LAST NAME:________________ **NAME:**________________ PENN ID # :________________

Please bubble-in the required information.

INSTRUCTOR'S NAME	T.A.'s NAME	
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DS David SANTOS	AP Andrei PAVELESCU	SV Shea VICK

Instructions: *You have 120 minutes to complete this exam.* **Do not detach this sheet from the body of the test.** *This is a multiple-choice test. Please show all your work. Answers with no supporting work will be given no points. If you change an answer, please either erase or cross out the answer you do not want considered; questions with more than one answer will be marked wrong. Bubble-in your choice both in the body of the test and in the grid below.*

Questions 1—7	Questions 8–14	Questions 15-20
1. A B C D E	8. A B C D E	15. A B C D E
2. A B C D E	9. A B C D E	16. A B C D E
3. A B C D E	10. A B C D E	17. A B C D E
4. A B C D E	11. A B C D E	18. A B C D E
5. A B C D E	12. A B C D E	19. A B C D E
6. A B C D E	13. A B C D E	20. A B C D E
7. A B C D E	14. A B C D E	

1. Let $\mathscr{C}$ be the curve in $\mathbb{R}^3$ defined by

$$x = \frac{t^2}{2}, \qquad y = \frac{4}{3}t^{3/2}, \qquad z = 2t, \qquad t \in [0; +\infty).$$

Calculate the distance along $\mathscr{C}$ from $(0,0,0)$ to $\left(\frac{1}{2}, \frac{4}{3}, 2\right)$.

(A) $\frac{\sqrt{217}}{6}$ (B) $\frac{5}{2}$ (C) $\frac{3}{2}$ (D) $\frac{\sqrt{2}}{2}$ (E) $2\sqrt{3} - \frac{4}{3}\sqrt{2}$

2. The two curves

$$C_1 : x = t, \quad y = t^2, \quad z = t^4$$

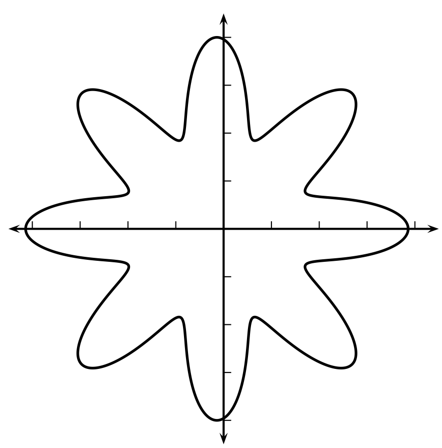
and

$$C_2 : x = (1+t)^2 - 3, \quad y = (1-t)^2 + 1, \quad z = 2 - t$$

intersect at $t = 1$. Find the angle between the tangent line to C_1 and the tangent line to C_2 at this point.

(A) 0 (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\cos^{-1}\left(\frac{8}{\sqrt{357}}\right)$ (E) $\sin^{-1}\left(\frac{8}{\sqrt{357}}\right)$

3. Find the area inside $r = 3 + \cos 8\theta$, which appears in the figure below.



(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$ (C) $\frac{19}{4}\pi$ (D) $\frac{19}{2}\pi$ (E) π

4. Find the slope of the polar curve $r = \sin(5\theta)$ at the point specified by $\theta = \frac{\pi}{2}$.

(A) $\frac{1}{2}$ (B) 0 (C) 1 (D) -1 (E) $-\frac{\sqrt{2}}{2}$

5. If $\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{c}} - \overrightarrow{\mathbf{a}}$ are parallel and it is known that $\overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{j}}$ and $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{j}} + \overrightarrow{\mathbf{k}}$, find $\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}}$.

(A) $\overrightarrow{\mathbf{j}} + \overrightarrow{\mathbf{k}}$ (B) $-\overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{k}}$ (C) $\overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{j}}$ (D) $\overrightarrow{\mathbf{i}} + \overrightarrow{\mathbf{j}}$ (E) $\overrightarrow{\mathbf{j}} - \overrightarrow{\mathbf{k}}$

6. The point on the plane $x + 2y + 3z = 7$ that is closest to the origin is

(A) $\left(1, 2, \frac{2}{3}\right)$ (B) $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$ (C) $(1, 3, 0)$ (D) $\left(\frac{1}{2}, 2, \frac{2}{3}\right)$ (E) $\left(\frac{7}{6}, \frac{7}{3}, \frac{7}{2}\right)$

7. Find the curvature κ and the torsion τ of the parameterized curve

$$x(t) = \cos(t) + t\sin(t), \qquad y(t) = \sin(t) - t\cos(t), \qquad z(t) = 0.$$

Then compute their difference.

