# Midterm Exam III for Math 110, Spring 2015

## Solutions

1. (a) Find the general solution to the differential equation

$$e^{kt} z' = \frac{a}{z}\,.$$

**Solution:** Move the $e^{kt}$ to the right-hand side and the $z$ to the left to see that it is a separable equation:

$$zz' = a\, e^{-kt}\,.$$

Integrate to get $\frac{1}{2}z^2 = -\frac{a}{k}e^{-kt} + c$, and solve for $z$ yielding

$$z = \sqrt{C - \frac{2a}{k}e^{-kt}}$$

where this $C$ is twice the old contant $c$, therefore can still be any real number.

(b) If the units of $t$ are seconds and the units of $z$ are feet, what are the units of $k$ and $a$?

**Solution:** The exponent must be unitless so the time constant $k$ has units of inverse seconds. Multiplying through by $z$ we see that $a = e^{kt} zz'$ which has units of (unitless) times feet times feet per second; therefore the answer is square feet per second.

2. (a) Circle the number of the correct solution to the initial value problem

$$y' = y\frac{\sin x}{x^2}\ ;\ \ y(2) = 3\,.$$

(i) $y = 3e^{2-x}$

(ii) $y = \ln\left(e^3 + \int_2^x \frac{\sin t}{t^2}\,dt\right)$

(iii) $y = e^{3+\int_2^x \frac{\sin t}{t^2}\,dt}$

(iv) $y = 3\int \frac{\sin x}{x^2}\,dx$

(v) $y = e^{\int \frac{\sin x}{x^2}\,dx + C}$

(vi) $y = 3 + e^{\int_2^x \frac{\sin t}{t^2}\,dt}$

(vii) $y = 3e^{\int_2^x \frac{\sin t}{t^2}\,dt}$

**Solution:** This is separable. The general solution is obtained by integrating $y'/y = \sin x/x^2$ to obtain

$$\ln|y| = \int \frac{\sin x}{x^2}\,dx\,.$$

Because the initial value occurs at $x = 2$ we turn the indefinite integral into a definite integral starting at $x = 2$, making sure to add in the arbitrary constant:

$$\ln|y| = C + \int_2^x \frac{\sin t}{t^2}\,dt\,.$$

Solving for $y$ gives

$$y = C_1 e^{\int_2^x \frac{\sin t}{t^2}\,dt}$$

where $C_1$ is $\pm e^C$ for the previous constant, $C$. When $x = 2$ the integral is zero and the expression evaluates to $C_1 e^0 = C_1$. Therefore $C_1 = 3$ and the correct answer is $(vii)$.

(b) Precisely one of the following statements is true; please circle the corresponding number.

(i) $y(x) \to \infty$ as $x \to \infty$.

(ii) $y(x)$ is defined for all $x \geq 2$ and has a horizontal asymptote.

(iii) $y(x)$ is defined for all $x \geq 2$ and has a vertical asymptote.

(iv) $y(x)$ has a vertical asymptote and is not defined past that point.

(v) The solution may have a vertical or horizontal asymptote, depending on the initial condition.

(vi) $y(x) \to -\infty$ as $x \to \infty$.

**Solution:** The integral of $\frac{dy}{y}$ from any positive constant to infinity diverges. The integral of $\frac{\sin x}{x^2}\,dx$ from any positive constant to infinity converges. The blow-up test then shows that $x$ grows without bound, while $y$ approaches a limiting value. The answer that describes this is $(ii)$.

3. A benefactor sets up a trust fund by giving a steady stream of money, with the rate of giving increasing over time according to the formula "rate = $100,000t$ per year at time $t$ years," starting at time $t = 0$ years with no money in the account. The account also grows by return on investment at the rate of 2% per year.

(a) Write an initial value problem for the value after time $t$ years.

**Solution:** Let $V(t)$ be the value at time $t$. Then

$$V' = 0.02V + 100,000t \text{ with } V(0) = 0\,.$$

(b) Solve this initial value problem.

**Solution:** This is a linear first order equation: $V' - 0.02V = 100,000t$. Multiply both sides by the integrating factor $e^{-0.02t}$ and integrate yielding the indefinite integral solution

$$e^{-0.02t}V = \int 100,000\, t\, e^{-0.02t}\, dt\,.$$

Integrating by parts on the right-hand side gives

$$e^{-0.02t}V = -5,000,000(t+50)e^{-0.02t} + C$$

Solving for $V$ and writing $0.02t$ as $t/50$ gives the general solution

$$V(t) = Ce^{t/50} - 5,000,000\,t - 250,000,000\,.$$

Setting $V(0) = 0$ yields $C = 250,000,000$, therefore

$$V(t) = 250,000,000(e^{t/50} - 1) - 5,000,000\,t = 250,000,000\left(e^{t/50} - 1 - \frac{t}{50}\right)\,.$$

(c) Write an exact expression for the value after one year.

