Name: ____

Midterm Exam I for Math 110, Spring 2015

Solutions

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

- You have ninety minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

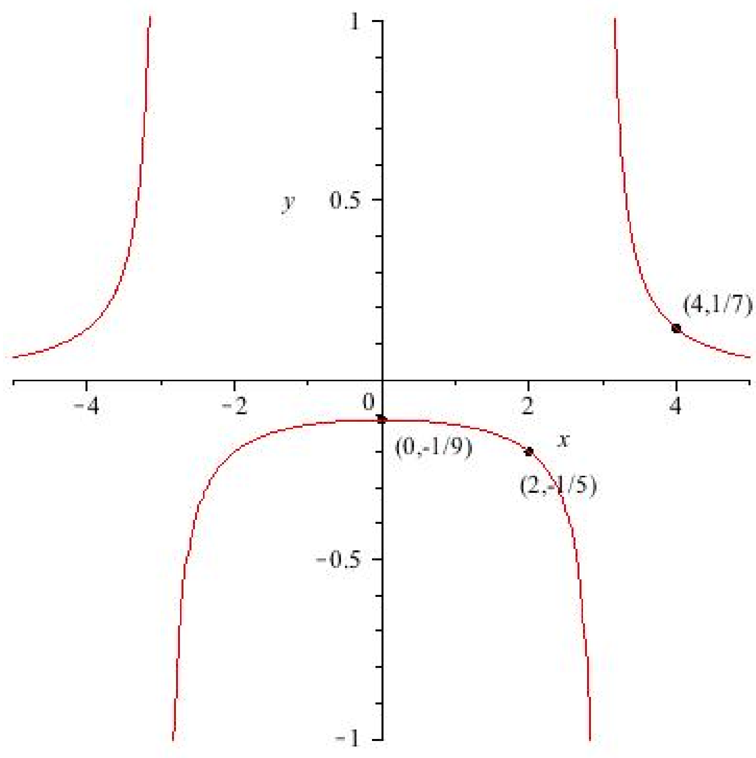
My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ________________________________

Date: ________________________________

1. Graph the function $\dfrac{1}{x^2-9}$. Please choose scales on the x- and y-axes that are neither too small nor too large to show the shape, and point out any maxima, minima and asymptotes, as well as exact coordinates of a few points on the graph.

Solution: There are vertical asymptotes at $x = \pm 3$, where the denominator goes to zero. As $x \to \infty$, the function goes to zero and this also happens as $x \to -\infty$, so there is a horizontal asymptote at height zero both to the right and to the left. By symmetry, the local maximum as at $x = 0$; the y-value there is $-1/9$; this point is marked on the graph along with a couple of others.



2. What is the relation between x and y if $\log_{10} x = 1 - \log_{10} y$?

Solution: The inverse function of $\log_{10} x$ is 10^x. To invert the log, raise 10 to the power of both sides, yielding

$$\begin{aligned} 10^{\log_{10} x} &= 10^{1-\log_{10} y} \\ \\ x &= 10^1 - 10^{\log_{10} y} \\ &= \frac{10}{y}. \end{aligned}$$

Therefore, the relation is $x = 10/y$ which can also be expressed as $xy = 10$.

3. The number of calories consumed per day by a mammal is roughly proportional to the 2/3 power of its volume.

   (a) Write an equation expressing this. Separately, state the interpretation of each variable or constant and its units.

   Solution:
   $$C = kV^{2/3}\,.$$
   Here, C represents the consumption in Calories per day, V represents the volume in cubic centimeters, and k is a constant of proportionality. In order to make the units balance, the units of k must be the units of C divided by the units of $V^{2/3}$. The units of C are calories per day. The units of V are cubic centimeters so the units of $V^{2/3}$ are square centimeters. Hence the units of k are $\dfrac{\text{Cal}}{\text{day cm}^2}$.

   (b) By what factor does the consumption increase if the animal quadruples in volume? Please leave this as an exact expression, not a decimal approximation.

   Solution: If we use the subscripts 0 and 1 to denote before and after, then $C_0 = kV_0^{2/3}$, $C_1 = kV_1^{2/3}$ and $V_1 = 4V_0$. Therefore
   $$C_1 = kV_1^{2/3} = k(4V_0)^{2/3} = k4^{2/3}V_0^{2/3} = 4^{2/3}C_0\,.$$
   Thus the consumption increases by a factor of $4^{2/3}$.

   (c) Give an approximate decimal value by using your log cheatsheet. It comes out easiest if you use base-ten logs.

   Solution: Use the fact that $\log_{10} 4 = 2\log^{10} 2 \approx 0.6$. Then $\log_{10} 4^{2/3} = (2/3)\log_{10} 4 \approx 0.4$. Therefore
   $$4^{2/3} = 10^{\log_{10} 4^{2/3}} \approx 10^{0.4}\,.$$
   To finish this off,
   $$10^{0.4} = 10^{1-0.6} = \frac{10}{10^{0.6}} \approx \frac{10}{4}\,.$$

4. Find a function $g(x) = cx^p$ such that $f(x) \sim g(x)$ as $x \to \infty$, where

$$f(x) = \sqrt{25x^3 + x^2 + e^{-x}}\,.$$

You do not have to prove your answer.