(A) $\kappa - \tau = t^2 - t$
(B) $\kappa - \tau = \dfrac{1}{|t|}$
(C) $\kappa - \tau = (\sin(t))^2 - \cos(t)$
(D) $\kappa - \tau = (\cos(t))^2 - (1 - t)$
(E) $\kappa - \tau = (\sin(t))^2 - (1 - t)$

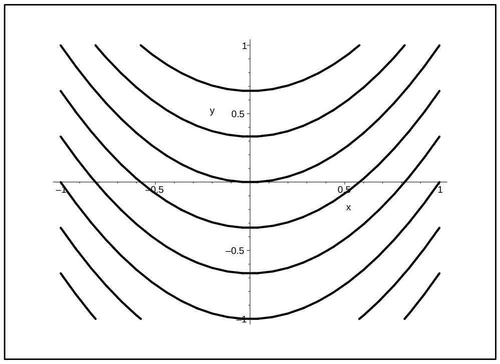
8. I throw a ball into the air with initial speed $10\sqrt{2}$ meters per second at angle 45 degrees from the horizontal. The ball is two meters off the ground when I let go of it. How far away from me does it land? Give your answer to the nearest meter. (Assume that the gravitational force is 10 meters per second per second.)

(A) 17 meters (B) 18 meters (C) 19 meters (D) 20 meters (E) 22 meters

9. Consider the following four functions

$$a(x,y) = x^2 - y, \quad b(x,y) = x - y^2, \quad c(x,y) = 8x^2 + y^2, \quad d(x,y) = x^2 + 8y^2,$$

and the four contour plots below. Which of the following matches is correct?



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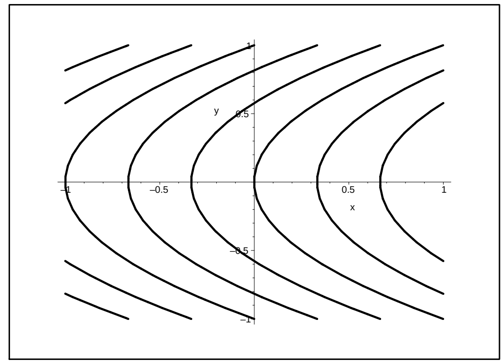

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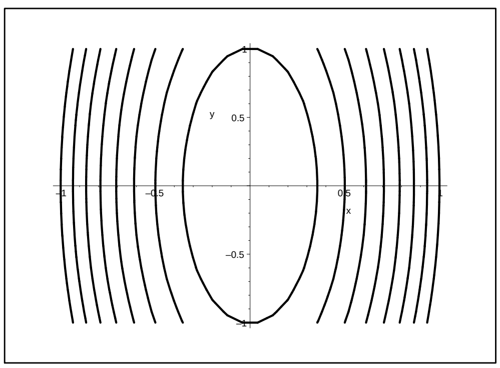

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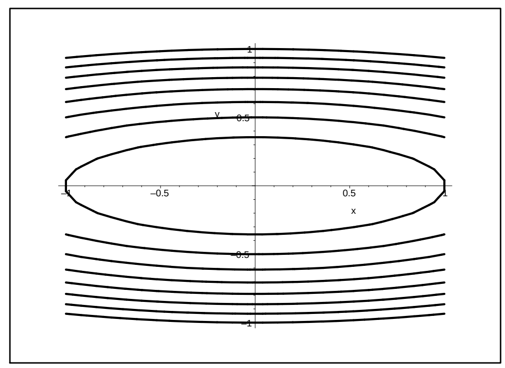


: IV

A $(a, \text{II}), (b, \text{I}), (c, \text{III}), (d, \text{IV})$

B $(a, \text{I}), (b, \text{III}), (c, \text{II}), (d, \text{IV})$

C $(a, \text{III}), (b, \text{I}), (c, \text{IV}), (d, \text{II})$

D $(a, \text{II}), (b, \text{I}), (c, \text{IV}), (d, \text{III})$

E $(a, \text{I}), (b, \text{II}), (c, \text{III}), (d, \text{IV})$

10. The linearization of the function

$$f(x, y) = x^2 + bxy - y^2$$

at the point $(1, 2)$ is

$$L(x, y) = 4x - 3y + 1.$$

Find b.

