Math 110

Fall 2014

Exam III

Solutions

1. Find the general solution to the differential equation

$$y' = \frac{1}{\sqrt{xy}}\,.$$

Solution: This is a separable equation. Beginning with $\frac{dy}{dx} = \frac{1/\sqrt{x}}{\sqrt{y}}$, cross-multiply and then integrate:

$$\begin{aligned} \sqrt{y}\,dy &= \frac{dx}{\sqrt{x}} \\ \frac{2}{3}y^{3/2} &= 2\sqrt{x} + C\,. \end{aligned}$$

Solving for y then gives

$$y(x) = \left(3\sqrt{x} + C\right)^{2/3}\,.$$

The constant C can be any real number (it is 3/2 times the arbitrary constant first introduced).

2. Find the general solution to the differential equation

$$y' = 2x - y\,.$$

Solution: Moving y to the left-hand side gives $y' + y = 2x$ which is a linear first-order equation with $P(x) = 1$ and $Q(x) = 2x$. The integrating factor is $v(x) = e^{\int P(x)\,dx} = e^x$. Multiplying through by the integrating factor gives

$$e^x y' + e^x y = 2x\,e^x$$

and integrating both sides gives with respect to x (using integration by parts on the right-hand side) gives

$$e^x y = (2x - 2)e^x + C\,.$$

Finally, dividing by the integrating factor e^x gives

$$y(x) = 2x - 2 + Ce^{-x}\,.$$

3. Solve the initial value problem

$$\frac{dy}{dx} = x - xy\,, \quad y(0) = 11\,.$$

Solution 1: Moving xy to the left-hand side, this is a linear first-order equation $y' + xy = x$. It has $P(x) = x$ and $Q(x) = x$. The integrating factor is $v(x) = e^{\int P(x)\,dx} = e^{x^2/2}$. Multiplying by $v(x)$ and integrating gives

$$e^{x^2/2} y = \int x e^{x^2/2}\, dx\,.$$

Because the initial condition is the value at $x = 0$ we evaluate the indefinite integral by a definite integral beginning at 0:

$$e^{x^2/2} y = \int_0^x t e^{t^2/2}\, dt + C\,.$$

Substituting $x = 0$ the initial condition says $y = 11$ therefore

$$11 = \int_0^0 t e^{t^2/2}\, dt + C$$

and thus $C = 11$. We have, finally,

$$e^{x^2/2} y = \int_0^x t e^{t^2/2}\, dt + 11 = e^{x^2/2} - 1 + 11$$

and therefore

$$y(x) = 1 + 10e^{-x^2/2}\,.$$

Solution 2: Writing the right-hand side as $x(1 - y)$ we see this is a separable equation and cross-multiply to get $\frac{dy}{1-y} = x\,dx$. Integrating, then simplifying,

$$\begin{aligned} -\ln|1 - y| &= \frac{x^2}{2} + C \\ |1 - y| &= C_1 e^{-x^2/2} \end{aligned}$$

where $C_1 = e^C$ is an positive number. This is equivalent to $1 - y = C_2 e^{-x^2/2}$ and hence $y(x) = 1 - C_2 e^{-x^2/2}$. where C_2 is any real number. By the initial condition, $11 = 1 - C_2$ therefore $C_2 = -10$ and

$$y(x) = 1 + 10e^{-x^2/2}\,.$$

4. A population grows by net births at an annual rate equal to 2% of the population and by net immigration at an annual rate of 10,000 people per year[1].

(a) Write a differential equation for the population as a function of time.

Solution: Denote the population at time t years by $N(t)$ (units of people). The instantaneous rate of change of N due to net births is $0.02N(t)$ people per year. The instantaneous rate of change of N due to immigration is the constant 10000 people per year. Adding these gives

$$N'(t) = 0.02N(t) + 10000\,.$$

Note: if you interpreted 0.02 as an *annualized rate*, this would mean that the actual instantaneous net birth rate was $\ln 1.02$ instead of 0.02; we accepted that answer as well.

(b) Find the general solution to this equation.

Solution: We will solve this as a linear first order equation. Note: you could use the off the shelf result about exponential growth and decay for equations of the form $y' = C(L - y)$ with $C = -0.02$ and $L = -5000000$. The equation is put in the form

$$N' - 0.02N = 10000$$

with a constant $P(x) = -0.02$ and also a constant $Q(x) = 10000$. The integrating factor is $e^{-0.02x}$. Multiplying and integrating gives

$$e^{-0.02t}N(t) = \int 10000e^{-0.02t}\,dt = C - 500000e^{-0.02t}\,.$$

Multiplying by $e^{0.02t}$ to cancel the $e^{-0.02t}$ yields

$$N(t) = Ce^{0.02t} - 500,000\,.$$

[1] "net births" means births minus deaths, so you don't have to worry about the two separately; similarly, "net immigration" means immigration minus emmigration."

(c) Find the population at time $t = 10$ years if the population at time $t = 0$ is one million people. Leave this in the form of an exact expression; do not evaluate as a decimal number.

