Math 110

Spring 2015

Final Exam

Solutions

1. Let $f(x, y) = x^2y - y^2$.

   (a) Compute ∇f.

   Solution: $\nabla f = f_x\,\hat{\mathbf{i}} + f_y\,\hat{\mathbf{j}} \;=\; 2xy\,\hat{\mathbf{i}} + (x^2 - 2y)\,\hat{\mathbf{j}}\,.$

   (b) From the point $(1, 1)$, which direction should you move in order to increase f the fastest? Give a unit vector in this direction.

   Solution: You should go in the direction of the gradient. The gradient at $(1, 1)$ is $2\hat{\mathbf{i}} - \hat{\mathbf{j}}$. The magnitude of the gradient is $\sqrt{(2^2) + (-1)^2} = \sqrt{5}$. A unit vector in the direction of the gradient is $\frac{1}{\sqrt{5}}(2, -1) = (\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$.

   (c) How fast does f increase per unit movement in this direction?

   Solution: The rate of increase of any function when moving in the direction of the gradient is always equal to the magnitude of the graadient. In this case, it is equal to $\sqrt{5}$.

   (d) At what rate does f increase per unit moved in a direction that differs from the previous direction by $45°$?

   Solution: The rate of change in direction $\mathbf{u}$ is $\nabla f \cdot \mathbf{u}$. This is equal to $|\nabla f| \times |\mathbf{u}| \times \cos\theta$ where θ is the angel between ∇f and $\mathbf{u}$. We plug in $|\nabla f| = \sqrt{5}$, $|\mathbf{u}| = 1$ and $\theta = 45°$ to get $\sqrt{5}\cos(45°) = \sqrt{5/2}$.

2. Find the location and the value of the maximum of the function

$$f(x, y) = 3x^2 + y^2$$

over the triangle $x \geq 0, y \geq 0, x + 2y \leq 13$. You need to justify why this, and not some other point, is the maximum.

Solution: The function f is increasing in both x and y so the maximum must occur at a point where it is not possible to move up or to the right. This means the maximum occurs on the diagonal $x + 2y = 13$ between $(13, 0)$ and $(0, 13/2)$. We need to find critical points on this line segment and to compare to the values at the two endpoints. We do not need to check for interior points where $\nabla f = 0$ because none of these can be the maximum.

Compute $\nabla f = 6x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}}$. The gradient of the constraint function $H(x, y) = x + 2y - 13$ is $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. These gradients are parallel when $12x - 2y = 0$, in other words, $y = 6x$. The point on $x + 2y = 13$ satisfying $y = 6x$ is $(1, 6)$. The value at $(1, 6)$ is $3 + 36 = 39$. The values at the endpoints are $13^2 = 169$ and $2(13/2)^2 = 169/2$. The first of these is the greatest and is greater than 39. Therefore, the maximum is 169 at $(13, 0)$.

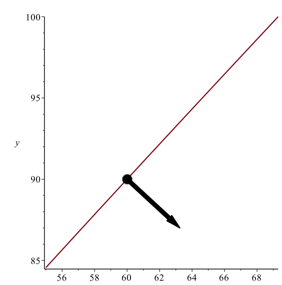
3. Survey data shows that satisfaction for purchasers of jetskis can be modeled by $u(x, y) = x - y^2/x$ where x is the maximum speed attainable in MPH and y is the number of accidents and breakdowns reported per year per 1,000 owners.

(a) What is the gradient of the utility function $u(x, y)$ at the point $(60, 90)$?

Solution: The gradient of u is given by $\nabla u(x, y) = (1+y^2/x^2)\hat{\mathbf{i}} - (2y/x)\hat{\mathbf{j}}$. Evaluating at $(60, 90)$ gives $(1 + (3/2)^2)\hat{\mathbf{i}} - 2(3/2)\hat{\mathbf{j}} = (13/4)\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$.

(b) Compute the slope of the level curve through of $u(x, y)$ through the point $(60, 90)$ and give a sketch showing the point $(60, 90)$, the gradient there, and the level curve as it passes through $(60, 90)$.

