors.

47. Let A be a square matrix. In the following, a sequence of matrices C_j *converges* if all of its elements converge.
 Prove that the following are equivalent:
 (i) $A^k \to 0$ as $k \to \infty$ [each of the elements of A^k converge to zero].
 (ii) All the eigenvalues λ_j of A have $|\lambda_j| < 1$.
 (iii) The matrix geometric series $\sum_0^\infty A^k$ converges to $(I - A)^{-1}$.

Inner Products and Quadratic Forms

48. Let V, W be vectors in the plane $\mathbb{R}^2$ with lengths $\|V\| = 3$ and $\|W\| = 5$. What are the maxima and minima of $\|V + W\|$? When do these occur?

49. Let V, W be vectors in $\mathbb{R}^n$.

a) Show that the Pythagorean relation $\|V+W\|^2 = \|V\|^2 + \|W\|^2$ holds if and only if V and W are orthogonal.

b) Prove the *parallelogram identity* $\|V+W\|^2 + \|V-W\|^2 = 2\|V\|^2 + 2\|W\|^2$ and interpret it geometrically.

50. Let $A = (-6,3)$, $B = (2,7)$, and C be the vertices of a triangle. Say the altitudes through the vertices A and B intersect at $Q = (2,-1)$. Find the coordinates of C.

[The *altitude* through a vertex of a triangle is a straight line through the vertex that is perpendicular to the opposite side — or an extension of the opposite side. Although not needed here, the three altitudes always intersect in a single point, sometimes called the orthocenter of the triangle.]

51. Find all vectors in the plane (through the origin) spanned by $\mathbf{V} = (1, 1-2)$ and $\mathbf{W} = (-1,1,1)$ that are perpendicular to the vector $\mathbf{Z} = (2,1,2)$.

52. Let U, V, W be orthogonal vectors and let $Z = aU + bV + cW$, where a, b, c are scalars.

a) (Pythagoras) Show that $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2 + c^2\|W\|^2$.

b) Find a formula for the coefficient a in terms of U and Z only. Then find similar formulas for b and c. [Suggestion: take the inner product of $Z = aU + bV + cW$ with U].

Remark The resulting simple formulas are one reason that orthogonal vectors are easier to use than more general vectors. This is vital for Fourier series.

c) Solve the following equations:

$$\begin{aligned}
x + y + z + w &= 2 \\
x + y - z - w &= 3 \\
x - y + z - w &= 0 \\
x - y - z + w &= -5
\end{aligned}$$

[Suggestion: Observe that the columns vectors in the coefficient matrix are orthogonal.]

53. Let A be a square matrix of real numbers whose columns are (non-zero) orthogonal vectors.

a) Show that A^TA is a diagonal matrix — whose inverse is thus obvious to compute.

b) Use this observation (or any other method) to discover a simple general formula for the inverse, A^{-1} involving only its transpose, A^T, and $(A^TA)^{-1}$. In the special case where the columns of A are orthonormal, your formula should reduce to $A^{-1} = A^T$.

c) Apply this to again solve the equations in Problem (52c).

54. [Gram-Schmidt Orthogonalization]

a) Let $A := \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Briefly show that the bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $(x, y) \mapsto x^t Ay$ gives a scalar product.

b) Let $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be the linear functional $\alpha : (x_1, x_2, x_3) \mapsto x_1 + x_2$ and let $v_1 := (-1, 1, 1)$, $v_2 := (2, -2, 0)$ and $v_3 := (1, 0, 0)$ be a basis of $\mathbb{R}^3$. Using the scalar product of the previous part, find an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ with $e_1 \in \operatorname{span}\{v_1\}$ and $e_2 \in \ker \alpha$.

55. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map defined by the matrix A. If the matrix B satisfies the relation $\langle AX, Y\rangle = \langle X, BY\rangle$ for all vectors $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^k$, show that B is the transpose of A, so $B = A^T$. [This basic property of the transpose,

$$\langle AX, Y\rangle = \langle X, AY\rangle,$$

is the *only* reason the transpose is important.]

56. In $\mathbb{R}^4$, compute the distance from the point $(1, -2, 0, 3)$ to the hyperplane $x_1 + 3x_2 - x_3 + x_4 = 3$.

57. Find the (orthogonal) projection of $\mathbf{x} := (1, 2, 0)$ into the following subspaces:

a) The line spanned by $\mathbf{u} := (1, 1, -1)$.

b) The plane spanned by $\mathbf{u} := (0, 1, 0)$ and $\mathbf{v} := (0, 0, -2)$

c) The plane spanned by $\mathbf{u} := (0, 1, 1)$ and $\mathbf{v} := (0, 1, -2)$

d) The plane spanned by $\mathbf{u} := (1, 0, 1)$ and $\mathbf{v} := (1, 1, -1)$

e) The plane spanned by $\mathbf{u} := (1, 0, 1)$ and $\mathbf{v} := (2, 1, 0)$.

f) The subspace spanned by $\mathbf{u} := (1, 0, 1)$, $\mathbf{v} := (2, 1, 0)$ and $\mathbf{w} := (1, 1, 0)$.

58. Let $\mathcal{S} \subset \mathbb{R}^4$ be the vectors $X = (x_1, x_2, x_3, x_4)$ that satisfy $x_1 + x_2 - x_3 + x_4 = 0$.

a) What is the dimension of $\mathcal{S}$?

b) Find a basis for the orthogonal complement of $\mathcal{S}$.

59. Let $\mathcal{S} \subset \mathbb{R}^4$ be the subspace spanned by the two vectors $v_1 = (1, -1, 0, 1)$ and $v_2 = (0, 0, 1, 0)$ and let $\mathcal{T}$ be the orthogonal complement of $\mathcal{S}$.

a) Find an orthogonal basis for $\mathcal{T}$.

b) Compute the orthogonal projection of $(1, 1, 1, 1$ into $\mathcal{S}$.

60. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with the property that $L\mathbf{v} \perp \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^3$. Prove that L cannot be invertible.
Is a similar assertion true for a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$?

61. In a complex vector space (with a hermitian inner product), if a matrix A satisfies $\langle X,\, AX\rangle = 0$ for all vectors X, show that $A = 0$. [The previous problem shows that this is false in a real vector space].

62. Using the inner product $\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$, for which values of the real constants α, β, γ are the polynomials $p_1(x) = 1$, $p_2(x) = \alpha + x$ $p_3(x) = \beta + \gamma x + x^2$ orthogonal?

63. Let $\mathcal{P}_2$ be the space of polynomials $p(x) = a + bx + cx^2$ of degree at most 2 with the inner product $\langle p,\, q\rangle = \int_{-1}^{1} p(x)q(x)\, dx$. Let ℓ be the functional $\ell(p) := p(0)$. Find $h \in \mathcal{P}_2$ so that $\ell(p) = \langle h,\, p\rangle$ for all $p \in \mathcal{P}_2$.

64. Let $C[-1, 1]$ be the real inner product space consisting of all continuous functions $f : [-1, 1] \to \mathbb{R}$, with the inner product $\langle f,\, g\rangle := \int_{-1}^{1} f(x)g(x)\, dx$. Let W be the subspace of odd functions, i.e. functions satisfying $f(-x) = -f(x)$. Find (with proof) the orthogonal complement of W.

65. a) Let $V \subset \mathbb{R}^n$ be a subspace and $Z \in \mathbb{R}^n$ a given vector. Find a unit vector X that is perpendicular to V with $\langle X,\, Z\rangle$ as large as possible.

    b) Compute $\max \displaystyle\int_{-1}^{1} x^3 h(x)\, dx$ where $h(x)$ is any continuous function on the interval $-1 \le x \le 1$ subject to the restrictions

$$\int_{-1}^{1} h(x)\, dx = \int_{-1}^{1} xh(x)\, dx = \int_{-1}^{1} x^2 h(x)\, dx = 0; \qquad \int_{-1}^{1} |h(x)|^2\, dx = 1.$$

    c) Compute $\displaystyle\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\, dx$.

66. [Dual variational problems] Let $V \subset \mathbb{R}^n$ be a linear space, $Q : R^n \to V^\perp$ the orthogonal projection into $V^\perp$, and $x \in \mathbb{R}^n$ a given vector. Note that $Q = I - P$, where P in the orthogonal projection into V

    a) Show that $\displaystyle\max_{\{z \perp V,\, \|z\|=1\}} \langle x,\, z\rangle = \|Qx\|$.

b) Show that $\min\limits_{v\in V}\|x-v\| = \|Qx\|$.

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

67. [Completing the Square] Let

$$\begin{aligned} Q(x) &= \sum a_{ij}x_ix_j + \sum b_ix_i + c \\ &= \quad \langle x,\, Ax\rangle + \langle b,\, x\rangle + c \end{aligned}$$

be a real quadratic polynomial so $x = (x_1,\dots,x_n)$, $b = (b_1,\dots,b_n)$ are real vectors and $A = a_{ij}$ is a real symmetric $n\times n$ matrix. Just as in the case $n=1$, if A is invertible show there is a change of variables $y == x - v$ (this is a translation by the vector v) so that in the new y variables Q has the form

$$Q = \langle y,\, Ay\rangle + \gamma \quad \text{that is,} \quad Q = \sum a_{ij}y_iy_j + \gamma,$$

where γ involves A, b, and c.

As an example, apply this to $Q(x) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

68. Let A be a positive definite $n\times n$ real matrix, β a real vector, and N a real unit vector.

a) For which value(s) of the real scalar c is the set

$$E := \{\, x\in\mathbb{R}^3 \mid \langle x,\, Ax\rangle + 2\langle\beta,\, x\rangle + c = 0\,\}$$

(an ellipsoid) non-empty? [Answer: $\langle\beta,\, A^{-1}\beta\rangle \ge c$. If $n=1$, this of course reduces to a familiar condition.]

b) For what value(s) of the scalar d is the plane $P := \{\, x\in\mathbb{R}^3 \mid \langle N,\, x\rangle = d\,\}$ tangent to the above ellipsoid E (assumed non-empty)?

[Answer:

$$d = -\langle N,\, A^{-1}\beta\rangle \pm \sqrt{\langle N,\, A^{-1}N\rangle}\sqrt{\langle\beta,\, A^{-1}\beta\rangle - c}.$$

For $n=1$ this is just the solution $d = \frac{-\beta\pm\sqrt{\beta^2-ac}}{a}$ of the quadratic equation $ax^2 + 2\beta x + c = 0$.]

[Suggestion: First discuss the case where $A = I$ and $\beta = 0$. Then show how by a change of variables, the general case can be reduced to this special case.]

69. a) Compute

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+4x^2+9y^2)^2},\ \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+2xy+5y^2)^2},\ \iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+5x^2-4xy+5y^2)^2}.$$

b) Compute $\iint_{\mathbb{R}^2} \frac{dx_1\, dx_2}{[1+\langle x,\, Cx\rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

c) Let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\, dt = \alpha$. If C be a positive definite 2×2 matrix. Show that

$$\iint_{\mathbb{R}^2} h(\, \langle x,\, Cx\rangle \,)\; dA = \frac{\pi\alpha}{\sqrt{\det C}}.$$

d) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2)}\, dx\, dy$.

e) Compute $\iint_{\mathbb{R}^2} e^{-(5x^2-4xy+5y^2-2x+3)}\, dx\, dy$.

f) Generalize part c) to $\mathbb{R}^n$ to obtain a formula for

$$\iint_{\mathbb{R}^n} h(\, \langle x,\, Cx\rangle \,)\; dV,$$

where now C be a positive definite $n \times n$ matrix. The answer will involve some integral involving h and also the "area" of the unit sphere $S^{n-1} \hookrightarrow \mathbb{R}^n$.

