# University *of* Pennsylvania

Mathematics Department

# Mathematics 114 – Final Examination
# Spring 2004

***Your Name:***________________________ **Penn** ***ID#***____________________

**Professor (Check one): Crotty ❑ 001 ❑ 601 Pemantle ❑ 002 Stovall ❑ 003 ❑ 004**

**You have 2 hours to complete this examination.**

***Please show all work in the space provided on your test paper and write your answers in the appropriate place below.***

***DO NOT DETACH THIS SHEET FROM YOUR TEST***

***Part I: True or False***

| | | | |
|---|---|---|---|
| 1 | | 5 | |
| 2 | | 6 | |
| 3 | | 7 | |
| 4 | | 8 | |

***Part II: Multiple Choice***

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | 11 | |
| 2 | | 7 | | 12 | |
| 3 | | 8 | | 13 | |
| 4 | | 9 | | 14 | |
| 5 | | 10 | | 15 | |

***Part III: Free Response***

***1.***

$h(t) =$ ____________________

$\lim_{t \to \infty} h'(t) =$ ____________________

***2.***

$y(x) =$ ____________________

***3.***

$y(e) =$ ____________________

***4.***

***5.***

$x =$ ____________ $y =$ ____________

----------------------------------------------*Please do not write below this line*----------------------------------------------

**Scores:**

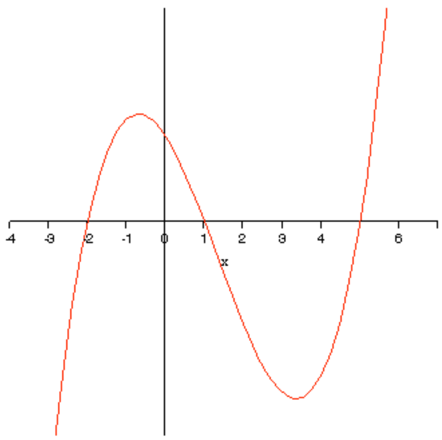
I. ____________ II. ____________ III. ____________ TOTAL: __________

***True/False questions***

In the appropriate space on your answer sheet, label each statement as *True* or *False*. No justification is required.

1. If $f(x, y)$ and $f_y(x, y)$ are continuous on $0 < x < 3$ and $0 < y < 3$ then the initial value problem $y' = f(x, y)$ and $y(1) = 2$ has a unique solution for $0 < x < 3$.

Questions 2, 3, 4, 5: Consider the differential equation $y' = g(y)$ where $g(y)$ is given in the graph below:




2. $y = -2$, $y = 1$ and $y = 5$ are constant solutions of $y' = g(y)$.

3. If the initial value $y(0) = 4$, the corresponding solution is increasing with a horizontal asymptote at 5.

4. If the initial value $y(0)$ is greater than 5, the corresponding solution will be an increasing function.

5. If $y(0) = -1$, then the corresponding solution is increasing.

6. If $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}$ for all points $(x, y)$ then $f(x, y) = \text{constant}$.

7. If $f(x, y, z) = x^2 + y^2 + z^2$ then the highest point where $f = 1$ will have $\nabla f = 0$.

8. The triangle $ABC$ with vertices $A = 2\mathbf{i} + 4\mathbf{j}$, $B = 5\mathbf{i} - 2\mathbf{j}$ and $C = -3\mathbf{i} - \mathbf{j}$ is a right triangle.

***Multiple Choice***:

Work each problem in the space provided. Write the letter corresponding to your answer in the appropriate space on your answer sheet.

1. If $R$ is the region $x^2 + y^2 \le 4$, then $\iint_R x\sqrt{x^2 + y^2}\, dA$ is equivalent to

a) $\int_0^2 \int_0^{2\pi} r^2 \; d\theta dr$ b) $\int_0^2 \int_0^{2\pi} r^2 \cos\theta \, d\theta dr$ c) $4\int_0^2 \int_0^{2\pi} r^3 \sin\theta \, d\theta dr$

d) $\int_{-2}^2 \int_{-2}^2 x\sqrt{x^2 + y^2} \; dydx$ e) $\int_0^2 \int_0^{2\pi} r^3 \cos\theta \; d\theta dr$ f) $4\int_0^2 \int_0^2 r^2 r \, dr d\theta$

