Name: ____

Midterm Exam II for Math 110, Spring 2015

Solutions

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

- You have ninety minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ________________________________

Date: ________________________________

1. (a) Write this integral as a limit or a sum of limits.

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{1+x^2+x^4}}$$

Solution: There are no discontinuities but both limits of integration are infinite so two limits are required, meaning the integral must be split somewhere. We let b denote such a choice; the value of b in this case can be anything between $-\infty$ and ∞ (because of the symmetry many would choose zero). Letting b denote a choice of where to split the integral, the integral then becomes

$$\lim_{M\to\infty} \int_{-M}^{b} \frac{dx}{\sqrt{1+x^2+x^4}} + \lim_{M\to\infty} \int_{b}^{M} \frac{dx}{\sqrt{1+x^2+x^4}}\,.$$

(b) State whether the integral converges and give a justification of your answer.

Solution: As $x \to \pm\infty$, $1+x^2+x^4 \sim x^4$; we know this because when adding terms of different orders, the largest prevails. Therefore $1/\sqrt{1+x^2+x^4} \sim 1/\sqrt{x^4} = x^{-2}$. By the p-power test, $\int_b^\infty x^{-2}\,dx$ and $\int_\infty^b x^{-2}\,dx$ both converge (because $p = -2 < -1$). Hence the original integral is convergent.

2. (a) For what value of C is $f(x) = Cxe^{-x/2}$ a probability density on the half line $[0, \infty)$? Please show you you evaluate any integrals.

Solution: The indefinite integral of $xe^{-x/2}$ is easily computed by parts. It is equal to $-(2x+4)e^{-x/2}$. Evaluating this at ∞ and 0, we see that

$$-(2x+4)e^{-x/2}\Big|_0^\infty = 0 - (-4) = 4\,.$$

Therefore the mean is 4. [To see why the value at infinity was zero, recall it is reall the limit $\lim_{x\to\infty} -(2x+4)e^{-x/2}$. To see that this is zero, use L'Hôpital's rule or the fact that exponentials outgrow any powers.]
Multiplying by C makes the integral equal to 1, so $C = 1/4$.

(b) What is the mean of this density?

Solution: The mean is given by $\int_0^\infty x \cdot C \cdot xe^{-x/2}\,dx = \int_0^\infty \frac{1}{4}x^2 e^{-x/2}\,dx$. Integrating by parts twice gives

$$\int \frac{x^2}{4}e^{-x/2}\,dx = -\frac{x^2+4x+8}{2}e^{-x/2}\,.$$

Evaluating this at ∞ and 0 and subtracting gives $0 - (-8/2) = 4$. So the mean is 4.

3. (a) Compute the quadratic Taylor polynomial $P_2(x)$ for the function $f(x) = \dfrac{1}{\sqrt{x}}$ about the point $x = 9$.

Solution: The first two derivatives of f are $-(1/2)x^{-3/2}$ and $(3/4)x^{-5/2}$. Evaluating f and its first two derivatives at $x = 9$ gives $f(9) = 1/3$, $f'(9) = -1/54$, $f''(9) = 1/324$. The quadratic Taylor polynomial is therefore given by

$$f(9) + f'(9)(x-9) + \frac{f''(9)}{2}(x-9)^2 = \frac{1}{3} - \frac{1}{54}(x-9) + \frac{1}{648}(x-9)^2 .$$

(b) What approximation does this give for $\dfrac{1}{\sqrt{10}}$? Write the answer in the box, simplifying fractions when possible:

Solution: Using $x - a = 10 - 9 = 1$, we get

$$P_2(10) = \frac{1}{3} - \frac{1}{54} + \frac{1}{648} = \frac{216 - 6 + 1}{648} = \frac{205}{648} .$$

$P_2(10) =$ $\boxed{\dfrac{205}{648}}$

(c) State what Taylor's theorem with remainder says about the remainder $R_2 = \dfrac{1}{\sqrt{10}} - P_2(10)$. You do not need to compute anything or find bounds, just state the conclusion of the theorem applied to this case.

Solution: The third derivative of f is $-(15/8)x^{-7/2}$. The remainder R_2 must therefore satisfy

$$\begin{aligned} R_2(10) &= -\frac{15}{8}u^{-7/2}\frac{(10-9)^3}{3!} \\ &= \frac{5}{16}u^{-7/2} \end{aligned}$$

for some u in the interval $[9, 10]$. FYI, that makes the remainder about -0.0001.

4. Write an initial value problem for this scenario. You do not have to solve the differential equation but you must give the interpretation of all variables and constants, their units, and indicate which is the dependent and the independent variable.

