Math 110

Fall 2014

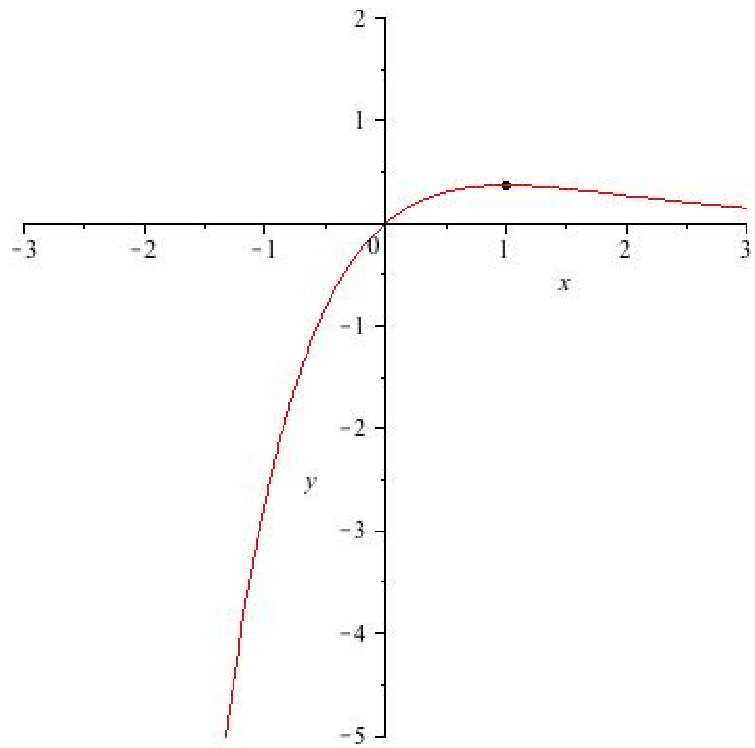
Exam I

Solutions

1. Graph the function xe^{-x} from $-\infty$ to ∞. Be sure to indicate any maxima, minima or asymptotes.

Solution:

Using L'Hôpital's rule, $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0$, therefore there is a horizontal asymptote at height zero. As $x \to -\infty$, the function goes to $-\infty$. The function has no discontinuities. Differentiating gives $e^{-x} - xe^{-x} = e^{-x}(1-x)$. This vanishes at $x = 1$, therefore $x = 1$ is a critical point, which must be a maximum. This point, $(1, 1/e)$, is marked on the graph by a dot.



2. (a) Write an equation stating that the surface area of a microchip is proportional to the 2/3 power of its volume. Be sure to state the units and the meaning of each variable and constant.

Solution: Let A denote the area in cm^2 and V the volume in cm^3. Then

$$A = kV^{2/3}$$

where k is a constant of proportionality. The units of $V^{2/3}$ are $(\text{cm}^3)^{2/3} = \text{cm}^2$, which is the same as the units of A, therefore k is unitless.

(b) The new microchip has one tenth the volume of the old microchip. What is the ratio of the surface area of the new chip to that of the old chip.

Please leave the answer in the form of a power; a decimal value is not needed.

Solution: Use subscripts of 1 to denote the new values and 0 to denote the old values. Then

$$\begin{aligned} A_1 &= V_1^{2/3} \\ &= \left(\frac{V_0}{10}\right)^{2/3} \\ &= 10^{-2/3} V_0^{2/3} \\ &= 10^{-2/3} A_0. \end{aligned}$$

The ratio in question is therefore $10^{-2/3}$. Although the problem did not ask for a numerical approximation, here it is anyway.

$$10^{-2/3} = 10^{-1+1/3} \approx \frac{1}{10} 10^{.333} = \frac{1}{10} 10^{0.3} 10^{0.033} \approx \frac{1}{5} e^{0.033 \cdot 2.3} \approx \frac{1}{5}(1+0.033 \cdot 2.3) \approx \frac{1.075}{5} = 0.215\,.$$

Alternate solution: Area scales as volume to the 2/3 power, so multiplying the volume by 1/10 multiplies the area by $(1/10)^{2/3}$.