Solution: Because $x^2 \ll 25x^3$ and $e^{-x} \ll 25x^3$ we know that $25x^3 + x^2 + e^{-x} \sim 25x^3$. Thus

$$f(x) \sim \sqrt{25x^3} = 5x^{3/2}\,.$$

Therefore $c = 5$ and $p = 3/2$.

5. True or false?

(a) $\ln x \ll x^{1/8}$ as $x \to \infty$

Solution: TRUE. The natural log grows slower than any positive power of x.

(b) $x^{1/2} = o(x^{1/3})$ as $x \to 0^+$

Solution: TRUE. The ratio of the two functions is $x^{1/2}/x^{1/3} = x^{1/6}$. As $x \to 0^+$ this ratio goes to zero.

(c) $x^{1/2} = o(x^{1/3})$ as $x \to \infty$

Solution: FALSE. The ratio, $x^{1/6}$ does not go to zero as $x \to \infty$.

(d) $\sqrt{1+x^4} \sim x^2$ as $x \to \infty$

Solution: TRUE. Dividing one by the other gives $\sqrt{1+x^4}/x^2 = \sqrt{1/x^4+1}$. Taking the limit as $x \to \infty$, the term $1/x^4$ goes to zero while the constant term goes to (remains at) 1. The limiting ratio is therefore $\sqrt{0+1} = 1$, which is exactly the definition of $\sim$.

6. In each case, write the sum (call it S) in Σ notation, then evaluate it (leave in exact form, do not use decimal approximations).

(a) $M + (M-13) + (M-26) + \cdots + (M-130)$

Solution: The sum has eleven terms. It seems easiest in this case to start numbering at zero and go up to 10. Writing it this way gives $S = \sum_{n=0}^{10} M - 13n$. This is an arithmetic series, so the value is the number of terms multiplied by the average of the first and last:

$$S = 11\frac{M + M - 130}{2} = 11M - \frac{1430}{2}.$$

(b) $\frac{3}{5} - \frac{3}{10} + \frac{3}{20} - \frac{3}{40} \cdots$ going on forever

Solution: This is an infinite geometric series with first term $A = 3/5$ and ratio $r = -1/2$. Therefore,

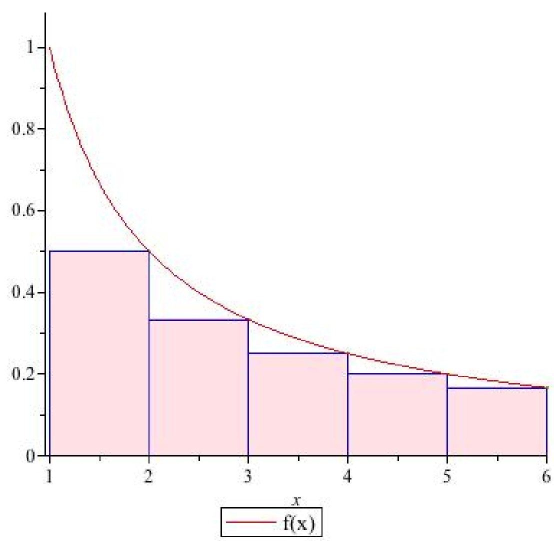
$$S = \sum_{n=0}^{\infty} \frac{3}{5}\left(\frac{-1}{2}\right)^n = \frac{3}{5}\frac{1}{1-(-1/2)} = \frac{2}{5}.$$

7. Let $A = \displaystyle\int_1^6 \frac{1}{x}\,dx$. Write, in Σ notation, a right-Riemann sum for this integral that has five terms. Evaluate this as a fraction. Draw the Riemann sum and the function on the same graph and use this drawing to say whether the Riemann sum is greater or less than A.

Solution: The intervals have width 1. The n^{th} interval is $[n, n+1]$. The rectangles in the right-Riemann sum have height corresponding to the right endpoint, namely $f(n+1)$. Therefore, the correct Riemann sum is

$$\sum_{n=1}^{5} 1 \cdot \frac{1}{n+1}\,.$$

This is equal to $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{20}{9}$ in lowerst terms.



The right-Riemann sum isa lower Riemann sum because the function is decreasing (or just look at the diagram), therefore the empirically computed value of 20/9 is an underestimate of the integral A (in this case a pretty severe underestimate of 1.22 compared to the true value of 1.79).

8. Compute the indefinite integral

$$\int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx \, .$$

Solution: Let $u = 1 + \sin x$ so that $du = \cos x \, dx$. Then

$$\int \frac{\cos x}{\sqrt{1 + \sin x}} \, dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{1 + \sin x} \, .$$

Therefore the general answer is $2\sqrt{0 + \sin x} + C$.