   > Toxins accumulate at a toxic waste site at an average rate of 1.9 kg per hour. Natural processes are able to break down the toxins at a rate proportional to the amount of toxins present, the time constant being 0.10 yr^{-1}. The site opened in 1995 when an area containing 4,800 kg of toxins was dedicated to toxic waste storage.

   Solution: Both hours and years are mentioned in the problem. Choose one of these for the time unit, say years, so that we can let t be the time, in years; we can start the problem at $t = 1995$ or we can measure the time in years elapsed since 1995. I will choose the former but either is fine. Time is the independent variable. Let $M(t)$ be the quantity of toxins in the site, measured in kilograms. This is the dependent variable. Then

$$M'(t) = 1.9 \times 24 \times 365 \frac{\text{kg}}{\text{hr}} - 0.1M(t)$$

   where the time constant 1/10 is measured in inverse years and the multiplication by 24×365 converts from kg/hr to kg/year. We are given that there were 4800 kg of toxins when the site opened. Thus the initial condition is $M(1995) = 4800$.

5. In each case, say whether the series converges and justify your answer.

(a)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

Solution: Yes, by the alternating series test. The three conditions are all true: (1) the terms alternate in sign; (2) $a_n = f(n)$ where $f(x) = 1/\ln x$ is a decreasing function; (3) $\lim_{n\to\infty} a_n = 0$.

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

Solution: Use the integral test. The sum converges if and only if the integral

$\displaystyle\int_b^{\infty} \frac{1}{x(\ln x)^{1/2}}\, dx$ converges. There is a test for this, the "power of log" test, which says that $\int (\ln x)^p/x\, dx$ converges if and only if $p < -1$. In this case $p = -1/2$ so there is no convergence. You can also work this out by hand. A simple substitution gives

$$\int_b^M \frac{(\ln x)^{1/2}}{x}\, dx = \frac{2}{3}(\ln x)^{3/2}\Big|_b^M = (\ln M)^{3/2} - (\ln b)^{3/2}\,.$$

As $M \to \infty$ this goes to infinity, hence the integral diverges.

6. Roberts' law for executive compensation says that an executive of a firm of size S earns an amount proportional to $S^{1/3}$. In order to study this further, an economist compiles a list of 1,000 firms, one of each size from 1 to 1,000.

(a) Write an expression in Σ-notation for the total executive compensation among the firms in the study. State which variables in your sum are free and which are bound.

Solution: Let k be the constant of proportionality, so the earnings of the executive in a firm of size S are $kS^{1/3}$. The total being asked for is therefore $k1^{1/3} + \cdots + k1000^{1/3}$ which may be written as

$$\sum_{n=1}^{1000} kn^{1/3}\,.$$

The constant k is a free variable: a value on which the sum depends. The index of summation n is a bound variable: the sum does not depend on any value of n.

(b) Give an approximate value for this sum by approximating it by an integral; free variables should remain unevaluated.

Solution: The sum is approximately $\int_0^{1000} kx^{1/3}\,dx$. This is easy to evaluate, and comes out to $\left.\frac{3}{4}kx^{4/3}\right|_0^{1000} = 7500k$. Alternatively we could use the integral approximation $\int_1^{1000} kx^{1/3}\,dx$ or $\int_1^{1001} kx^{1/3}\,dx$. These come out to values that are similar but a little messier.

(c) Say whether the actual sum is greater or less than your integral approximation.

Solution: The integral is a lower Riemann approximation for the sum, hence the sum is bigger than the integral.

7. Write an initial value problem for this integral equation.

$$y(t) = 3t + \int_2^t \frac{1}{s + y(s)}\, ds\,.$$

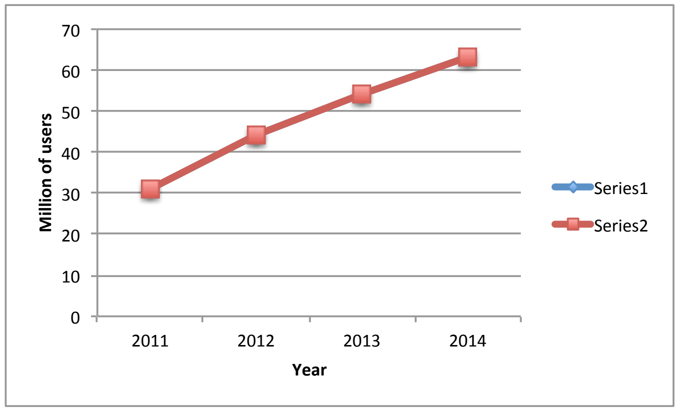
Solution: Differentiating both sides,

$$y'(t) = 3 + \frac{1}{t + y(t)}\,.$$

The known initial value will be at $t = 2$ because there the integral is over an interval of length zero and so is zero. Plugging in $t = 2$ gives $y(2) = 6$, which is therefore the initial value.