2. What is the area enclosed by one loop of the curve $r^2 = 2\sin\theta$.

a) $\pi$ b) 3 c) 2 d) 1 e) 1/2 f) 4

3. Find the length of the curve $\mathbf{r}(t) = 2t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\ 0 \le t \le 2\pi$.

a) $\pi\sqrt{12}$ b) $\pi\sqrt{20}$ c) $4\pi$ d) $\pi\sqrt{6}$ e) $\pi\sqrt{18}$ f) $\pi\sqrt{8}$

4. The value of $\alpha$ for which the vector $\mathbf{v} = -6\mathbf{i} + \alpha\mathbf{j} + 3\mathbf{k}$ is parallel to the plane $z = 2x - 5y + 7$ is

a) –3 b) –9/5 c) 0 d) 8/5 e) 2 f) 12

5. Let $g(x, t, z) = \frac{1}{t}\left(x - z^2 t\right)$. What is the value of $\frac{\partial^2 g}{\partial z \partial t}$ when $x = 1$, $t = 2$ and $z = 3$?

a) –27/3 b) –17/2 c) –12 d) 8 e) –2 f) 0

6. The function $f(x, y) = x^3 - y^2 - 3x + y + 5$ has critical points at $P_1 = \left(1, \frac{1}{2}\right)$ and $P_2 = \left(-1, \frac{1}{2}\right)$. The nature of these critical points is:

a) $P_1$ = maximum; $P_2$ = maximum
b) $P_1$ = minimum; $P_2$ = minimum
c) $P_1$ = minimum; $P_2$ = saddle
d) $P_1$ = saddle; $P_2$ = maximum
e) $P_1$ = saddle; $P_2$ = minimum
f) $P_1$ = maximum ; $P_2$ = saddle

7. Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.
a) $\pi/6$ b) $\pi/4$ c) $\pi/3$ d) $\pi/2$ e) $2\pi/3$ f) $5\pi/6$

8. Use a linear approximation of the function $f(x,y) = e^{x\cos 2y}$ at (0, 0) to estimate $f(0.1, -0.2)$.
a) 1.2 b) 1.1 c) 1 d) 0.9 e) 0.3 f) 0

9. In solving the differential equation $y' + x^4 y = 0$ by use of a power series $\sum_{n=0}^{\infty} a_n x^n$, what is the first value of $n$ beyond $n = 0$ for which the coefficient $a_n$ can be non-zero?

a) 1 b) 2 c) 3 d) 4 e) 5 f) 6

10. The initial value problem $y' = x^3(1 + e^y)$ subject to $y(0) = A$ where $A$ is a constant:

a) always increases without bound as x increases to infinity
b) always increases to a finite limit as x increases to infinity
c) always increases to infinity at some finite x value
d) can exhibit more than one of behaviors a), b) and c), depending on the value of $A$
e) has no solution in a neighborhood of zero
f) has only the constant solution $y$= A

11. The general solution of the differential equation $(3x^2+2y^2)dx+(4xy+6y^2)dy=0$ is

a) $y-x^3+2x^2+x=C$ b) $x^3+2x^2y^2=C$ c) $xy^3+2x^2y=C$

d) $x^3+4xy^2+y^2=C$ e) $x^3+2xy^2+2y^3=C$ f) $x^2+xy+y^2=C$

12. Compute the third Picard iterate, $y_3$, for $y'=x+y$ subject to $y(0)=1$.

a) $y_3 = 1$ b) $y_3=1+x+x^2+\frac{x^3}{3}+\frac{x^4}{4!}$ c) $y_3=1+x^2+\frac{x^4}{4!}$

d) $y_3=1+x+x^2+\frac{x^3}{3!}$ e) $y_3=1+x+\frac{x^2}{2}$ f) $y_3=\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$

13. The general solution of the differential equation $y'' + 4y' + 4y = 0$ is

a) $y = c_1 e^{2x} + c_2 e^{2x}$ b) $y = c_1 e^{-2x} + c_2 e^{2x}$ c) $y = c_1 + c_2 x^4$