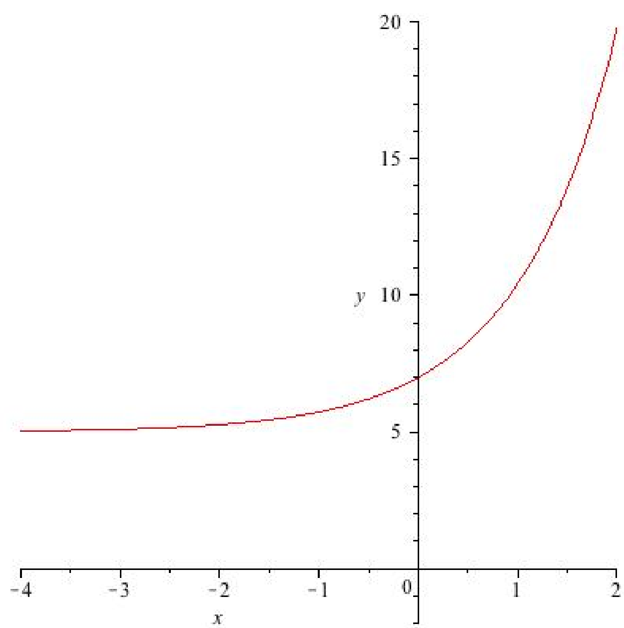
3. Compute the inverse function for $f(x) = 5 + e^{2x}$ and give the domain and range of the inverse function.

Solution: If $y = 5 + 2e^x$ then solving for x gives $y - 5 = 2e^x$ hence $(y - 5)/2 = e^x$ hence $x = \ln((y - 5)/2)$. We can write this as

$$f^{-1}(y) = \ln \frac{y - 5}{2}\,.$$

The name of the variable does not matter – we could x or u or anything we like instead of y, e.g., $f^{-1}(x) = \ln((x - 5)/2)$.

The argument of the natural log must be postive, hence the domain of the inverse function is $y > -5$. The range is all real numbers because the domain of f is al real numbers.



4. (a) Use the linearization of $\log_{10} x$ near $a = 1$ to estimate $\log_{10} 1.046$.

Solution: The derivative of $\log_{10} x$ is $\dfrac{1}{x \ln 10}$. Evaluating at $x = 1$ gives $\dfrac{1}{\ln 10}$, thus,

$$L(x) = f(a) + f'(a)(x - a) = \frac{x-1}{\ln 10}\,.$$

Plugging in $x = 1.046$ gives

$$\log_{10} 1.046 \approx L(1.046) = \frac{0.046}{\ln 10} \approx \frac{0.046}{2.3} = 0.02\,.$$

(b) Use this to estimate $\log_{10}\left(1.046^{65}\right)$.

Solution:

$$\log_{10}(1.046)^{65} = 65 \log_{10} 1.046 \approx 65 \cdot 0.02 = 1.3\,.$$

(c) Use this to estimate 1.046^{65}.

Solution: For any number y, $10^{\log_{10} y}$ is y. We have just estimated that the base-ten log of 1.046^{65} is about 1.3. Therefore,

$$1.046^{65} \approx 10^{1.3} = 10 \cdot 10^{0.3} \approx 10 \cdot 2 = 20.$$

5. How many digits does the number e^{47} have? Circle the correct answer. If you answer with no justification you get all or nothing. If you provide justification you may get partial credit, but to get full credit it has to be correct.

(i) 19

(ii) 20

(iii) 21

(iv) 46

(v) 47

(vi) 48

Solution: To determine the number of digits a number has, we need to estimate its base-ten logarithm. The base-ten logarithm of e^{47} is estimated like this:

$$\log_{10} e^{47} = 47 \log_{10} e = 47 \frac{1}{\ln 10} \approx \frac{47}{2.3}\,.$$

We know that $46/2.3$ is precisely 20, so $47/2.3$ is between 20 and 21, meaning that e^{47} has 21 digits.

6. Find a function g of the form $g(x) = cx^p$ such that $f(x) \sim g(x)$ as $x \to \infty$ if f is the function defined by

$$f(x) = \frac{3x^2 + 1}{6x^5 + 2x^4 + 7x^3 - 9x + 9}\,.$$

Solution: The leading terms of the numerator and denominator are respectively $3x^2$ and $6x^5$. Dividing these yields the answer: $(1/2)x^{-3}$. Therefore $c = 1/2$ and $p = -3$. To prove that this works, evaluate the limit of $f(x)/(x^{-3}/2)$ as $x \to \infty$ and check if you get 1.