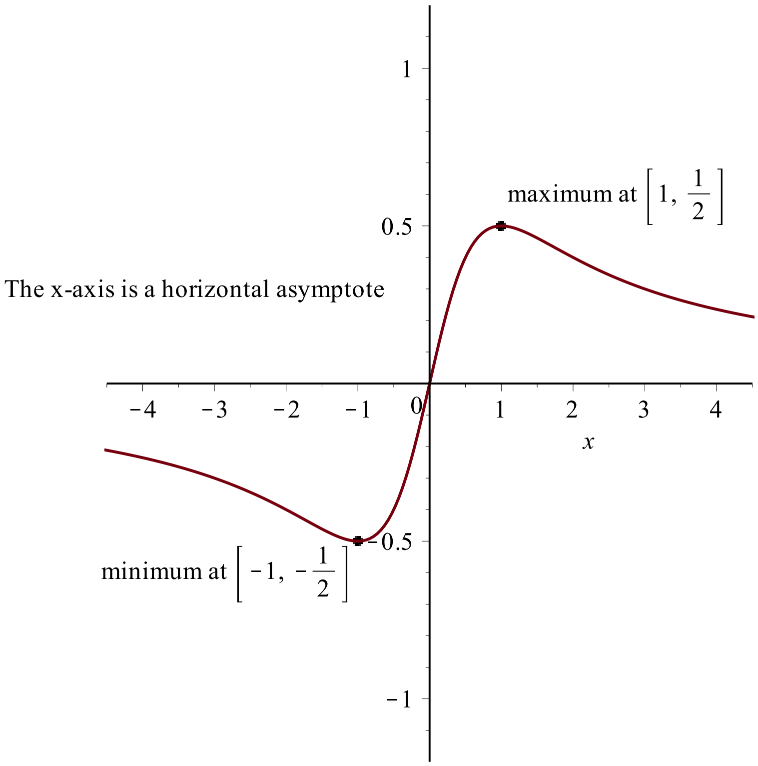
Solution: The slope is given by $-u_x/u_y$. This is equal to $(1 + y^2/x^2)/(2y/x) = (x^2 + y^2)/(2xy) = x/(2y) + y/(2x)$. Evaluating at the point $(60, 90)$ gives $60/180 + 90/120 = 1/3 + 3/4 = 13/12$.



(c) How many more MPH must next year's model go for each one accident or breakdown per thousand increase over this year's model if it is to maintain the same satisfaction and this year's model goes 60 MPH with 90 reported accidents or breakdowns per 1,000 owners?

Solution: The question asks for the marginal rate of substitution of MPH versus breakdowns, which is x versus y. This is the inverse of the slope or $12/13$. To see whether it is positive or negative, check the working of the problem: "how many more MPH per more breakdown?" When breakdowns increase, so must MPH, therefore the answer to the question is a positive number.

4. Graph the function $y = \dfrac{x}{1+x^2}$. Please give a scale, choosing one that allows you to show important features. Please any maxima, minima, asymptotes or discontinuities.



5. Circle the best approximation to 1.03^{90}.

(i) 2.7

(ii) 3.7

(iii) 15

(iv) 27

(v) 37

(vi) $1,000,000$

(vii) $1,000,000,000,000,000,000,000,000,000$

Solution: It is (iii). To see this, simplify to $e^{90 \cdot \ln 1.03}$. Approximating $\ln 1.03$ by the linear estimate $\ln(1+x) \approx x$ gives $\ln 1.03 \approx 0.03$, hence

$$1.03^{90} \approx e^{90 \cdot 0.03} = e^{2.7}\,.$$

There are a number of ways to see that 15 is the closest value. One is to compute $2.7^3 \approx 19.6$. Another is to convert to base 10:

$$e^{2.7} = 10^{2.7/2.3} \approx 10^{1.2} = 10 \times 10^{0.2} \approx 10 \times \sqrt[3]{10^{0.6}} \approx 10 \times \sqrt[3]{4} \approx 10 \times 1.6 = 16\,.$$

6. True or false?

(a) $\ln x \ll x^{1/8}$ as $x \to \infty$

Solution: TRUE. The natural log grows slower than any positive power of x.

(b) $x^{1/2} = o(x^{1/3})$ as $x \to 0^+$

Solution: TRUE. The ratio of the two functions is $x^{1/2}/x^{1/3} = x^{1/6}$. As $x \to 0^+$ this ratio goes to zero.

(c) $\dfrac{x^2}{\sqrt{1+x^3}} \ll x$ as $x \to \infty$

Solution: TRUE. Seeing that $1 \ll x^3$, the denominator is asymptotically equivalent to $\sqrt{x^3} = x^{3/2}$. That makes the whole expression asymptotically equaivalent to $x^{1/2}$ and we know $x^{1/2} \ll x$ as $x \to \infty$.

(d) $\sqrt{1+x^4} \sim x^2$ as $x \to \infty$

Solution: TRUE. Dividing one by the other gives $\sqrt{1+x^4}/x^2 = \sqrt{1/x^4+1}$. Taking the limit as $x \to \infty$, the term $1/x^4$ goes to zero while the constant term goes to (remains at) 1. The limiting ratio is therefore $\sqrt{0+1} = 1$, which is exactly the definition of $\sim$.