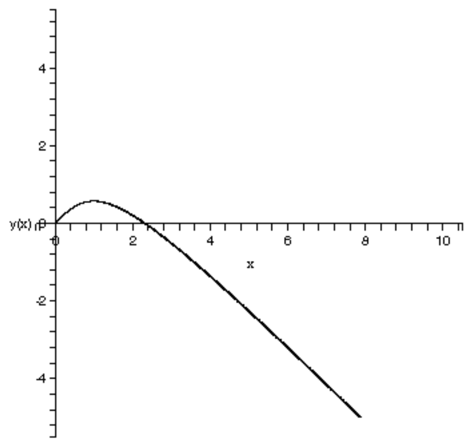
d) $y = c_1 e^{-2x} + c_2 x e^{-2x}$ e) $y = c_1 e^{-x} + c_2 e^{3x}$ f) $y = c_1 \cos 2x + c_2 \sin 2x$

14. Solve the initial value problem $y' - \frac{1}{x} y = xe^x$ subject to $y(1) = 0$. From your solution, evaluate $y(2)$.

a) $\frac{1}{2}e^2$ b) $e^2$ c) $2(e^2 - e)$ d) $e^2 \ln 2$ e) $e^2 + \ln 2$ f) $e^2 + 2$

15. Shown below are graphs of the solutions of three differential equations. The graphs are drawn using the initial condition $y(0) = 0$ and also $y'(0) = 1$ for the second order equations.

(i) y" + y = 1/(10+x^2)
(ii) y" + y = sin(x)
(iii) y' = cos (x + y)

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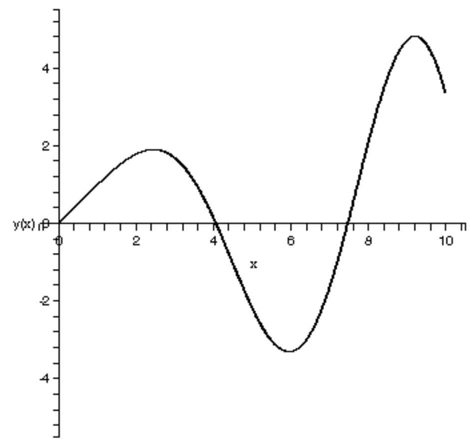


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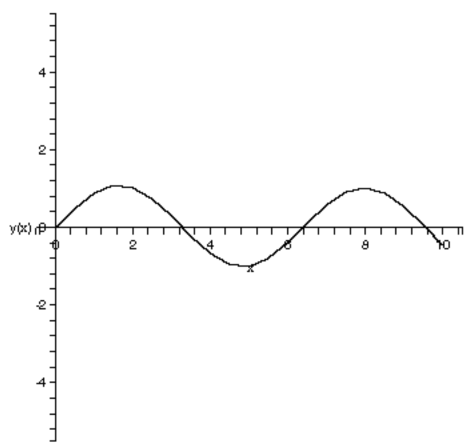


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The solution graphs shown match the differential equations (i), (ii) and (iii) in the order:

(a) 1, 2, 3 (b) 1, 3, 2 (c) 2, 1, 3 (d) 2, 3, 1 (e) 3, 1, 2 (f) 3, 2, 1

***Free Response Questions***:

Work each problem in the space provided. Write your answers in the appropriate spaces on your answer sheet.

1. A falling object is acted upon by gravity and air resistance; its height at time $t$ is denoted $h(t)$. There is acceleration due to gravity and air resistance, respectively -$g$ and $-k\frac{dh}{dt}$, so that the particle satisfies

$$h'' + kh' + g = 0 \ .$$

(a) What is the distance fallen, starting from rest, after time $t$?

(b) What is the limit as $t$ goes to infinity of $h'(t)$ ?

[NOTE: answers should be given in terms of $g$ and $k$.]

2. Solve $y'' - xy' + 2y = 0$ *subject to* $y(0) = 1,\ y'(0) = 1$ by power series expansion about $x = 0$. Write out all non-zero terms in the solution through the term in $x^5$.

3. Solve the initial value problem $y'' + 2y' + y = \frac{e^{-x}}{x}$ *subject to* $y(1) = 0,\ y'(1) = e^{-1}$. From your solution, compute $y(e)$.

4. Find the location of all absolute maxima of $f(x,y) = x^3 - 3xy^2$ on the unit disk $\{x^2 + y^2 \le 1\}$.

5. A business produces two products, A and B. Let $x$ and $y$ denote respectively the quantity of product A and B produced. Limitations on the company's resources require that $500x^2 + 100y$ be at most 100,000. Each unit of A produced yields a profit of \$5,000 and each unit of B produced yields a \$500 profit. Given the constraint, what should $x$ and $y$ be to maximize profit?