Math 110

Fall 2014

Exam II

Solutions

1. Write the following indefinite integrals as limits. Do not attempt to evaluate them.

(a)

$$\int_0^1 \ln x \, dx \, .$$

Solution: The function $\ln x$ has a discontinuity at $x = 0$, diverging to $-\infty$ as $x \to 0^+$. Therefore

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx.$$

(b)

$$\int_{-\infty}^{\infty} e^{1/(1+x^2)} \, dx$$

Solution: The integrand is continuous everywhere but the integral is type-I improper in both its lower andupper limit. The limits must be evaluated separately. We may break at any point, b. Choosing, for example, $b = 7$ gives

$$\int_{-\infty}^{\infty} e^{1/(1+x^2)} \, dx \; = \; \lim_{t \to -\infty} \int_t^7 e^{1/(1+x^2)} \, dx + \lim_{M \to \infty} \int_7^M e^{1/(1+x^2)} \, dx \, .$$

2. (a) For what value of C is Cx^{-5} a probability distribution on $[1, \infty)$?

Solution: $\int_1^\infty Cx^{-5}\,dx = -\frac{C}{4}x^{-4}\Big|_0^\infty = \frac{C}{4}$. Setting this equal to 1 we find that $C = 4$.

(b) What is the mean of this distribution?

Solution: The mean is $\int_1^\infty x4x^{-5}\,dx = -\frac{4}{3}x^{-3}\Big|_0^\infty = \frac{4}{3}$.

3. (a) Compute the quadratic Taylor polynomial $P_2(x)$ for the function $f(x) = x^{1/3}$ about the point $x = 1$.

Solution: Computing derivatives gives $f'(x) = (1/3)x^{-2/3}$ and $f''(x) = (-2/9)x^{-5/3}$. Therefore, at $a = 1$, the values of f and its first two derivatives are

$$\begin{aligned} f(a) &= 1 \\ f'(a) &= \frac{1}{3} \\ f''(a) &= -\frac{2}{9}. \end{aligned}$$

Remembering to divide by 2! in the last term, the quadratic Taylor polynomial is

$$P_2(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2.$$

(b) Use this to estimate $\sqrt[3]{1.3}$ and write the answer in the box:

Plug in $x - a = 0.3$ to get $1 + \frac{0.3}{3} - \frac{0.09}{9}$, so

$P_2(1.3) =$ 1.09

(c) State what Taylor's theorem with remainder says about the remainder $R_2 = 1.3^{1/3} - P_2(1.3)$. You do not need to compute anything or find bounds.

Solution: Taylor's theorem says that

$$R_2 = \frac{f'''(u)}{3!}(x-a)^3$$

for some u between a and x. Plugging in $a = 1$ and $x = 1.3$ and computing $f'''(x) = (10/27)x^{-8/3}$, this says that

$$R_2 = \frac{10}{27}\frac{1}{6u^{8/3}}(0.3)^3 = \frac{1}{600u^{8/3}}$$

for some u between 1 and 1.3. In particular the remainder is positive but less than $1/600$.

4. Compute the quadratic Taylor polynomial $P_2(x)$ for the function

$$f(x) = \int_3^x \ln(1 + t^2)\, dt$$

about the point $x = 3$. Do not evaluate fractions, radicals, logarithems and so forth as decimals: you should simplify if possible, leaving expressions such as $\sqrt{2}/2$, $\ln 5$, etc.

Solution: We need to compute f and its first two derivatives evaluated at 3. Clearly $f(3) = 0$ because the limits of the integral are both 3. By the Fundamental theorem of calculus, $f'(x) = \ln(1 + x^2)$, so $f'(3) = \ln 10$. Differentiaing again, $f''(x) = 2x/(1 + x^2)$ so $f''(3) = 6/10$. The quadratic Taylor polynomial is therefore

$$(x - 3)\ln 10 + \frac{3}{10}(x - 3)^2 .$$

5. For which values of x does the series $\sum_{n=1}^{\infty} 3^{n/2} x^n$ converge?

Solution 1: The series is geometric with ratio $a_{n+1}/a_n = x\sqrt{3}$. A geometric series with ratio r converges precisely when $|r| < 1$, which in this case means $|x| < 1/\sqrt{3}$, so the values of x making the series convergent are $-1/sqrt3 < x < 1/\sqrt{3}$.

Solution 2: For a fixed value of x, the ratio a_{n+1}/a_n is equal to $x\sqrt{3}$, which is a constant independent of n, therefore $\lim_{n\to\infty} a_{n+1}/a_n$ is also equal to the constant $x\sqrt{3}$. The ratio test tells us this converges with $|x\sqrt{3}| < 1$ and diverges when $|x\sqrt{3} > 1$. To see what happens at the border, when $x = \pm 1/\sqrt{3}$, plug in the value of x to see that the series is either $1 + 1 + 1 + \cdots$ or $1 - 1 + 1 - 1 + \cdots$, neither of which converges. Therefore the series converges if and only if $|x| < 1/\sqrt{3}$, which is the same as saying $-1/\sqrt{3} < x < \sqrt{3}$.

6. (a) Compute the first four nonzero terms of the Taylor polynomial for $e^{x^2/2}$ around $x = 0$. [Hint: substituion is easier than computing the derivatives directly.]