Solution: Plugging in $N(0) = 1,000,000$ gives $1,000,000 = C - 500,000$ so $C = 1,500,000$. Thus

$$N(t) = 1,500,000e^{0.02t} - 500,000\,.$$

(d) Give an approximate value for the population at time $t = 10$ years by using the quadratic Taylor approximation to evaluate the exponential.

Solution: The exact solution at $t = 10$ is $1,500,000e^{.2} - 500,000$. The quadratic Taylor polynomial for e^x is $P_2(x) = 1 + x + x^2/2$. Evaluating this at $x = 0.2$ gives $P_2(0.2) = 1 + 0.2 + (0.2)^2/2 = 1.22$. Using $P_2(0.2)$ in place of $e^{0.2}$ gives the approximation

$$N(t) \approx 1,500,000P_2(0.2) - 500,000 = 1,500,000(1.22) - 500,000 = 1,330,000\,.$$

FYI, though it is not part of the problem, the true value of $P(0.2)$ is equal to $1,332,104$; the quadratic approximation is off by a couple thousand people out of a million; the cubic approximation, $1,332,000$, is off by about a hundered.

5. Compute the volume above the unit square $0 \le x, y \le 1$ and underneath a canopy whose height at the point (x, y) is given by the function xe^{xy}.

Solution: The volume is given by the integral

$$\int_R \int xe^{xy}\, dA$$

where R is the unit square. We can integrate in either order. Integrating in vertical strips gives

$$\int_0^1 \int_0^1 xe^{xy} dy\, dx = \int_0^1 \left(e^{xy}|_{y=0}^{y=1} \right) dx = \int_0^1 (e^x - 1)\, dx = e^x - x|_0^1 = e - 2\,.$$

<u>Note:</u> you may also have tried the horizontal strip integral. If you try, you find that the inner integral can still be computed but not the outer integral. The computation begins as follows. The inner integral $\int xe^{xy}\, dx$ needs to be integrated by parts, giving

$$\int xe^{xy}\, dx = x\frac{e^{xy}}{y} - \int \frac{e^{xy}}{y}\, dx = \frac{xe^{xy}}{y} - \frac{e^{xy}}{y^2} = \left(\frac{x}{y} - \frac{1}{y^2} \right) e^{xy}\,.$$

Evaluating at $x - 0$ and 1 yielding

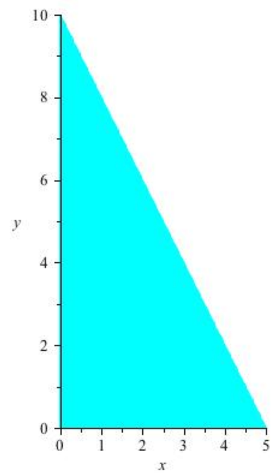
$$\int_0^1 \left[\left(\frac{1}{y} - \frac{1}{y^2} \right) e^y - \frac{1}{y^2} \right] dy$$

which cannot be done except by reversing the order of integration.

6. Sketch the region of integration and evaluate the double integral.

$$\int_0^5 \int_0^{10-2x} \left(1 + \frac{xy}{50}\right)\, dy\, dx$$

Solution: The region is bounded above by the segment of the line $y = 10 - 2x$ that goes from $(0, 10)$ to $(5, 0)$.



Evaluating the inner integral gives

$$\begin{aligned}\int_0^{10-2x} \left(1 + \frac{xy}{50}\right) dy &= \left(y + \frac{xy^2}{100}\right)\Bigg|_0^{10-2x} \\ &= 10 - 2x + \frac{x}{100}(10-2x)^2 \\ &= 10 - 2x + x - \frac{2}{5}x^2 + \frac{x^3}{25}\end{aligned}$$

Integrating from 0 to 5 gives

$$\int_0^5 10 - x - \frac{2}{5}x^2 + \frac{1}{25}x^3 = \left(10x - \frac{x^2}{2} - \frac{2}{15}x^3 + \frac{x^4}{100}\right)\Bigg|_0^5$$

which comes out finally to be

$$\frac{312}{25}.$$

7. (a) For what value of C is Cye^{-x} a probability density on the infinite strip $0 \le y \le 1, 0 \le x < \infty$?

Solution: We integrate $\int\int Cye^{-x}\,dx\,dy$ over the strip. The inner integral is

$$\int_0^\infty Cye^{-x}\,dx = -Cye^{-x}\big|_0^\infty = Cy\,.$$

The outer integral is then $\frac{1}{2}Cy^2\Big|_0^1 = \frac{C}{2}$. Therefore $C = 2$.

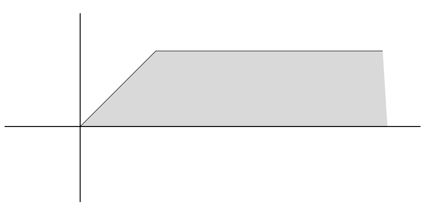
(b) What is the mean of the Y variable for this probability density?