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{\cdot} 2x^3 &= \lim_{x\to\infty} \frac{6x^5 + 2x^3}{6x^5 + 2x^4 + 7x^3 - 9x + 9} \\ &= \lim_{x\to\infty} \frac{6 + 2x^{-2}}{6 + 2x^{-1} + 7x^{-2} - 9x^{-4} + 9x^{-5}} \end{aligned}$$

Both numerator and denominator have limit 6, therefore the quotient has limit 1.

7. Write the infinite sum $\frac{1}{14} - \frac{1}{28} + \frac{1}{56} - \frac{1}{112} + \cdots$ in Sigma notation and evaluate it.

Please leave the result as a fraction.

Solution: This is a geometric series with initial term $1/14$ and ratio $a_{n+1}/a_n = -1/2$. We may therefore write it as

$$\sum_{n=1}^{\infty} 14 \cdot \left(-\frac{1}{2}\right)^n .$$

The sum $\sum Ar^n$ is $A/(1-r)$, as long as $|r| < 1$, this being necessary for convergence. Plugging in $A = 1/14$ and $r = -1/2$ gives a sum of

$$\frac{1/14}{1-(-1/2)} = \frac{1/14}{3/2} = \frac{2}{3 \cdot 14} = \frac{1}{21} .$$

8. On January 1, in each year from 2001 to 2014, my grandma gave me a Certificate of Deposit for \$1,000 that grows at the rate of 5% per year. On January 1, 2015 (she did not give me one in 2015) what is the total value of these CD's?

   Please leave the answer as an exact expression in powers, fractions, etc.

   Solution: Gransma deposited 14 installments of \$1,000 for me. The first one grew for 14 years, from 1/1/2001 to 1/1/2015, so it grew in value to 1000×1.05^{14}. The second one grew for 13 years and is now worth 1000×1.05^{13}. The pattern continued this way, so the total value on 1/1/2015 was

$$1000 \times 1.05^{14} + 1000 \times 1.01^{13} + \cdots + 1000 \times 1.05^{1}$$

   which we write in Sigma notation as

$$\sum_{n=1}^{14} 1000 \times 1.05^{n} .$$

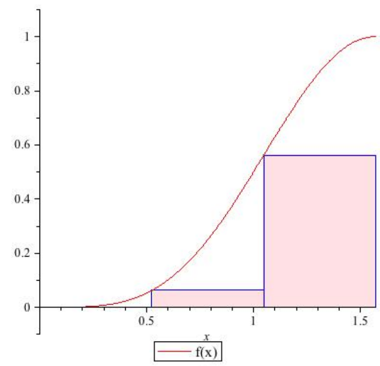
   The formula for a finite geometric sum shows this to be

$$1000\,\frac{1.05^{15} - 1}{1.05 - 1} \;=\; 20000(1.05^{15} - 1)\,.$$

9. Approximate $\int_0^{\pi/2} \sin^4 x \, dx$ in the following three ways. In parts (a)–(c) you should report the exact value (that is, leave in terms of π, fractions, trig functions, etc.).

(a) A lower Riemann sum with three rectangles

Solution: The first rectangle of the lower Riemann sum has height zero, so does not show up in the picture.

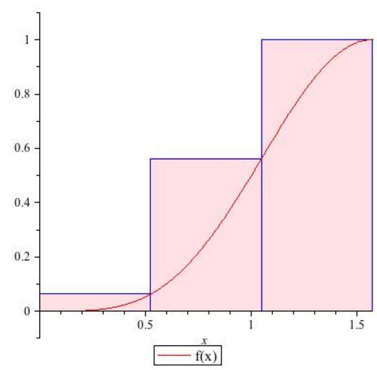


The rectangles all have width $\pi/6$ and have respective heights 0, $\sin^4(\pi/6)$ and $\sin^4(2\pi/6)$, that is, $0, 1/16$ and $9/16$. Adding the areas gives

$$\frac{\pi}{6} \times 0 + \frac{\pi}{6} \times \frac{1}{16} + \frac{\pi}{6}\frac{9}{16} = \frac{10\pi}{96}\,.$$

(b) An upper Rieman sum with three rectangles

Solution: The first two rectangles of the upper sum have the same height as the last two rectangles ofthe lower sum. This is always true when the function is monotone increasing or decreasing.