Solution: The Taylor series for e^x is $1 + x\frac{x^2}{2} + \frac{x^3}{6} + \cdots$. Substututing $x^2/2$ for x gives

$$1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 = 1 + \frac{x^2}{2} + \frac{x^4}{2!2^2} + \frac{x^6}{3!2^3}\,.$$

We can multiply out the denominators to get $1 + x^2/2 + x^4/8 + x^6/48$.

(b) Write the Taylor series as a sum in Sigma notation. Examples of such series are $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$ and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution: One way to write it is $\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^2}{2}\right)^n$. Multiplying out the power gives a not too different form: $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!\,2^n}$.

7. Write a differential equation or initial value problem for this scenario. Be sure to give interpretations and units for every variable and constant.

   > When a nuclear reactor becomes hotter than its "critical temperature", its temperature increases at a rate which remains proportional to the fourth power of the amount by which the critical temperature is exceeded.

Solution: We will need a variable for the temperature of the reactor, let's say T. We also need a notation for the critical temperature, let's say T_0. Both T and T_0 can be in units of degrees Celsius (or Farenheit if you prefer). As usual, with rates of change, we need a variable for time, say t, in any reasonable time unit, say seconds in this case because meltdowns happen fast. The equation then says

$$\frac{dT}{dt} = k(T - T_0)^4 \,.$$

The only thing left is to find the units of k. Units on the left are temperature per time and units on the right are temperature to the fourth power (because temperature minus temperature has units of temperature). Therefore, k must be in units of $t^{-1}T^{-3}$.

8. (a) Find the general solution of the differential equation

$$\frac{dy}{dt} = k \cdot (125 - y) \, .$$

Solution: The general solution of $y' = k(L - y)$ should probably be on your cheatsheet! Here, $L = 125$ and the solution is

$$y = 125 + Ce^{-kt} \, .$$

(b) If $y(1) = 50$ and $y(3) = 122$ then what is $y(0)$?

Solution 1: Solve for e^{-k}.

$$\begin{aligned} 50 &= 125 + Ce^{-k} \\ 122 &= 125 + C(e^{-k})^3 \end{aligned}$$

Thus $Ce^{-k} = -75$ and $C(e^{-k})^3 = -3$. Dividing the second by the first yields $(e^{-k})^2 = 1/25$. Thus $e^{-k} = 1/5$ and $C = -5 \times 75 = -375$. This means $y(0) = 125 - 375 = -250$.

Solution 2: The difference between y and the final value of 125 is 75 at time 1 and 2 at time 3. In two time units it went down by a factor of 25. Therefore each time unit it goes down by a factor of 5. Going back one unit in time multiplies this difference by 5, yielding 375, meaning that $y(0) = 125 - 375 = -250$.

9.

$$\frac{dy}{dx} = y - x, \quad y(0) = 1\,.$$

Approximate $y(1)$ by using Euler iteration with step size 1/3. Please leave everything in exact form (fractions or radicals rather than decimals).

Solution: Let $f(x, y) = y - x$ so that $dy/dx = f(x, y)$. Make a table of values until you reach the y-value corresponding to $x = 1$.

x	y	$f(x,y)$	Δy
0	1	1	1/3
1/3	4/3	1	1/3
2/3	5/3	1	1/3
1	2		

10. (a) Choose which differential equation is depicted in this slope field. You don't need to write anything, just circle one of the choices.

   (i) $y' = y - (x-2)^2$
   (ii) $y' = 4 - y - x$
   (iii) $y' = \dfrac{x}{1+y}$
   (iv) $y' = (x-2)^2 - y$
   (v) $y' = \dfrac{4}{1+y}$

   Solution: Only (*iv*) matches the figure. (*i*) would have negative slopes up the y-axis, (*ii*) would have the same slope along diagonal lines sloping down to the right, (*iii*) would not be zero at $(2,0)$ and (*v*) would not depend on x.

(b) If, in addition, you are given that $y(0) = 3$ then which of the choices best approximates $y(3)$? Justify your answer by sketching the solution onto the given slopefield.

   (i) 3.6
   (ii) 3.0
   (iii) 2.4
   (iv) 1.8
   (v) 1.2
   (vi) 0.6

   Solution: Clearly the solution dips below 1, closer to 0.6 than to ay of the other values.

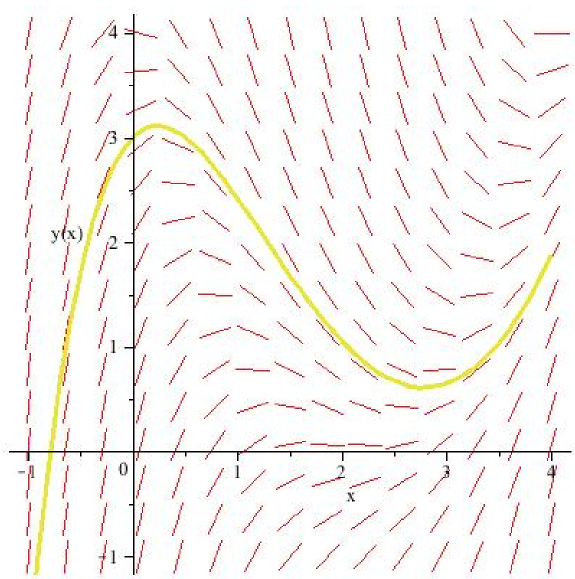

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (a >$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x >$