7. Write the sum $\frac{1}{6} - \frac{1}{12} + \frac{1}{24} - \frac{1}{48} + \cdots$ (don't overlook the negative signs) in Sigma notation and evaluate it.

Solution: This is a geometric series with the ratio between succesive terms equal to a negative number, $-1/2$ (it is the negative ratio that gives the alternation between positive and negative values). Thus we may write the general term as $(1/6)(-1/2)^n$ (starting at the $n = 0$ term). Thus the sum is equal to

$$\sum_{n=0}^{\infty} \frac{1}{6}\left(-\frac{1}{2}\right)^n = \frac{1/6}{1-(-1/2)} = \frac{1/6}{3/2} = \frac{1}{9}.$$

8. Uncle Sam has a debt of 18 trillion dollars (as of January 1, 2015). Senator Paul decides he needs to pay it off, and proposes an installment scheme, under which Uncle Sam will make payments of one trillion dollars on December 31, 2015 and every December 31 thereafter. Unfortunately, the debt increases by 5% during the course of every year.

(a) Write expressions for the amount owed by Uncle Sam on January 1 of 2015, 2016 and 2017.

Solution: Each year the new debt is the old debt multiplied by 1.05, with one trillion subtracted. This give the following table.

date	debt in trillions of dollars
Jan 1, 2015	18
Jan 1, 2016	$18 \times 1.05 - 1$
Jan 1, 2015	$18 \times 1.05^2 - 1.05 - 1$

(b) Write an expression using Sigma notation for how much Uncle Sam owes on January 1 of year n, counting 2015 as year 0.

Solution: After year n, Uncle Sam's debt is given, in trillions of dollars, by

$$18 \times 1.05^n - \sum_{k=0}^{n-1} 1.05^k\,.$$

(c) Evaluate the sum to get an algebraic expression.

Solution: The geometric series has first term 1 and $r = 1.05$. Plugging into the formula for a geometric series gives

$$\sum_{k=0}^{n-1} 1.05^k = \frac{1.05^n - 1}{1.05 - 1} = 20\left(1.05^n - 1\right) .$$

Therefore, on January 1 of year n, Uncle Sam owes

$$18 \times 1.05^n - 20\left(1.05^n - 1\right) \;=\; 20 - 2 \times 1.05^n \text{ trillion dollars.}$$

(d) Solve for the number of years, n, in which the debt will be paid off. Please leave this as an exact expression (it is OK if this leads to a value which is not a whole number).

Solution: From $20 - 2 \cdot 1.05^n = 0$ we get $1.05^n = 10$. We can write this as

$$n = \log_{1.05} 10 .$$

It may be somewhat easier to continue to simplify. An alternative expression is

$$n = \frac{\ln 10}{\ln 1.05} .$$

(e) Estimate the numerical value of n to the nearest whole number.

Solution: Estimate the value of $\ln 1.05$ by the linearization of $\ln x$ near 1 whicih is $x - 1$. Thus $\ln 1.05 \approx 0.05$. Using the log cheatsheet, $\ln 10 \approx 2.3$, therefore

$$\frac{\ln 10}{\ln 1.05} \approx \frac{2.3}{0.5} = 46 .$$

In other words, it will take Uncle Sam about 46 years to pay off this debt according to Senator Paul's schedule.

9. (a) Write the improper integral $\int_0^\infty xe^{-x}\,dx$ as a limit and determine whether it converges.

Solution: $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty}\int_0^M xe^{-x}\,dx$. There are several ways to see this converges. One is to do the integral, then evaluate the limit. Another is to observe that $x \ll e^{0.01x}$ (because x is asymptotically smaller than e^{cx} for any positive c) therefore $xe^{-x} \ll e^{0.01x}e^{-x} = e^{-0.99x}$ whose integral converges.

Because we will need it anyway, let's evaluate the integral. By parts,

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x} = (-x-1)e^{-x}\,.$$

Evaluating between zero an M gives

$$\int_0^M xe^{-x}\,dx = (-M-1)e^{-M} + e^{-0} = 1-(M+1)e^{-M}\,.$$

Checking that $(M+1)e^{-M}$ has a limit of zero as $M\to\infty$ (use L'Hôpital's rule or the fact that $M+1 \ll e^M$) shows that the limit exists and is equal to $e^0 = 1$.

(b) For what value of C is Cxe^{-x} a probability density on $[0,\infty)$?

Solution: The integral is already 1, so the constant C is just 1.

(c) What is the mean of this probability density?