(A) $b = 1$
(B) $b = 2$
(C) $b = 3$
(D) b is none of $1, 2$ or 3
(E) b cannot be determined from the information given

11. Find the point on the surface $2x^2 + xy + y^2 + 4x + 8y - z + 14 = 0$ for which the tangent plane is $4x + y - z = 0$.

(A) $(1, -4, 0)$ (B) $(1, -3, 2)$ (C) $(0, -4, -2)$ (D) $(0, -3, -1)$ (E) $(1, 2, 6)$

12. Let $f(x, y) = x^3 + 4xy - 2y^2 + 1$. Which of the following is true?

(A) f has a saddle point at $(0, 0)$ and a local minimum at $\left(-\frac{4}{3}, -\frac{4}{3}\right)$.
(B) f has a saddle point at $(0, 0)$ and a local maximum at $\left(-\frac{4}{3}, -\frac{4}{3}\right)$.
(C) f has a saddle points at $(0, 0)$ and at $\left(-\frac{4}{3}, -\frac{4}{3}\right)$.
(D) f has a local minimum $(0, 0)$ and a local maximum at $\left(-\frac{4}{3}, -\frac{4}{3}\right)$.
(E) f has a local maximum $(0, 0)$ and a local minimum at $\left(-\frac{4}{3}, -\frac{4}{3}\right)$.

13. Find $\iint_{\mathscr{R}} xy \, \mathrm{d}A$ where $\mathscr{R}$ is the region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant.

(A) $\frac{1}{6}$ (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{48}$ (E) $\frac{5}{12}$

14. Evaluate the following integral. (*Hint: Combine the sum into a single expression.*)

$$\int_0^2 \int_0^x \frac{\mathrm{d}y\mathrm{d}x}{\sqrt{x^2 + y^2}} \quad + \quad \int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \frac{\mathrm{d}y\mathrm{d}x}{\sqrt{x^2 + y^2}}.$$

(A) $\sqrt{2}\,\pi$ (B) $\frac{\sqrt{2}\,\pi}{2}$ (C) $\frac{\sqrt{2}\,\pi}{3}$ (D) $\frac{4\pi\sqrt{2}}{3}$ (E) $2\sqrt{2}\,\pi$

15. How many of the expressions below describe the integral of f over the solid bounded by the coordinate planes and the planes $z = 1 - x$ and $z = 1 - y$?

- $\int_0^1 \int_0^{1-y} \int_0^{1-z} f(x, y, z)\mathrm{d}x\mathrm{d}z\mathrm{d}y$
- $\int_0^1 \int_0^{1-x} \int_0^{1-z} f(x, y, z)\mathrm{d}y\mathrm{d}z\mathrm{d}x$
- $\int_0^1 \int_0^{z} \int_0^{1-z} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_0^1 \int_z^{1} \int_0^{y} f(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$
- $\int_0^1 \int_0^{x} \int_0^{1-x} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x + \int_0^1 \int_x^{1} \int_0^{1-y} f(x, y, z)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$

(A) All four do
(B) Exactly three of them do
(C) Exactly two of them do
(D) Just one of them does
(E) None of them does

16. If $i = \sqrt{-1}$, find the roots of the equation

$$z^3 = i.$$

(A) $-\frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad i$
(B) $-\frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad -i$
(C) $-\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -i$
(D) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -i$
(E) $-\frac{1}{2} - i\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

17. Let f be a continuously differentiable function such that

$$f'(x) + f(x) = x; \qquad f(2) = 2.$$

Find $f(1)$.

(A) $f(1) = 1$ (B) $f(1) = -1$ (C) $f(1) = e$ (D) $f(1) = 2e$ (E) $f(1) = e^{-1}$

18. The function y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0.$$

It passes through the origin with a slope of 1. What is the slope at $x = -1/3$?

(A) 3 (B) 7 (C) 0 (D) e (E) π

19. Find all the continuous functions f defined for all $x \geq 0$ and satisfying

$$2xf(x) = 3\int_0^x f(t)\mathrm{d}t$$

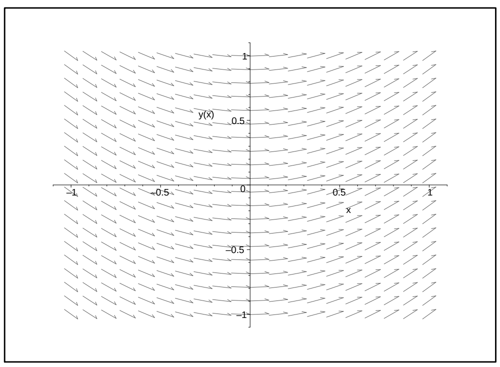
for all $x \geq 0$.