70. Let $v_1 \ldots v_k$ be vectors in a linear space with an inner product $\langle\ ,\ \rangle$. Define the *Gram determinant* by $G(v_1, \ldots, v_k) = \det(\langle v_i,\, v_j\rangle)$.

a) If the $v_1 \ldots v_k$ are orthogonal, compute their Gram determinant.

b) Show that the $v_1 \ldots v_k$ are linearly independent if and only if their Gram determinant is not zero.

c) Better yet, if the $v_1 \ldots v_k$ are linearly independent, show that the symmetric matrix $(\langle v_i,\, v_j\rangle)$ is positive definite. In particular, the inequality $G(v_1, v_2) \ge 0$ is the *Schwarz inequality.*

d) Conversely, if A is any $n \times n$ positive definite matrix, show that there are vectors $v_1, \ldots, v_n$ so that $A = (\langle v_i,\, v_j\rangle)$.

e) Let $\mathcal{S}$ denote the subspace spanned by the linearly independent vectors $w_1 \ldots w_k$. If X is any vector, let $P_{\mathcal{S}}X$ be the orthogonal projection of X into $\mathcal{S}$, prove that the distance $\|X - P_{\mathcal{S}}X\|$ from X to $\mathcal{S}$ is given by the formula

$$\|X - P_{\mathcal{S}}X\|^2 = \frac{G(X, w_1, \ldots, w_k)}{G(w_1, \ldots, w_k)}.$$

71. Let $L : V \to W$ be a linear map between the linear spaces V and W, both having inner products. Denote by $\operatorname{im} L$ the *image* of V under L, that is, all vectors $w \in W$ such that there is at least one solution $v \in V$ of the equation $Lv = w$.

a) Show that $(\operatorname{im} L)^\perp = \ker L^*$, where L^* is the adjoint of L.

b) Show that $\dim \operatorname{im} L = \dim \operatorname{im} L^*$. [Don't use determinants.] This assertion is equivalent to the row rank of L equals the column rank of L.

72. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that

$$\dim \ker(L) - \dim \ker(L^*) = n - k.$$

($\ker(L^*)$ is often called the *cokernel* of L).

73. Let U, V, and W be finite dimensional vector spaces with inner products. If $A : U \to V$ and $B : V \to W$ are linear maps with adjoints A^* and B^*, define the linear map $C : V \to V$ by

$$C = AA^* + B^*B.$$

If $U \xrightarrow{A} V \xrightarrow{B} W$ is *exact* [that is, image $(A) = \ker(B)$], show that $C : V \to V$ is invertible.

74. [Bilinear and Quadratic Forms] Let ϕ be a bilinear form over the finite dimensional real vector space V. ϕ is called *non-degenerate* if $\phi(x, y) = 0$ for all $y \in V$ implies $x = 0$.

True or False

a) If ϕ is non-degenerate, then $\psi(x, y) := \frac{1}{2}[\phi(x, y) + \phi(y, x)]$ is a scalar product.

b) If $\phi(x, y) = -\phi(y, x)$ for all $x, y \in V$, then $\phi(z, z) = 0$ for all $z \in V$.

c) If ϕ is symmetric and $\phi(x, x) = 0$ for all $x \in V$, then $\phi = 0$.

d) Assume the bilinear forms ϕ and ψ are both symmetric and positive definite. Then $\{z \in V \mid \phi(x, z)^3 + \psi(y, z)^3 = 0\}$ is a subspace of V.

e) If ϕ and ψ are bilinear forms over V, then $\{z \in V \mid \phi(x, z)^2 + \psi(y, z)^2 = 0\}$ is a subspace of V.

Norms and Metrics

75. Let $\mathcal{P}_n$ be the space of real polynomials with degree at most n. Write $p(t) = \sum_{j=0}^n a_j t^j$ and $q(t) = \sum_{j=0}^n b_j t^j$.

True or False

a) Define the distance $d(p, q)$ between the polynomials $p(t)$ and $q(t)$ by $d(p, q) := \sum_{j=0}^n |a_j - b_j|$. Then $\|p\| = d(p, 0)$ is a norm on $\mathcal{P}_n$.

b) For $p \in \mathcal{P}_n$ let $\|p\| := 0$ when $p = 0$ and $\|p\| := \max(0, NP(p))$ for $p \neq 0$. Here $NP(p)$ is the set of all the real zeroes of p. Claim: $\|p\|$ is a norm on $\mathcal{P}_n$.

c) Define a norm $\|\cdot\|$ on $\mathcal{P}_n$ by $\|p\| := \max_{t \in [0,1]} |p(t)|$. Then there is a bilinear form ϕ on $\mathcal{P}_n$ with $\phi(p, p) = \|p\|^2$ for all $p \in \mathcal{P}_n$.

d) Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathcal{P}_n$ and $\|\cdot\|$ the associated norm. If α is an endomorphism of $\mathcal{P}_n$ with the property that $\|\alpha(p)\| = \|p\|$ for all $p \in \mathcal{P}_n$, then α is orthogonal in this scalar product.

e) The real function $(p, q) \mapsto (pq)'(0)$, where f' is the derivative of f, defines a scalar product on the subspace $\{p \in \mathcal{P}_n \mid p(0) = 0\}$.

Projections and Reflections

76. a) Let $v \in \mathbb{R}^n$ be a unit vector and Px the orthogonal projection of $x \in \mathbb{R}^n$ in the direction of v, that is, if $x = \text{const.}\, v$, then $Px = x$, while if $x \perp v$, then $Px = 0$. Show that $P = vv^T$ (here v^T is the transpose of the column vector v). In matrix notation, $(P)_{ij} = v_i v_j$.

b) Continuing, let Q be the orthogonal projection into the subspace perpendicular to v. Show that $Q = I - vv^T$.

c) Let u and v be orthogonal unit vectors and let R be the orthogonal projection into the subspace perpendicular to both u and v. Show that $R = I - uu^T - vv^T$.