Solution: This is given by integrating

$$\int_0^1\int_0^\infty 2y^2e^{-x}\,dx\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}\,.$$

(c) (4 points extra credit, if you have time) What is the probability that $Y \le X$ for a pair of values (X, Y) drawn from this density? (Show your work on the facing page)

Solution: The region $y \le x$ is all but a triangle of the strip:



The probability is given by integrating over this region: $\displaystyle\int_0^1\int_y^\infty 2ye^{-x}\,dx\,dy$. The inner integral now evaluates to $2ye^{-y}$. Integrating this from 0 to 1 yields (by parts): $-(2+2y)e^{-y}\big|_0^1 = 2 - \frac{4}{e}$.

8. Let $f(x, y) = \ln(x + \sqrt{y})$.

(a) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{x + \sqrt{y}} \\ \frac{\partial f}{\partial y} &= \frac{1}{2\sqrt{y}} \frac{1}{x + \sqrt{y}} = \frac{1}{2x\sqrt{y} + 2y} \end{aligned}$$

(b) Evaluate these at $x = 1/2$ and $y = 9/4$.

Solution: at $x = 1/2, y = 9/4$, use the fact that $\sqrt{y} = 3/2$ to see that these evaluate to $1/2$ and $1/(3/2 + 9/2) = 1/6$ respectively.

(c) Use the Increment Theorem with $\Delta x = 1/10$ and $\Delta y = -1/4$ to estimate $\ln(0.6 + \sqrt{2})$.

Solution: We use the Increment Theorem at $(x, y) = (1/2, 9/4)$ to see that

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \ln 2 + \frac{1}{2}(0.1) + \frac{1}{6}\left(-\frac{1}{4}\right) = \ln 2 + \frac{1}{120}.$$

9. Use the chain rule to evaluate $\dfrac{\partial f}{\partial s}$ at the point $(s,t)=(4,7)$ if:

$$\begin{aligned} x &= s^2 - 2t + 1 \\ y &= \ln(2s - t) \\ f(x,y) &= \sqrt{x + e^y} \end{aligned}$$

Solution:

$$\begin{aligned} \frac{\partial x}{\partial s} &= 2s \\ \frac{\partial y}{\partial s} &= \frac{2}{2s - t} \\ \frac{\partial f}{\partial x} &= \frac{1}{2\sqrt{x + e^y}} \\ \frac{\partial f}{\partial y} &= \frac{e^y}{2\sqrt{x + e^y}} \end{aligned}$$

Evaluate $x(4,7)=3$ and $y(4,7)=0$. Plug these into the first two equations below and plug in $s=4, t=7$ to the last two to get

$$\begin{aligned} \frac{\partial x}{\partial s}(4,7) &= 8 \\ \frac{\partial y}{\partial s}(4,7) &= 2 \\ \frac{\partial f}{\partial x}(3,0) &= \frac{1}{4} \\ \frac{\partial f}{\partial y}(3,0) &= \frac{1}{4} \end{aligned}$$

Finally, by the chain rule,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{4}\cdot 8 + \frac{1}{4}\cdot 2 = \frac{5}{2}\,.$$

Note that we do not need to compute $\partial x/\partial t$ or $\partial y/\partial t$.

10. Suppose the price P that that the average buyer is willing to pay for a new sports car is proportional to $x^3/\sqrt{y}$ where x is the maximum speed the car can attain in MPH and y is the average number of fatalities per million hours driven. For each part (a)–(c), please state what quantity needs to be computed to answer the given question, then compute it.

(a) What is the amount that the sales price will increase per 1MPH increase in the maximum speed? This will depend on x and y and perhaps other variables or constants; you should start by writing down an equation satisfied by P, x and y.

Solution; Begin by writing the equation $P = kx^3/\sqrt{y}$ where k is a constant of proportionality. What is needed in this first part is the partial derivative of P with respect to x.

$$\frac{\partial P}{\partial x} = \frac{3kx^2}{\sqrt{y}} .$$

(b) Compute the amount that the sales price will increase for each increase of 1 fatality per million hours driven in the consumer safety data.

Solution: Similarly, here we need $\partial P/\partial y$.

$$\frac{\partial P}{\partial y} = \frac{kx^3}{2y^{3/2}} .$$

(c) Suppose that the actual maximum speed is 180 MPH and that the number of fatalities per million was estimated at 6.0. If new data causes this estimate to be revised upward to 6.4, roughly how much does the maximum speed need to increase in order for the manufacturer to be able to charge the same price as before?

Solution: The rate of change of x per unit movement in y on the level curve when P is constant is given by

$$\frac{dx}{dy} = -\frac{\partial P/\partial y}{\partial P \partial x} .$$

Plugging in gives

$$\frac{dx}{dy} = -\frac{-kx^3/2y^{3/2}}{3kx^2/\sqrt{y}} = \frac{x}{6y} .$$

At $x = 180$ and $y = 6$ this comes out to $180/6^2 = 5$. Therefore, if y increases by the small amount 0.4, then x should increase by roughly $0.4 \times dx/dy = 2$ in order to remain on the level curve for P.