(A) $f(x) = Cx$ (B) $f(x) = Cx^2$ (C) $f(x) = Cx^{1/3}$ (D) $f(x) = Cx^{1/2}$ (E) $f(x) = Cx^{1/4}$

20. Consider differential equations a, b, c, d given by

$$a : \frac{\mathrm{d}y}{\mathrm{d}x} = y, \quad b : \frac{\mathrm{d}y}{\mathrm{d}x} = x, \quad c : \frac{\mathrm{d}y}{\mathrm{d}x} = 1, \quad d : \frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

and the slope fields below. Which of the following matches is correct?



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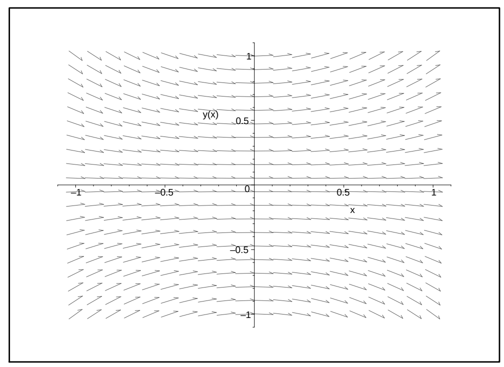

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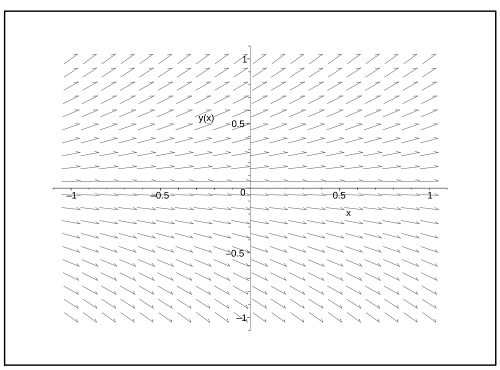

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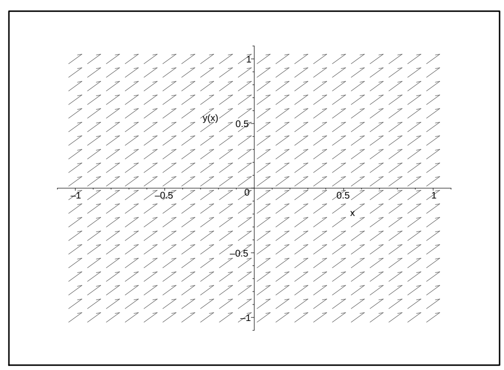


: IV

(A) (a, II), (b, I), (c, III), (d, IV)

(B) (a, I), (b, III), (c, II), (d, IV)

(C) (a, III), (b, I), (c, IV), (d, II)

(D) (a, II), (b, I), (c, IV), (d, III)

(E) (a, II), (b, I), (c, III), (d, IV)

1. B One has
$$\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2}=\sqrt{t^2+4t+4}\,\mathrm{d}t=(t+2)\mathrm{d}t,$$
whence the required length is
$$\int_0^1(t+2)\mathrm{d}t=\frac{5}{2}.$$
Remark: Straightforward.

2. C The tangent vector to C_1 is $\begin{bmatrix}1\\2t\\4t^3\end{bmatrix}$ and that to C_2 is $\begin{bmatrix}2(1+t)\\-2(1-t)\\-1\end{bmatrix}$. At $t=1$ they are respectively $\begin{bmatrix}1\\2\\4\end{bmatrix}$ and $\begin{bmatrix}4\\0\\-1\end{bmatrix}$. The angle between them is
$$\arccos\left(\frac{1\cdot 4+2\cdot 0+4\cdot(-1)}{\sqrt{1^2+2^2+4^2}\cdot\sqrt{4^2+0^2+(-1)^2}}\right)=\arccos 0=\frac{\pi}{2}.$$
Remark: Straightforward.

3. D The required area is plainly
$$\frac{1}{2}\int_0^{2\pi}(3+\cos 8\theta)^2\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}(9+6\cos 8\theta+\cos^2 8\theta)\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}(9+6\cos 8\theta+\frac{1}{2}+\frac{1}{2}\cos 16\theta)\mathrm{d}\theta=\frac{1}{2}\int_0^{2\pi}(9+\frac{1}{2})\mathrm{d}\theta=\frac{19\pi}{2}.$$
Remark: Straightforward.