Solution: The mean is $\displaystyle\int_0^\infty x^2e^{-x}\,dx$. This must be integrated by parts twice, but should be familiar. The indefinite integral is

$$\int x^2e^{-x}\,dx = -x^2e^{-x} + \int 2xe^{-x} = -x^2e^{-x} - \int 2xe^{-x} + \int 2e^{-x} = (-x^2-2x-2)e^{-x}\,.$$

These all evaluate to zero at infinity, so the mean is $2e^0 = 2$.

10. Let $f(x) = \int_1^x \frac{e^x}{x^2}\,dx$.

(a) Compute the linear and quadratic Taylor polynomials for f about the point $x = 1$.

Solution: First $f(1) = 0$ because the lower limit of the integral is 1. Next, by the Fundamental Theorem of Calculus, $f'(1) = e^x/x^2$ evaluated at $x = 1$, which is e. Next, the second derivative is the derivative of e^x/x^2, which by the product rule is $(1/x^2)e^x - (2/x^3)e^x$. Evaluating at $x = 1$ gives $-e$. We can now write down the Taylor polynomials.

$$\begin{aligned} L(x) &= 0 + e(x-1) \\ P_2(x) &= 0 + e(x-1) - \frac{e}{2}(x-1)^2 \end{aligned}$$

(b) Use these to give two estimates of $f(3/2)$. Leave as exact expresssions; do not evaluate numerically.

Solution:

$$\begin{aligned} L\left(\frac{3}{2}\right) &= \frac{e}{2} \\ P_2\left(\frac{3}{2}\right) &= \frac{e}{2} - \frac{e}{8} = \frac{3e}{8} \end{aligned}$$

(c) Now estimate the same integral by a trapezoidal approximation with just one trapezoid. Again, leave as an exact expression.

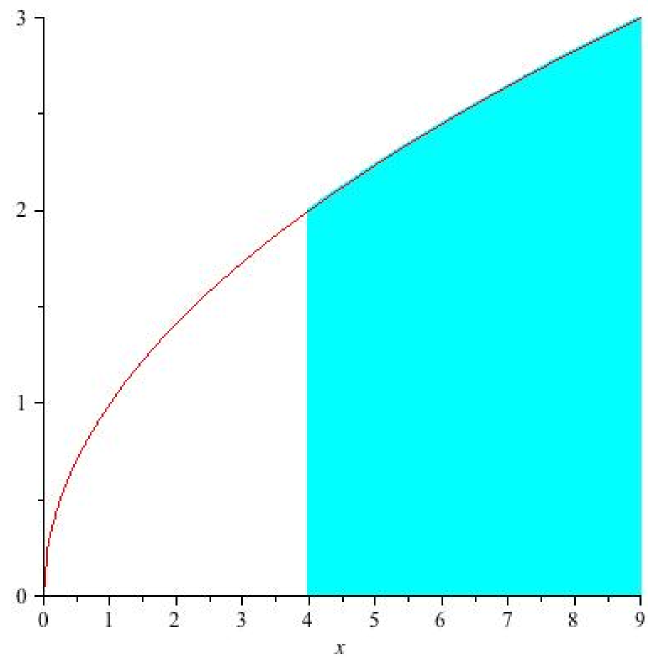
Solution: The trapezoid has base $1/2$ and heights e and $e^{3/2}/(9/4)$. The area is the base times the average of the heights, which is $(1/2)\times(e+(4/9)e^{3/2})/2 = e/4+e^{3/2}/9 = e(1/4 + e^{1/2}/9)$.

FYI, it's pretty easy to see that the trapezoidal approximation is between the other two, because $1/4 + \sqrt{e}/9$ is pretty clearly between $3/8$ and $1/2$. Therefore, one might tend to trust this one the most. On the other hand, it is hardest to compute because of the $\sqrt{e}$.

11. Sketch the region and evaluate the integral.

$$\int_4^9 \int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy\, dx$$

Solution:



The indefinite integral of $e^{y/\sqrt{x}}\, dy$ may be evaluated by the substitution $u = y/\sqrt{x}$; it is equal to $\sqrt{x}e^{y/\sqrt{x}}$. Therefore the inner integral comes out to

$$\int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy = \sqrt{x}e^{y/\sqrt{x}}\Big|_{y=0}^{\sqrt{x}} = \sqrt{x}(e-1)\,.$$

The outer integral then evaluates to

$$\int_4^9 \sqrt{x}(e-1)\, dx = (e-1)\,\frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{3}(27-8)(e-1) = \frac{38}{3}(e-1).$$

12. Five million barrels of oil are spilled in the Gulf of Mexico. Natural forces break the oil down continuously at a rate of 8% of the present amount per year. Unfortunately, the broken rig was not properly capped and continues to leak $100,000$ barrels per year.