77. A linear map $P\colon V \to V$ acting on a vector space V is called a *projection* if $P^2 = P$ (this P is not necessarily an orthogonal projection).

a) Show that the matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a projection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1, 2)$, $(-1, 0)$, (and $(0, 3)$ and their images under the map P. Also indicate both the image and nullspace of P. [Observe: since the image and nullspace of P are not orthogonal this is not an orthogonal projection.]

b) Repeat this for $Q := I - P$.

78. Mo re on general projections, so all one knows is that $P^2 = P$.

a) Show that for a projection P one can find complementary subspaces U, and W so that:

$$V = U \oplus W, \qquad Px = x \text{ for all } x \in U, \qquad Px = 0 \text{ for all } x \in W.$$

Thus, P is the projection into U.

b) Show that $Q := I - P$ is also a projection, but it projects into W.

c) If P is written as a matrix, it is similar to the block matrix $M = \begin{pmatrix} I_U & 0 \\ 0 & 0_W \end{pmatrix}$, where I_U is the identity map on U and 0_W the zero map on W.

d) Show that $\dim \operatorname{image}(P) = \operatorname{trace}(P)$.

e) If V has an inner product, show that the subspaces U and W are orthogonal if and only if $P = P^*$. Moreover, if $P = P^*$, then $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$. These are the *orthogonal projections*.

f) If two projections P and Q on V have the same rank, show they are similar.

79. Let P be a projection, so $P^2 = P$. If $c \neq 1$, compute $(I - cP)^{-1}$.

80. A linear map $R\colon V \to V$ acting on a vector space V is called a *reflection* if $R^2 = I$ (R is not necessarily an orthogonal reflection).

   a) Show that the matrix $R = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ is a reflection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, (and $(0,3)$ and their images under R. Also indicate both the subspaces of vectors that are map to themselves: $Ru = u$, and those that are mapped to their opposites: $Rw = -w$. [From your sketch it is clear that this R is not an orthogonal reflection.]

   b) More generally, show that for any reflection one can write $V = U \oplus W$ so that $Ru = u$ for all $u \in U$ and $Rw = -w$ for all $w \in W$. Thus, R is the reflection across U.

   c) Show that R is similar to the block matrix $M = \begin{pmatrix} I_U & 0 \\ 0 & -I_W \end{pmatrix}$, where I_U is the identity map on U.

   d) Moreover, if V has an inner product, show that U and W are orthogonal if and only if $R = R^*$. Then R is an *orthogonal reflection.*

81. Show that projections and reflections are related by the formula $R = 2P - I$. This makes obvious the relation between the above several problems.

82. Let A be an $n \times n$ real matrix with the property that $(A - \alpha I)(A - \beta I) = 0$, where $\alpha \neq \beta$ are real. Show that A is similar to the block matrix $\begin{pmatrix} \alpha I_k & 0 \\ 0 & \beta I_{n-k} \end{pmatrix}$, where I_k is the $k \times k$ identity matrix. [REMARK: one approach is to reduce the the special case where A is a projection: $A^2 = A$. This problem generalizes the above problems on projections ($P^2 = P$) and reflections ($R^2 = I$)].

83. Let $\mathbf{n} := (a, b, c) \in \mathbb{R}^3$ be a *unit* vector and $\mathcal{S}$ the plane of vectors (through the origin) perpendicular to $\mathbf{n}$.

   a) Show that the *orthogonal projection of* $\mathbf{x}$ *in the direction of* $\mathbf{n}$ can be written in the matrix form

$$\langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (\mathbf{n}\mathbf{n}^T)\mathbf{x} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

   where $\langle \mathbf{x}, \mathbf{n} \rangle$ is the usual inner product, $\mathbf{n}^T$ is the transpose of the column vector $\mathbf{n}$, and $\mathbf{n}\mathbf{n}^T$ is matrix multiplication.

b) Show that the *orthogonal projection* P of a vector $\mathbf{x} \in \mathbb{R}^3$ into $\mathcal{S}$ is

$$P\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (I - \mathbf{n}\mathbf{n}^T)\mathbf{x},$$

Apply this to compute the orthogonal projection of the vector $\mathbf{x} = (1, -2, 3)$ into the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

c) Find a formula similar to the previous part for the orthogonal *reflection* R of a vector across $\mathcal{S}$. Then apply it to compute the orthogonal reflection of the vector $\mathbf{v} = (1, -2, 3)$ across the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

d) Find a 3×3 matrix that projects a vector in $\mathbb{R}^3$ into the plane $x - y + 2z = 0$.

e) Find a 3×3 matrix that reflects a vector in $\mathbb{R}^3$ across the plane $x - y + 2z = 0$.

Similar Matrices

84. Let C and B be square matrices with C invertible. Show the following.

a) $(CBC^{-1})^2 = C(B^2)C^{-1}$

b) Similarly, show that $(CBC^{-1})^k = C(B^k)C^{-1}$ for any $k = 1, 2, \ldots$.

c) If B is also invertible, is it true that $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$? Why?

85. Let $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

a) Find an invertible matrix C such that $D := C^{-1}AC$ is a diagonal matrix. Thus, $A = CDC^{-1}$.

b) Compute A^{50}.

86. Prove that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ are similar.

87. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}$.

a) Are A and B similar? Why?

b) Are B and C similar? Why?

c) Show that B is not similar to any diagonal matrix.

88. Let $A(s) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ and let $M = A(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ If $s \neq 0$, show that $A(s)$ is similar to M. [Suggestion: Find some simple specific invertible matrix T –which will depend on s – so that $TA(s) = MT$.]

REMARK: In view of the first part of the previous problem, this a simple and fundamental counterexample to the assertion: "If $A(s)$ depends smoothly on the parameter s and is similar to M for all $s \neq 0$, then $A(0)$ is also similar to M."