4. B The slope at θ is
$$\frac{y'(\theta)}{x'(\theta)}=\frac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\sin\theta},$$
and so the required slope is
$$\frac{y'(\pi/2)}{x'(\pi/2)}=\frac{5(\cos 5\pi/2)\sin\pi/2+\sin 5\pi/2\cos\pi/2}{5(\cos 5\pi/2)\cos\pi/2-\sin 5\pi/2\sin\pi/2}=0,$$
which should also be obvious from the graph. **Remark:** Fairly easy.

5. B One has
$$(\overrightarrow{\mathbf{b}}-\overrightarrow{\mathbf{a}})\times(\overrightarrow{\mathbf{c}}-\overrightarrow{\mathbf{a}})=\overrightarrow{\mathbf{0}}\implies\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{b}}+\overrightarrow{\mathbf{b}}\times\overrightarrow{\mathbf{c}}+\overrightarrow{\mathbf{c}}\times\overrightarrow{\mathbf{a}}=\overrightarrow{\mathbf{0}}$$
This gives
$$\overrightarrow{\mathbf{b}}\times\overrightarrow{\mathbf{c}}=-(\overrightarrow{\mathbf{a}}\times\overrightarrow{\mathbf{b}}+\overrightarrow{\mathbf{c}}\times\overrightarrow{\mathbf{a}})=-(\overrightarrow{\mathbf{j}}+\overrightarrow{\mathbf{k}}+\overrightarrow{\mathbf{i}}-\overrightarrow{\mathbf{j}})=-\overrightarrow{\mathbf{i}}-\overrightarrow{\mathbf{k}}.$$
Remark: Fairly easy.

6. B The point sought lies on a line perpendicular to the plane and connecting with the origin. Since $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is perpendicular to the plane, the point sought must satisfy $t+4t+9t=7\implies t=\dfrac{1}{2}$, hence the point sought is $(\frac{1}{2},1,\frac{3}{2})$. **Remark:** Straightforward.

7. B One has
$$\overrightarrow{\mathbf{r}}'(t)=(t\cos t)\overrightarrow{\mathbf{i}}+(t\sin t)\overrightarrow{\mathbf{j}}\implies\left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|=|t|,$$

$$\overrightarrow{\mathbf{T}}(t) = \frac{\overrightarrow{\mathbf{r}}'(t)}{|t|} = (\frac{t}{|t|}\cos t)\overrightarrow{\mathbf{i}} + (\frac{t}{|t|}\sin t)\overrightarrow{\mathbf{j}} = (\text{signum}\,(t)\cos t)\overrightarrow{\mathbf{i}} + (\text{signum}\,(t)\sin t)\overrightarrow{\mathbf{j}},$$

$$\overrightarrow{\mathbf{T}}'(t) = (-\text{signum}\,(t)\sin t)\overrightarrow{\mathbf{i}} + (\text{signum}\,(t)\cos t)\overrightarrow{\mathbf{j}}, \implies \left|\left|\overrightarrow{\mathbf{T}}'(t)\right|\right| = 1,$$

whence

$$\kappa = \frac{\left|\left|\overrightarrow{\mathbf{T}}'(t)\right|\right|}{\left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right|} = \frac{1}{|t|}.$$

Furthermore $\tau 0$ as the curve is embedded on a plane. **Remark:** Students will probably assume $t > 0$ and deduce the correct expression then.

8. E From the projectile formulæ one gets

$$x = x_0 + v_0(\cos\alpha)t \implies x = 10t; \qquad y = y_0 + v_0(\sin\alpha)t - \frac{gt^2}{2} \implies y = 2 + 10t - 5t^2.$$

When the projectile lands, $y = 0$, whence

$$0 = 2 + 10t - 5t^2 \implies t = 1 + \frac{\sqrt{35}}{5} \approx 2.18,$$

upon rejecting the negative root. Thus $x = 10(1 + \frac{\sqrt{35}}{5}) \approx 22$ **Remark:** Straightforward.

9. E **Remark:** Too easy!!!

10. A One has $L(x, y) = 4x - 3y + 1 = -1 + 4(x - 1) - 3(y - 2)$ and

$$\begin{aligned} f(x, y) = x^2 + bxy - y^2 &= (x - 1 + 1)^2 + b(x - 1 + 1)(y - 2 + 2) - (y - 2 + 2)^2 \\ &= (2b - 3) + (2 + 2b)(x - 1) + (b - 4)(y - 2) + \text{quadratic part} \end{aligned}$$

Equating coefficients with $L(x, y)$ this necessitates $2b - 3 = -1$, $2b + 2 = 4$, $b - 4 = -3$. These are all met when $b = 1$ **Remark:** Work backwards to get the right L!!!