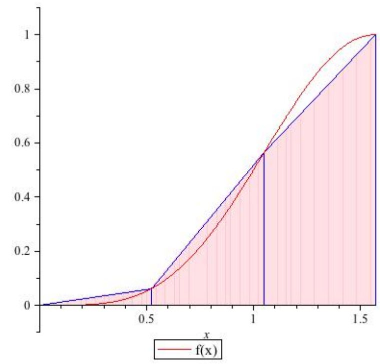
The rectangles all have width $\pi/6$ and have respective heights $\sin^4(\pi/6)$ and $\sin^4(2\pi/6)$, $\sin^4 \pi/2$, that is, $1/16, 9/16$, and 1. Adding the areas gives

$$\frac{\pi}{6} \times \frac{1}{16} + \frac{\pi}{6} \times \frac{9}{16} + \frac{\pi}{6}\frac{16}{16} = \frac{26\pi}{96}\,.$$

(c) A trapezoidal approximation with three trapezoids (triangles count as trapezoids)

Solution: The area of the trapezoids is excatly the average of the areas of the upper and lower Riemann sums (again, true for all monotone functions). Thus the trapezoidal approxmation yields

$$\frac{18\pi}{96} = \frac{3}{\pi}16\,.$$



(d) Circle which of these answers you believe to be closest to the numerical value.

- 0.2

- 0.4

- 0.6

- 0.8

- 1.0

- 1.2

Solution: $3\pi/16$ is a little bigger than $9/16$, so between 0.5 and 0.6. The closest answer is therefore 0.6.

10. Compute these integrals.

(a) $\displaystyle\int \cos x\sqrt{1+\sin x}\,dx$

Solution: Let $u = 1+\sin x$ so that $du = \cos x\,dx$. The integral then becomes

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+\sin x)^{3/2} + C\,.$$

(b) $\displaystyle\int_0^{\pi/4} e^{\tan\theta}\,\sec^2\theta\,d\theta$

Solution: Let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$. The integral then becomes

$$\int e^u\,du = e^u = e^{\tan\theta}\,.$$

Evaluating,

$$e^{\tan\theta}\Big|_0^{\pi/4} = e^1 - e^0 = e - 1\,.$$

11. Compute the indefinite integral $\int x \sec^2 x dx$.

Solution: Integrate by parts, taking advantage of the fact that $\sec^2 x$ is the derivative of $\tan x$. Letting $u = x, du = dx$, $v = \tan x, dv = \sec^2 x\, dx$, we have

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx\,.$$

You can look up the integral of $\tan x$ in the integral table, or else compute it using the substitution $u = \cos x$ to get

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| = -\ln|\cos x|\,.$$

The final answer is therefore

$$\int x \sec^2 x\, dx = x \tan x + \ln|\cos x|\,.$$

12. The arrival rate of a flash mob in people per minute is $100\,t\,e^{-t}$ where t is the time in minutes since the message went out. After ten minutes, the mob gives a final shout and disperses. Approximately how many people are present for the final shout?

Solution: The number present at time t is the integral of the arrival rate from time zero to time t. At time $t = 10$ minutes, this is

$$\int_0^{10} 100te^{-t}\,dt\,.$$

To evaluate this, integrate by parts, leading to

$$\begin{aligned} -100te^{-t}\big|_0^{10} - \int_0^{10} -100e^{-t}\,dt &= -100te^{-t}\big|_0^{10} - 100e^{-t}\big|_0^{10} \\ &= -1000e^{-10} + 100 - 100e^{-10}\,. \end{aligned}$$

The number e^{-10} is so small that this expression is very nearly equal to 100. In fact $e^{-10} = 10^{-10/2.3} = 10^{-4.\text{something}}$, meaning that $e^{-10} < 0.0001$ and even when multiplied by 1000 is less that 1/10.

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \end{aligned}$$

Here are some values of base-ten logs to three decimal places:

$$\begin{aligned} \log_{10}(2) &\approx 0.301 \\ \log_{10}(3) &\approx 0.477 \\ \log_{10}(4) &\approx 0.602 \\ \log_{10}(5) &\approx 0.699 \\ \log_{10}(6) &\approx 0.778 \\ \log_{10}(7) &\approx 0.845 \\ \log_{10}(8) &\approx 0.903 \\ \log_{10}(9) &\approx 0.954 \end{aligned}$$

Some other useful quantities to with 1% or so:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \\ \sqrt{3} &\approx 1.732 \\ \sqrt{5} &\approx 2.236 \end{aligned}$$

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)	12. $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	13. $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
3. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	14. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert +$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert +$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$	16. $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	17. $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x\,dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x\,dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$
10. $\int \sec x \tan x\,dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$