89. Say a matrix A is *similar* to the matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Give a proof or counterexample for each of the following assertions.

a). $A^2 = A$
b). $\det A = 0$.
c). $\operatorname{trace} A = 1$.
d). $\lambda = 0$ is an eigenvalue of A.
e). $\lambda = 1$ is an eigenvalue of A.
f). $V = (1, 0)$ is an eigenvector of A.

90. A square matrix M is *diagonalized* by an invertible matrix S if SMS^{-1} is a diagonal matrix. Of the following three matrices, one can be diagonalized by an orthogonal matrix, one can be diagonalized but not by any orthogonal matrix, and one cannot be diagonalized. State which is which — and why.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}.$$

91. Let A be the matrix

$$A = \begin{pmatrix} 1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Show that there exists a matrix B with $BAB^{-1} = A^t$ (here A^t is the transpose of A).

92. Let A be an $n \times n$ matrix with real or comples coefficients and let S be an invertible matrix.

a) If $SAS^{-1} = \lambda A$ for some scalar λ, show that either $\lambda^n = 1$ or $A$is nilpotent.

b) If n is odd and $SAS^{-1} = -A$, show that 0 is an eigenvalue of A.

c) If n is odd and $SAS^{-1} = A^{-1}$, show that 1 is an eigenvalue of A.

Symmetric and Self-adjoint Maps

93. Let A and B be symmetric matrices with A positive definite.

a) Show there is a change of variables $y = Sx$ (so S is an invertible matrix) so that $\langle x,\, Ax\rangle = \|y\|^2$ (equivalently, $S^T AS = I$). One often rephrases this by saying that a positive definite matrix is *congruent* to the identity matrix.

b) Show there is a linear change of variables $y = Px$ so that *both* $\langle x,\, Ax\rangle = \|y\|^2$ and $\langle x,\, Bx\rangle = \langle y,\, Dy\rangle$, where D is a diagonal matrix.

c) If A is a positive definite matrix and B is positive semi-definite, show that

$$\operatorname{trace}(AB) \geq 0$$

with equality if and only if $B = 0$.

94. [Congruence of Matrices] Two $n\times n$ real symmetric matrices A, B are called *congruent* if there is an $n \times n$ invertible matrix T with $A = T^*BT$ (here T^* is the *adjoint* (or *transpose*) of T); equivalently, if

$$\langle Tx,\, ATy\rangle = \langle X,\, BY\rangle \qquad \text{for all vectors } x, y,$$

so T is just a change of coordinates.

True or False?

a) Over $\mathbb{R}$ the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) If A and B are congruent over $\mathbb{C}$, then A and B are similar over $\mathbb{C}$.

c) If A is real and all of its eigenvalues are positive, then over $\mathbb{R}$ A is congruent to the identity matrix.

d) Over $\mathbb{R}$ if A is congruent to the identity matrix, then all of its eigenvalues are positive.

95. Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$.

a) Show that

$$\lambda_1 = \min_{x\neq 0} \frac{\langle x,\, Ax\rangle}{\|x\|^2} \qquad \text{and} \qquad \lambda_n = \max_{x\neq 0} \frac{\langle x,\, Ax\rangle}{\|x\|^2}.$$

b) Show that

$$\lambda_2 = \min_{x\perp v_1,\ x\neq 0} \frac{\langle x,\, Ax\rangle}{\|x\|^2}.$$

96. Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and let $C = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{smallmatrix}\right)$ be the upper-left 2×2 block of A with eigenvalues $\mu_1 \leq \mu_2$.

a) Show that $\lambda_1 \leq \mu_1$ and $\lambda_n \geq \mu_2$.

b) Generalize.

97. Suppose that A is a real $n \times n$ symmetric matrix with two equal eigenvalues. If v is any vector, show that the vectors v, Av,..., $A^{n-1}v$ are linearly dependent.

98. Let Q and M be symmetric matrices with Q invertible. Show there is a matrix A such that $AQ + QA^* = M$.

99. Let the real matrix A be *anti-symmetric* (or *skew-symmetric*), that is, $A^* = -A$.

   a) Give an example of a 2×2 anti-symmetric matrix.

   b) Show that the diagonal elements of any $n \times n$ anti-symmetric matrix must all be zero.

   c) Show that every square matrix can (uniquely?) be written as the sum of a symmetric and an anti-symmetric matrix.

   d) Show that the eigenvalues of a real anti-symmetric matrix are purely imaginary.

   e) Show that $\langle \mathbf{V}, A\mathbf{V} \rangle = 0$ for every vector $\mathbf{V}$.

   f) If A is an $n \times n$ anti-symmetric matrix and n is odd, show that $\det A = 0$ — and hence deduce that A cannot be invertible.

   g) If n is even, show that $\det A \geq 0$. Show by an example that A may be invertible.

   h) If A is a real invertible $2n \times 2n$ anti-symmetric matrix, show there is a real invertible matrix S so that
   $$A = SJS^*,$$
   where $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$; here I_k is the $k \times k$ identity matrix. [Note that $J^2 = -I$ so the matrix J is like the complex number $i = \sqrt{-1}$.

Orthogonal and Unitary Maps

100. Let the real $n \times n$ matrix A be an *isometry*, that is, it preserves length:
$$\|Ax\| = \|x\| \quad \text{for all vectors} \quad x \in \mathbb{R}^n. \tag{1}$$
These are the *orthogonal transformations*.

   a) Show that (1) is equivalent to $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all vectors x, y, so A preserves inner products. HINT: use the *polarization* identity:
   $$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right). \tag{2}$$
   This shows how, in a real vector space, to recover a the inner product if you only know how to compute the (euclidean) length.

b) Show that (1) is equivalent to $A^{-1} = A^*$.

c) Show that (1) is equivalent to the columns of A being unit vectors that are mutually orthogonal.

d) Show that (1) implies $\det A = \pm 1$ and that all eigenvalues satisfy $|\lambda| = 1$.

e) If $n = 3$ and $\det A = +1$, show that $\lambda = 1$ is an eigenvalue.

f) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ have the property (1), namely $\|F(x)\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Then F is an orthogonal transformation. Proof or counterexample.

g) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a *rigid motion*, that is, it preserves the distance between any two points: $\|F(x) - F(y)\| = \|x - y\|$ for all vectors $x, y \in \mathbb{R}^n$. Show that $F(x) = F(0) + Ax$ for some orthogonal transformation A.

101. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector. Find a formula for the 3×3 matrix that determines a rotation of $\mathbb{R}^3$ through an angle θ with $\mathbf{n}$ as axis of rotation (assuming the axis passes through the origin). Here we outline one approach to find this formula — but before reading further, try finding it on your own.

a) (Example) Find a matrix that rotates $\mathbb{R}^3$ through the angle θ using the vector $(1, 0, 0)$ as the axis of rotation.

b) More generally, let $\mathbf{u}$ and $\mathbf{w}$ be orthonormal vectors in the plane perpendicular to $\mathbf{n}$. Show that the map

$$R_{\mathbf{n}} : \mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\, \mathbf{u} + \sin\theta\, \mathbf{w}$$

rotates $\mathbf{x}$ through an angle θ with $\mathbf{n}$ as axis of rotation. [Note: one needs more information to be able to distinguish between θ and $-\theta$].

c) Using Problem 102 to write $\mathbf{u}$ and $\mathbf{w}$, in terms of $\mathbf{n}$ and $\mathbf{x}$, show that one can rewrite the above formula as

$$\begin{aligned} R_{\mathbf{n}}\mathbf{x} &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\, [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] + \sin\theta\, (\mathbf{n} \times \mathbf{x}) \\ &= \mathbf{x} + \sin\theta\, (\mathbf{n} \times \mathbf{x}) + (1 - \cos\theta)[(\mathbf{x} \cdot \mathbf{n})\mathbf{n} - \mathbf{x}]. \end{aligned}$$

Thus, using Problem 31, if $\mathbf{n} = (a, b, c) \in \mathbb{R}^3$ deduce that:

$$R_{\mathbf{n}} = I + \sin\theta \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + (1 - \cos\theta) \begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix}.$$

d) Let $A_{\mathbf{n}}$ be as in Problem 31 (but using $\mathbf{n}$ rather than $\mathbf{v}$). Show that

$$R_{\mathbf{n}} = I + \sin\theta\, A_{\mathbf{n}} + (1 - \cos\theta)\, A_{\mathbf{n}}^2$$

e) Use this formula to find the matrix that rotates $\mathbb{R}^3$ through an angle of θ using as axis the line through the origin and the point $(1, 1, 1)$.

102. Recall (see Problem 83) that $\mathbf{u} := \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ is the projection of $\mathbf{x}$ into the plane perpendicular to the unit vector $\mathbf{n}$. Show that in $\mathbb{R}^3$ the vector

$$\mathbf{w} := \mathbf{n} \times \mathbf{u} = \mathbf{n} \times [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] = \mathbf{n} \times \mathbf{x}$$

is perpendicular to both $\mathbf{n}$ and $\mathbf{u}$, and that $\mathbf{w}$ has the same length as $\mathbf{u}$. Thus $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{w}$ are orthogonal with $\mathbf{u}$, and $\mathbf{w}$ in the plane perpendicular to the axis of rotation $\mathbf{n}$.

103. Show that the only real matrix that is orthogonal, symmetric and positive definite is the identity.

104. Let V be a finite dimensional vector space over $\mathbb{R}$and W a finite dimensional vector space over $\mathbb{C}$.

True or False

a) Let α be an endomorphism of W. In a unitary basis for W say M is a diagonal matrix all of whose eigenvalues satisfy $|\lambda| = 1$. Then α is a unitary matrix.

b) The set of orthogonal endomorphisms of V forms a ring under the usual addition and multiplication.

c) Let $\alpha \neq I$ be an orthogonal endomorphism of V with determinant 1. Then there is no $v \in V$ (except $v = 0$) satisfying $\alpha(v) = v$.

d) Let α be an orthogonal endomorphism of V and $\{v_1, \dots, v_k\}$ a linearly independent set of vectors in V. Then the vectors $\{\alpha(v_1), \dots, \alpha(v_k)\}$ are linearly independent.

e) Using the standard scalar product for $\mathbb{R}^3$, let $v \in \mathbb{R}^3$ be a unit vector, $\|v\| = 1$, and define the endomorphism $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ using the cross product: $\alpha(x) := v \times x$. Then the subspace $v^\perp$ is an invariant subspace of α and α is an orthogonal map on this subspace.

105. Let $R(t)$ be a family of real orthogonal matrices that depend smoothly on the real parameter t.

a) If $R(0) = I$, show that the derivative, $A := R'(0)$ is anti-symmetric, that is, $A^* = -A$.

b) Let the vector $x(t)$ be a solution of the differential equation $x' = A(t)x$, where the matrix $A(t)$ is anti-symmetric. Show that its (Euclidean) length is constant, $\|x(t)\|$ =const. In other words, using this $x(t)$ if we define the map $R(t)$ by $R(t)x(0) := x(t)$, then $R(t)$ is an orthogonal transformation.

c) Let $A(t)$ be an anti-symmetric matrix and let the square matrix $R(t)$ satisfy the differential equation $R' = AR$ with $R(0)$ an orthogonal matrix. Show that $R(t)$ is an orthogonal matrix.

Least Squares

106. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If the equation $Lx = b$ has no solution, instead frequently one wants to pick x to minimize the error: $\|Lx - b\|$ (here we use the Euclidean distance). Assume that the nullspace of L is zero.

    a) Show that the desired x is a solution of the *normal equations* $L^*Lx = L^*b$ (here L^* is the adjoint of L.). Note that since the nullspace of L is zero, $L^*L : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (why?).

    b) Apply this to find the optimal horizontal line that fits the three data points $(0,1)$, $(1,2)$, $(4,3)$.

    c) Similarly, find the optimal straight line (not necessarily horizontal) that fits the same data points.

107. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If A is not one-to-one, but the equation $Ax = y$ has some solution, then it has many. Is there a "best" possible answer? What can one say? Think about this before reading the next paragraph.

    If there is some solution of $Ax = y$, show there is exactly one solution x_1 of the form $x_1 = A^*w$ for some w, so $AA^*w = y$. Moreover of all the solutions x of $Ax = y$, show that x_1 is closest to the origin (in the Euclidean distance). [Remark: This situation is related to the case where where A is not onto, so there may not be a solution — but the method of least squares gives an "best" approximation to a solution.]

108. Let P_1, P_2,..., P_k be k points (think of them as *data*) in $\mathbb{R}^3$ and let $\mathcal{S}$ be the plane

$$\mathcal{S} := \left\{X \in \mathbb{R}^3 \,:\, \langle X,\, N\rangle = c\right\},$$

where $N \neq 0$ is a unit vector normal to the plane and c is a real constant. This problem outlines how to find the plane that *best approximates the data points* in the sense that it minimizes the function

$$Q(N,c) := \sum_{j=1}^{k} \operatorname{distance}(P_j, \mathcal{S})^2.$$

    a) Show that for a given point P

$$\operatorname{distance}(P, \mathcal{S}) = |\langle X,\, N\rangle - c| = |\langle X,\, N\rangle - c\,|\,.$$

    b) First do the special case where the center of mass $\bar{P} := \frac{1}{k}\sum_{j=1}^{k} P_j$ is at the origin, $\bar{P} = 0$, showing the plane is determined by letting N be an eigenvector of the matrix

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it's lowest eigenvalue (here we view the P_j as column vectors, P_j^T is the transpose so $P_j P_j^T$ is a square matrix). What is c?

c) Reduce the general case to the previous case by letting $V_j = P_j - \bar{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1, 3)$, $(0, 1)$, $(1, -1)$, $(2, -3)$.

e) Let $P_j := (p_{j1}, \ldots, p_{j3})$, $j = 1, \ldots, k$ be the coordinates of the j^{th} data point and $Z_\ell := (p_{1\ell}, \ldots, p_{k\ell})$, $\ell = 1, \ldots, 3$ be the vector of ℓ^{th} coordinates. If a_{ij} is the ij element of A, show that $a_{ij} = \langle Z_i, Z_j \rangle$. Note that this exhibits A as a *Gram matrix* (see Problem 70).

f) Generalize to where P_1, P_2,..., P_k are k points in $\mathbb{R}^n$.

Derivatives

109. Let $A(t) = (a_{ij}(t))$ be a square real matrix whose elements are smooth functions of the real variable t and write $A'(t) = (a'_{ij}(t))$ for the matrix of derivatives. [There is an obvious equivalent coordinate-free definition of the derivative of a matrix using $\lim_{h \to 0}[A(t+h) - A(t)]/h$].

a) Compute the derivative: $dA^3(t)/dt$.

b) If $A(t)$ is invertible, find the formula for the derivative of $A^{-1}(t)$. Of course it will resemble the 1×1 case $-A'(t)/A^2(t)$.

110. Let $A(t)$ be a square real matrix whose elements are smooth functions of the real variable t. Assume $\det A(t) > 0$.

a) Show that $\dfrac{d}{dt} \log \det A = \operatorname{trace}(A^{-1}A')$. [Suggestion: First reduce to the special case where $A(0) = I$]

b) Conclude that for any invertible matrix $A(t)$

$$\frac{d \det A(t)}{dt} = \det A(t) \operatorname{trace}[A^{-1}(t)A'(t)].$$

c) If $\det A(t) = 1$ for all t and $A(0) = I$, show that the matrix $A'(0)$ has trace zero.

d) Compute: $\dfrac{d^2}{dt^2} \log \det A(t)$.

Block Matrices

The next few problems illustrate the use of block matrices. (See also Problems 78 , 80.)

111. Let $M = \left(\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array}\right)$ be an $(n+k)\times(n+k)$ block matrix partitioned into the $n\times n$ matrix A, the $n\times k$ matrix B, the $k\times n$ matrix C and the $k\times k$ matrix D.

a) Show that matrices of the above form but with $C=0$ are a ring.

b) If $C=0$, find a formula for M^{-1} involving A^{-1}, etc.

c) If $B=0$ *and* $C=0$, show that $\det M = (\det A)(\det D)$. [SUGGESTION: One approach begins with $M = \left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & X \end{smallmatrix}\right)$ for some appropriate matrix X.]

d) If $B=0$ *or* $C=0$, show that $\det M = \det A \det D$. [SUGGESTION: If $C=0$, observe that $\det M = \det\left[\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)\right]$ and pick the matrix X cleverly.]

e) If A is invertible, show that $\det M = \det A\ \det(D - CA^{-1}B)$. If M is 2×2, this of course reduces to $ad-bc$. There is of course a similar formula only assuming D is invertible.

112. Let $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ be a square block matrix, where A is also a square matrix.

a) Find the relation between the non-zero eigenvalues of M and those of A. What about the corresponding eigenvectors?

b) Proof or Counterexample: M is diagonalizable if and only if A is diagonalizable.

113. If a unitary matrix M has the block form $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, show that $B=0$ while both A and D must themselves be unitary.

114. Let the square matrix M have the block form $M :=. \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$.

a) If B and C are square, show that M is invertible if and only if both B and C are, and find an explicit formula for M^{-1}. [ANSWER: $M^{-1} := \begin{pmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{pmatrix}$].

b) Even if B and C are *not* square but both A and $T := CA^{-1}B$ are invertible, find a formula for M^{-1} of the form $M^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$. [ANSWER:

$$M^{-1} = \begin{pmatrix} A^{-1} - A^{-1}BT^{-1}CA^{-1} & A^{-1}BT^{-1} \\ T^{-1}CA^{-1} & -T^{-1} \end{pmatrix} \quad].$$

115. Let $L: V \to V$ be a linear map acting on the finite dimensional linear vector space mapping V and say for some subspace $U \in V$ we have $L: U \to U$, so U is an L-invariant subspace. Pick a basis for U and extend it to a basis for V. If in this basis

$A: U \to U$ is the square matrix representing the action of L on U, show that in this basis the matrix representing L on V has the block matrix form

$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$

where 0 is a matrix of zeroes having the same number of columns as the dimension of U and $*$ represent other matrices.