**Solution:** Plug in $t = 1$ to get $250,000,000(e^{1/50} - 1 - 1/50)$.

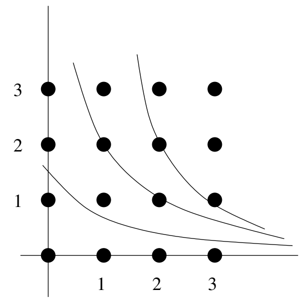
(d) Give an approximate numerical value to the expression in part ($c$) by using the quadratic Taylor approximation at $t = 0$ to the expression computed in part ($b$) at $t = 1$.

**Solution:** The quadratic Taylor approximation to $e^{t/50}$ is $1 + t/50 + t^2/(2 \cdot 50^2)$, leading to

$$P_2(t) = 250,000,000\left(1 + \frac{t}{50} + \frac{t^2}{2 \cdot 50^2} - 1 - \frac{t}{50}\right) = 250,000,000\frac{t^2}{5,000} = 50,000t^2\,.$$

Plugging in $t = 1$ gives \$50,000 for the value after one year. Note: this the same amount the benefactor has put in! The effect of the investment return does not show up until you take the cubic Taylor polynomial: the extra term of $250,000,000(t/50)^3/3! = \$333.33$ approximates the true interest of \$335.00 quite well.

4. A contour plot is shown for the function $u(x, y)$. For each item please circle T for true or F for false.




F: $u$ is changing faster near $(1, 2)$ than it is near $(2, 1)$.

The contours are farther apart near $(1, 2)$ so the function changes more slowly there.

F: It is possible that $u(x, y)$ is $\dfrac{1}{1 + x + y}$.

Two reasons why not. One is that any function that depends only on $x + y$ would have straight contours sloping at 45 degrees, because those are the level curves of $x + y$. (Same logic as in hwk10 #1.) Another is that $1/(1+x+y)$ is symmetric in $x$ and $y$ so the contours should be symmetric about the line $y = x$.

T: If $u(x, y)$ represents my utility then I am indifferent between the outcomes at $(1, 2)$ and $(2, 1)$.

There is a contour passing through both of these points, therefore the utility value is equal at the two points, which is the definition of indifference.

F: It is possible that $u(x, y) = ye^{-x}$.

This increases when $y$ increases and $x$ decreases, contradicting the fact that there are level curves along which $y$ increases and $x$ decreases.

5. Compute $\displaystyle\int_1^5 \int_0^2 \frac{x+1}{y}\, dx\, dy\,.$

**Solution:** The easiest route is to use the magic product formula to see that

$$\int_1^5 \int_0^2 \frac{x+1}{y}\, dx\, dy = \left(\int_1^5 \frac{dy}{y}\right) \times \left(\int_0^2 (x+1)\, dx\right)\,.$$

Evaluating these gives

$$(\ln 5 - \ln 1) \times \left(2 + \frac{1}{2}2^2\right) = 4 \ln 5\,.$$

6. Compute the average temperature over the upper half of the unit disk in the $x$-$y$ plane if the temperature is given by $T(x,y) = 2y$.

**Solution:** The area if the half disk, $R$, is $\pi/2$ so the average is $(2/\pi)$ times the integral of $T$ over $R$. The region $R$, described in horizontal strips is

$$\{(x,y) : 0 \leq y \leq 1 \text{ and } -\sqrt{1-y^2} \leq y \leq \sqrt{1-y^2}\}\,.$$

Therefore the average temperature is

$$\frac{2}{\pi}\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2y\,dx\,dy\,.$$

In the inner integral $y$ is constant and we get $4y\sqrt{1-y^2}$. Thus, the average temperature is given by

$$\frac{2}{\pi}\int_0^1 4y\sqrt{1-y^2}\,dy\,.$$

The substitution $u = 1 - y^2$ allows us to evaluate this as

$$\frac{2}{\pi}\left(-\frac{4}{3}(1-y^2)^{3/2}\right)\Big|_0^1 = \frac{8}{3\pi}\,.$$

7. A trapezoid $T$ has corners at $(\pm 2, 0)$ and $(\pm 1, 3)$.

(a) Describe the region $T$ in horizontal strips:

$$T = \{(x, y) \, : \, \cdots\} \, .$$

**Solution:** $T = \{(x, y) : 0 \leq y \leq 3 \text{ and } -2 + y/3 \leq x \leq 2 - y/3\}$.

(b) If the dart is thrown at the region $T$ and lands in a random location, distributed uniformly over $T$, what density describes this probability distribution?

**Solution:** The area of $T$ is $(1/2)(3)(2 + 4) = 9$. The density of the uniform distribution is a constant which is the reciprocal of the area. Therefore, the density of the uniform distribution on $T$ is the constant $1/9$ over the region $T$.

(c) What is the mean of the $Y$ value chosen by the dart?