8. The number of iPhone users in the US in the years 2011–2014 is given by the following chart[1].

2011	31 million
2012	44 million
2013	54 million
2014	63 million



(a) (8 points) Circle which of the following differential equations you think best models this information, if t is time in years and y is users in millions? This part will be graded on an all or nothing basis unless you choose to provide a justification.

(i) $\frac{dy}{dt} = y + 10.3$

(ii) $\frac{dy}{dt} = y + 10.3t$

(iii) $\frac{dy}{dt} = y + 10.3(t - 2011)$

(iv) $\frac{dy}{dt} = 30 + 10.3(t - 2011)$

(v) $\frac{dy}{dt} = y - t$

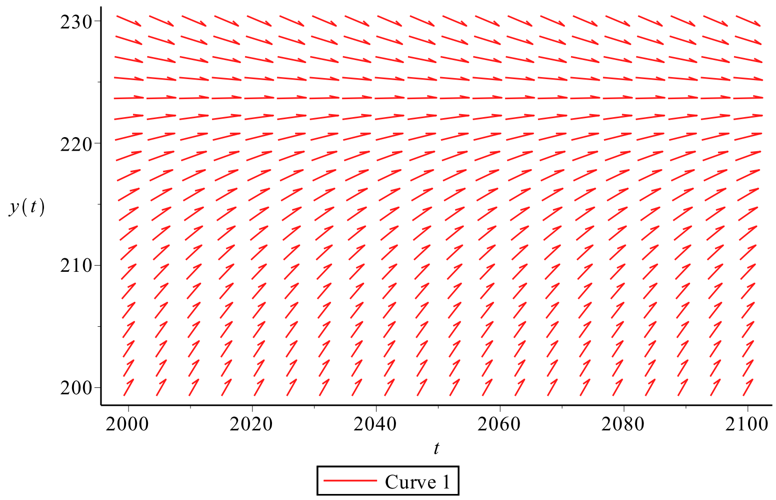
(vi) $\frac{dy}{dt} = 14 - \sqrt{y - 28}$

Solution: We have data at the following (t, y) pairs: $(2011, 31), (2012, 44), (2013, 54)$ and $(2014, 63)$. The six equations give different values for $y'(t)$. The last one gives the values $14 - \sqrt{3}, 10$, a little under 9, and a little under 8. This is a reasonable fit for the amount of growth taking place in the consecutive years: 13, 10 and 9 millions in the intervals from year 1 to 2, 2 to 3 and 3 to 4 respectively. None of the other equations gives anything close to this.

[1] Data is taken from http://www.statista.com/statistics/232790/forecast-of-apple-users-in-the-us

(b) (4 points) For your choice of differential equation, will $y(t)$ grow forever or approach a limiting value? Justify your answer by saying where the slope field slopes up versus down.

Solution: The slope field points upward whenever $\sqrt{y-28} < 14$ and downward whenever $\sqrt{y-28} > 14$. We can check that this means it points upward whenever $y < 224$ and downward when $y > 224$. Therefore, it looks something like this:



The y value cannot exceed 224 because as y passes through 224 the slope becomes negative, which would push the y value back downward.

9. Use Euler iteration with a step size of 1/3 to approximate $y(2)$ where $y(t)$ is the solution to the initial value problem

$$y' = 3x - y\,; \quad y(1) = 2\,.$$

Solution: We make a table of x values by increments of 1/3 from the initial condition $x = 1$ to the desired final condition $x = 2$. The increment Δx will be 1/3 each time. Filling in the Δy values, y values and re-computed y' values, we get this. Note that the 1 and the 2 in the top row come from the initial condition, the next 1 is $3 \cdot 1 - 2$, and the Δy is this times Δx, where $\Delta x = 1/3$; further rows of the table are computed from these as we did in class.

x	y	y'	Δy
1	2	1	1/3
4/3	7/3	5/3	5/9
5/3	26/9	19/9	19/27
2	97/27		

The answer is $y(2) = 97/27$.

10. (a) Compute the MacLaurin series for $e^{-x^2/2}$ and write the result in Σ-notation.

Solution: Plugging in $-x^2/2$ for x in the series for e^x, which is $\sum_{n=0}^{\infty} x^n/n!$, yields

$$\sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!} \, .$$

(b) For what x does this series converge?

Solution:

The ratio test gives

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^2}{2(n+1)} \, .$$

As $n \to \infty$ this goes to zero no matter what the value of x. Therefore the series converges for all x.

(c) What is the quartic Taylor polynomial $P_4(x)$?

Solution: The series has only even powers, so there are only three terms:

$$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} \, .$$

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k)$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a >$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x >$

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \\ \log_{10}(3) &\approx 0.48 \end{aligned}$$

Some other useful quantities to with 1%:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \end{aligned}$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)