11. A Put $f(x, y, z) = 2x^2 + xy + y^2 + 4x + 8y - z + 14$. Then

$$\nabla f(x, y, z) = \begin{bmatrix} 4x + y + 4 \\ x + 2y + 8 \\ -1 \end{bmatrix}.$$

Since $\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ is normal to the plane, one needs

$$4x + y + 4 = 4; \qquad x + 2y + 8 = 1 \implies x = 1, \quad y = -4.$$

Furthermore, $z = 4x + y = 0$. The point sought is $(1, -4, 0)$.

12. B We $f(x, y) = x^3 + 4xy - 2y^2 + 1$.. We find

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 + 4y \\ 4x - 4y \end{bmatrix}.$$

The critical points are gotten by setting

$$3x^2 + 4y = 0; \quad 4x - 4y = 0 \implies x(3x + 4) = 0 \implies (x, y) = (0, 0), (x, y) = \left(-\frac{4}{3}, -\frac{4}{3}\right).$$

We have

$$\mathscr{H}_{f(x,y)} = \begin{bmatrix} 6x & 4 \\ 4 & -4 \end{bmatrix} \implies \det \mathscr{H}_{f(x,y)} = -24x - 16.$$

Now, $\det \mathscr{H}_{f(0,0)} < 0$, whence $(0,0)$ is a saddle point. Also, $f_{xx}\left(-\frac{4}{3}, -\frac{4}{3}\right) < 0$ and $\mathscr{H}_{f\left(-\frac{4}{3},-\frac{4}{3}\right)} = 16 > 0$, whence $\left(-\frac{4}{3}, -\frac{4}{3}\right)$ is a local maximum.

13. D The desired integral is plainly

$$\begin{aligned}\int_0^1 \int_{y^{1/2}}^{y^{1/3}} xy\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_0^1 y(y^{2/3} - y)\mathrm{d}y \\ &= \frac{1}{2}\left(\frac{3}{8} - \frac{1}{3}\right) \\ &= \frac{1}{48}.\end{aligned}$$

Remark: Straight forward.

14. B The region of integration is plainly

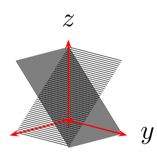
$$\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2\sqrt{2},\ 0 \leq y \leq \sqrt{8 - x^2}.\}$$

The integral may be evaluated via polar cö-ordinates, obtaining

$$\int_0^{\pi/4} \int_0^{2\sqrt{2}} \frac{r}{r}\mathrm{d}r\mathrm{d}\theta = \frac{\pi\sqrt{2}}{2}.$$

Remark: Straightforward.

15. B The region of integration is shewn below.



Remark: The region consists of two tetrahedra split along the plane $x = y$.

16. B $i = e^{i\pi/2}$. Hence $z^3 = e^{i\pi/2}e^{2\pi ik}$, $k = 0, 1, 2$, and

$$z_0 = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}; \qquad z_1 = e^{i\pi/6}e^{2\pi i/3} = e^{5\pi i/6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}; \qquad z_2 = e^{i\pi/6}e^{4\pi i/3} = e^{3\pi i/2} = -i.$$

Remark: Straight forward.

17. C The equation is equivalent to

$$(f(x)e^x)' = xe^x \implies f(x)e^x = xe^x - e^x + C \implies f(x) = x - 1 + Ce^{-x}.$$

Now $f(2) = 2 \implies 2 = 2 - 1 + Ce^{-2} \implies C = e^2$. Hence $f(x) = x - 1 + e^{-x+2} \implies f(1) = e$.

18. C We have $y(x) = ae^{3x} + bxe^{3x}$, $y(0) = 0$, and $y'(0) = 1$. Hence $0 = a$ and $1 = b$. This gives $y(x) = xe^{3x}$, $y'(x) = e^{3x} + 3xe^{3x}$ and $y(-1/3) = e^{-1} - e^{-1} = 0$. **Remark:** Straight forward.

19. D Differentiating both sides

$$2f(x) + 2xf'(x) = 3f(x) \implies f'(x) = \frac{1}{2}f(x) \implies f(x) = C\sqrt{x}.$$

Remark: Straight forward.

20. C **Remark:** Too easy!