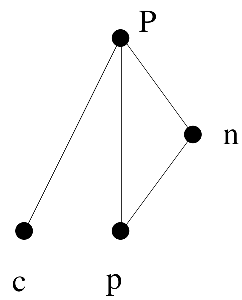
**Solution:** The mean is $\int_T y \, dA$. As a double integral, using $(1/9) \, dx \, dy$ for $dA$ and the limits of integration given by the answer to part $(a)$, this becomes

$$\int_0^3 \int_{-2+y/3}^{2-y/3} \frac{1}{9} \, dx \, dy \, .$$

The inner integral is $\frac{1}{9}\left(2 - \frac{y}{3}\right)$ and integrating this from 0 to 3 gives $\frac{4}{9}y - \frac{1}{27}y^2$ evaluated between 0 and 3, which comes out to $12/9 - 9/27 = 1$.

8. Suppose that profit depends on price, cost per unit, and number sold according to the formula $P = n(p - c)$. The number sold is a function of the price.

   (a) Make a branch diagram for this.




   (b) State which of $P, p, c$ and $n$ are independent variables, which are dependent variables and which are intermediate variables.

   **Solution:** Profit, $P$, is the dependent variable. The cost, $c$, and the price, $p$ are independent variables. The number, $n$ is an intermediate variable because it depends on $p$ and $P$ depends on it.

   (c) Suppose that the present values of $p, c$ and $n$ are respectively 3, 1 and 10,000. Write a formula for the increase in profit per unit increase in price assuming that cost is held constant. Your formula may contain symbols for functions and derivatives not explicitly known.

   **Solution:** By the multivariate chain rule, the rate of change of $P$ with respect to $p$ is obtained by summing products along the $P$-$p$ branch and the $P$-$n$-$p$ branch. This yields

$$\frac{\partial P}{\partial p} = \frac{\partial P}{\partial p} + \frac{\partial P}{\partial n}\frac{dn}{dp}\,.$$

   Note the ambiguous use of the term $\partial P/\partial p$ occurring on both sides with different meanings! Computing partial derivatives of $P = n(p - c)$ gives $P_n = p - c$ and $P_p = n$. At the given values, $P_p = 10,000$ and $P_n = 2$, so the right-hand side is $10,000 + 2\,dn/dp$. This is the answer sought.

9. A customer's satisfaction with her hotel room in Barbados is modeled by the utility function $u = 100 - (T-70)^2 - \dfrac{c}{10}$ where $T$ is temperature in degrees Farenheit and $c$ is cost per night in dollars. If the room is presently $78°F$ and costs \$200 per night, how many more dollars per night would she be willing to pay per extra degree of lower temperature in the room?

**Solution:** The slope of the tangent line to the level curve of $u(T, c)$ through the point $(78, 200)$ is $-u_T/u_c$. The marginal rate of substitution is the negative of this slope, so it is $u_T/u_c$. Here, $u_T = -2(T-70) = -16$ and $u_c = -1/10$, so the slope of the tangent in the $T$-$c$ plane is 160. This means she would be willing to pay \$160 per night extra per degree cooler at this point.

10. Use the increment theorem for the function $f(x, y) = \sqrt{x + \ln y}$ to give a numerical estimate of $\sqrt{99 + \ln 1.3}$.

You don't need to state the theorem, but if you want to be eligible for partial credit you should state the values of $\Delta x$ and $\Delta y$ and show how the relevant partial derivatives are evaluated.

**Solution:** We will approximate near $x = 100, y = 1$ with $\Delta x = -1$ and $\Delta y = 0.3$. Then

$$\Delta f \approx f_x \Delta x + f_y \Delta y$$

where $f_x = 1/(2\sqrt{x + \ln y})$ which evaluates to $1/20$ and $f_y = (1/y)/(2\sqrt{x + \ln y})$ which also evaluates to $1/20$. Thus $\Delta f \approx (-1 + 0.3)/20 = -0.035$ leading to

$$\sqrt{99 + \ln 3} \approx 10 - 0.035 = 9.965\,.$$

TABLE 8.1 Basic integration formulas

1. $\int k\, dx = kx + C$ (any number $k$)

2. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C$ $(n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\, dx = e^x + C$

5. $\int a^x\, dx = \dfrac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$

6. $\int \sin x\, dx = -\cos x + C$

7. $\int \cos x\, dx = \sin x + C$

8. $\int \sec^2 x\, dx = \tan x + C$

9. $\int \csc^2 x\, dx = -\cot x + C$

10. $\int \sec x \tan x\, dx = \sec x + C$

11. $\int \csc x \cot x\, dx = -\csc x + C$

12. $\int \tan x\, dx = \ln|\sec x| + C$

13. $\int \cot x\, dx = \ln|\sin x| + C$

14. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\, dx = \cosh x + C$

17. $\int \cosh x\, dx = \sinh x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C$ $(a >$

22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C$ $(x >$