9. The predicted yield of an oil rig in barrels per day, after x days in operation, is $f(x) = x^2 e^{-x/100}$.

   (a) Compute the exact value of $\int_{100}^{400} f(x)\, dx$.

   Solution: The indefinite integral of $x^2 e^{-x/100}$ is obtained by integrating by parts twice. Letting $u = x^2$ and $dv = e^{-x/100} dx$ gives $du = 2x\, dx$ and $v = -100e^{-x/100}$. Thus

$$\int x^2 e^{-x/100}\, dx = -100x^2 e^{-x/100} + \int 200xe^{-x/100}\, dx\,.$$

   Integrating by parts again gives

$$\begin{aligned} \int x^2 e^{-x/100}\, dx &= -100x^2 e^{-x/100} - 20,000xe^{-x/100} + \int 20000e^{-x/100}\, dx \\ &= -100x^2 e^{-x/100} - 20,000xe^{-x/100} - 2,000,000e^{-x/100} \\ &= -100e^{-x/100}\left[x^2 + 200x + 20,000\right]\,. \end{aligned}$$

   Evaluating at $x = 400$ and 100 yields the definite integral

$$100e^{-1}(50,000) - 100e^{-4}(260,000) = 5,000,000e^{-1} - 26,000,000e^{-4}\,.$$

   (b) State what physical quantity this represents.

   Solution: This is the total number of barrels produced between day 100 and day 400.

   (c) To the nearest million (a very crude estimate, no pun intended), how many barrels is this?

   Solution: The value of $e^{-1} = 1/2.718\ldots$ which is a bit less than $1/2.5 = 0.4$. Therefore $5,000,000e^{-1}$ is a little under two million. The value of e^{-4} is a little under $1/2.5^4$ or around $1/40$, but one could obtain a better estimate with logs: $e^{-4} \approx 10^{-4/2.3} \approx 10^{1.8} = 10 \times 10^{0.8}$ which is a little more than 60. In any case, $26,000,000 \times e^{-4}$ is around half a million, so when you subtract you still get between one and two million barrels. Either one million or two million counts as correct.

10. Joe's credit card charges 2.0% interest per month. There is a minimum payment of \$100 per month to avoid a further finance charge. Suppose Joe charges \$300 per month and never pays more than the minimum.

   (a) How much does he owe after three months of this? **Solution**: AFter the first month, Joe has charged \$300, paid \$100, and owes \$200. After the second month, he gets charged interest, so this \$200 has grown to $\$200 \times 1.02$ and he has spent another net \$200 (spent\$300 and paid \$100); therefore he owes $\$200 \times 1.02 + \$200$. After the third month, this has grown by a factor of 1.02 and he owes another \$200, for total of

$$200(1 + 1.02 + 1.02^2)\,.$$

   (b) Write an expression in Sigma notation for what Joe owes after five years. **Solution**: Five years is 60 months, so Joe will owe

$$\sum_{k=0}^{59} 200 \times 1.02^k\,.$$

   (c) Evaluate this expression analytically – that means you will have expressions with powers in them. **Solution**: Using the first version and the formula for summing a finite geometric series with $A = 200, r = 1.02$ and $n = 59$ gives

$$A\frac{1 - r^{n+1}}{1 - r} = 200\frac{1.02^{60} - 1}{0.02}\,.$$

(d) Find a numerical approximation to this expression by using ln and its linearization near 1.

Solution: First we approximate $\ln 1.02$. The linear approximation for $\ln x$ near $x = 1$ is $L(x) = x - 1$. Thus

$$\ln 1.02 \approx 1.02 - 1 = 0.02\,.$$

Using this, we then have

$$\ln\left(1.02^{60}\right) = 60 \ln 1.02 \approx 60 \times 0.02 = 1.2$$

and hence

$$\frac{200}{0.02}(1.02^{60} - 1) = \frac{200}{0.02}(e^{60 \ln 1.02} - 1) \approx \frac{200}{0.02}(e^{1.2} - 1)\,.$$

To get a numerical value, the linearization of e^x is $1 + x$ so $e^{0.2} \approx 1.2$, giving

$$\approx 10,000\,(e{\cdot}1.2{-}1) \approx 10,000(2.7{\times}1.2{-}1) \approx 10,000(3.2{-}1) = 22,000\,.$$

So the approximate answer is \$22,000. FYI, an accurate computation yields \$22,810.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ $(n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C$ (*a* >

22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C$ (*x* >

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$
\begin{aligned}
e &\approx 2.7 \\
\ln(2) &\approx 0.7 \\
\ln(10) &\approx 2.3 \\
\log_{10}(2) &\approx 0.3 \\
\log_{10}(3) &\approx 0.48
\end{aligned}
$$

Some other useful quantities to with 1%:

$$
\begin{aligned}
\pi &\approx \frac{22}{7} \\
\sqrt{10} &\approx \pi \\
\sqrt{2} &\approx 1.4 \\
\sqrt{1/2} &\approx 0.7
\end{aligned}
$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)