Interpolation

116. a) Find a cubic polynomial $p(x)$ with the properties that $p(0) = 1$, $p(1) = 0$, $p(3) = 2$, and $p(4) = 5$. Is there more than one such polynomial?

  b) Given any points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) with the x_i's distinct, show there is a unique cubic polynomial $p(x)$ with the properties that $p(x_i) = y_i$.

117. Let a_0, a_1, $\ldots$, a_n be $n+1$ distinct real numbers and b_0, b_1, $\ldots$, b_n be given real numbers. One seeks an *interpolating polynomial* $p(x) = c_k x^k + \cdots + c_1 x + c_0$ that passes through these points $(a_j,\, b_j)$; thus $p(a_j) = b_j$, $j = 0, \ldots, n$.

  a) If $k = n$ show there exists such a polynomial and that it is unique.

  b) If p has the special form $p(x) = c_{n+1}x^{n+1} + \cdots + c_1 x$ (so $k = n+1$ and $c_0 = 0$), discuss both the existence and uniqueness of such a polynomial.

  c) If p has the special form $p(x) = x^{n+1} + c_n x^n + \cdots + c_1 x + c_0$, discuss both the existence and uniqueness of such a polynomial.

Miscellaneous Problems

118. A *tridiagonal matrix* is a square matrix with zeroes everywhere except on the main diagonal and the diagonals just above and below the main diagonal.

  Let T be a real anti-symmetric tridiagonal matrix with elements $t_{12} = c_1$, $t_{23} = c_2, \ldots$, $t_{n-1n} = c_{n-1}$. If n is even, compute $\det T$.

119. [Difference Equations] One way to solve a second order linear difference equation of the form $x_{n+2} = ax_n + bx_{n+1}$ where a and b are constants is as follows. Let $u_n := x_n$ and $v_n := x_{n+1}$. Then $u_{n+1} = v_n$ and $v_{n+1} = au_n + bv_n$, that is,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

which, in obvious matrix notation, can be written as $U_{n+1} = AU_n$. Consequently, $U_n = A^n U_0$. If one can diagonalize A, the problem is then straightforward. Use this approach to find a formula for the Fibonacci numbers $x_{n+2} = x_n + x_{n+1}$ with initial conditions $x_0 = 0$ and $x_1 = 1$.

120. Let P be the vector space of all polynomials with real coefficients. For any fixed real number t we may define a linear functional L on P by $L(f) = f(t)$ (L is "evaluation at the point t"). Such functionals are not only linear but have the special property that $L(fg) = L(f)L(g)$. Prove that if L is any linear functional on P such that $L(fg) = L(f)L(g)$ for all polynomials f and g, then either $L = 0$ or there is a c in $\mathbb{R}$ such that $L(f) = f(c)$ for all f.

121. Let $\mathcal{M}$ denote the vector space of real $n \times n$ matrices and let ℓ be a linear functional on $\mathcal{M}$. Write C for the matrix whose ij entry is $(1/\sqrt{2})^{i+j}$. If $\ell(AB) = \ell(BA)$ for all $A, B \in \mathcal{M}$, and $\ell(C) = 1$, compute $\ell(I)$.

122. [Rank One Matrices] Let $A = (a_{ij})$ be an $n \times n$ matrix having rank 1.

a) Show that $a_{ij} = b_i c_j$ for some $a_1, \ldots a_n$ and $b_1, \ldots b_n$.

b) If A has a non-zero eigenvalue λ_1, show that $\lambda_1 = \operatorname{trace}(A)$.

c) If $\operatorname{trace}(A) = 1$, show that A is a projection: $A^2 = A$.

d) If $\operatorname{trace}(A) \neq 0$, prove that A is similar to the $n \times n$ matrix

$$c \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $c = \operatorname{trace}(A)$

e) What can you say if $\operatorname{trace}(A) = 0$?

123. a) Let $L : V \to V$ be a linear map on the vector space V. If L is nilpotent, so $L^k = 0$ for some integer k, show that the map $M := I - L$ is invertible by finding an explicit formula for $(I - L)^{-1}$.

b) Apply the above result to find a particular solution of $y' - y = 5x^2 - 3$. [Hint: Let V be the space of quadratic polynomials and $L := d/dx$].

c) Similarly, find a particular solution of $y'' + y = 1 - x^2$.

124. [Wine and Water] You are given two containers, the first containing one liter of liquid A and the second one liter of liquid B. You also have a cup which has a capacity of

r liters, where $0 < r < 1$. You fill the cup from the first container and transfer the content to the second container, stirring thoroughly afterwords.

Next dip the cup in the second container and transfer k liters of liquid back to the first container. This operation is repeated again and again. Prove that as the number of iterations n of the operation tends to infinity, the concentrations of A and B in both containers tend to equal each other. [Rephrase this in mathematical terms and proceed from there].

Say you now have three containers A, B, and C, each containing one liter of different liquids. You transfer one cup form A to B, stir, then one cup from B to C, stir, then one cup from C to A, stir, etc. What are the long-term concentrations?

125. Snow White distributed 21 liters of milk among the seven dwarfs. The first dwarf then distributed the contents of his pail evenly to the pails of other six dwarfs. Then the second did the same, and so on. After the seventh dwarf distributed the contents of his pail evenly to the other six dwarfs, it was found that each dwarf had exactly as much milk in his pail as at the start.

What was the initial distribution of the milk?
Generalize to N dwarfs.
[From: K. Splinder *Abstract Algebra with Applications*, Vol. 1, page 192, Dekker (1994)]

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