(a) Write an initial value problem for this.

Solution: Let $A(t)$ be the amount of oil in the Gulf at time t years fromn the time of the spill. The initial condition is $A(0) = 5 \times 10^6$. The evolution equation is

$$\frac{dA}{dt} = -0.08A(t) + 10^5 \,.$$

(b) Solve the initial value problem.

Solution: Moving the $0.08A(t)$ to the other side, we see that this is a first order linear equation with $P(t) = 0.08$, and $Q(t) = 10^5$ both constant. The integrating factor is $e^{\int P(t)\,dt} = e^{0.08t}$. Multiplying by this and integrating leads to

$$e^{0.08t}A(t) = \int 10^5 e^{0.08t}\,dt = C + \frac{10^5}{0.08}e^{0.08t} = C + 1.25 \times 10^6 e^{0.08t} \,.$$

Multiplying by $e^{-0.08t}$ then gives

$$A(t) = Ce^{-0.08t} + 1.25 \times 10^6 \,.$$

Use the initial condition to see that $5 \times 10^6 = C + 1.25 \times 10^6$, hence $C = 3.75 \times 10^6$. The solution is therefore

$$A(t) = 3.75 \times 10^6\, e^{-0.08t} + 1.25 \times 10^6 \,.$$

(c) How much oil is present in the gulf after ten years? Please give an exact answer.

Solution: Plugging in $t = 10$ and writing the answer in millions of barrels gives $A(10) = 3.75 \times e^{-0.8} + 1.25$ million barrels.

(d) Give a numerical approximation to this exact value.

Solution: $e^{-0.8} \approx 10^{-0.8/2.3} \approx 10^{-0.35} = 10^{0.65-1}$ which is between $10^{0.6}/10$ and $10^{0.7}/10$ which are respectively roughly 0.4 and 0.5. Using 0.45 for $e^{-0.8}$ gives, in millions of barrels, $1.25 + 0.45 \times 3.75 = 1.6875 + 1.25 = 2.9375$. So after ten years, it's a little less than three million barrels.

FYI, a more precise answer is $A(10) = 2.93498\ldots$ million barrels. Looks like the 2.9375 was closer than one might have thought! Also, FYI, as $t \to \infty$, the spilled oil approaches a steady state of 1.25 million barrels, which is the dynamic equilibrium where the new spillage balances the biodegrading.

13. Let $f(x) = \sqrt[3]{1+x}$.

(a) Compute the quadratic MacLaurin polynomial $P_2(x)$.

Solution: $f'(x) = (1/3)(1+x)^{-2/3}$ and $f''(x) = (-2/9)(1+x)^{-5/3}$. Evaluating at $x = 0$ gives $1/3$ and $-2/9$ respectively. Also $f(0) = 1$. Therefore,

$$P_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2\,.$$

(b) Evaluate this at $x = 1/2$, giving both an exact expression and a numerical estimate.

Solution: $P_2\left(\frac{1}{2}\right) = 1 + \frac{1}{3}\cdot\frac{1}{2} - \frac{1}{9}\cdot\frac{1}{4} = 1\,\frac{5}{36}$. Long division gives the repeating decimal $1.138888\ldots$. A good approximation is 1.14.

(c) Use Taylor's Remainder Theorem to give bounds on the difference between $f(1/2)$ and $P_2(1/2)$.

Solution: The remainder is $f'''(u)\frac{(1/2)^3}{3!} = \frac{1}{48}f'''(u)$ for some u between 0 and $1/2$. The third derivative of f is $(10/27)(1+x)^{-8/3}$ which is $10/27$ at $x = 0$ and less but still positive at $x = 1/2$. Therefore, reasonable upper and lower bounds on the difference are

$$0 \le f(1/2) - P_2(1/2) \le \frac{10}{27}\cdot\frac{1}{48} = \frac{10}{1296}\,.$$

In other words, the true value is a little higher than $P_2(1/2)$ but by less than 0.01.

14. Suppose that $f(x, y)$ is a function and that x and y depend on parameters s and t by the formulas $x = \ln(1 - s + t^2)$ and $y = \sqrt{t - 4}$. Which of the following expressions correctly describes the rate of change of f with respect to t when (s, t) starts at the value $(11, 5)$ and then t is varied while s is held constant? You need only circle the correct number from (i) to (v).

(i) $\frac{1}{15}\frac{\partial f}{\partial x}(15, 1) + \frac{\partial f}{\partial y}(15, 1)$

(ii) $\frac{1}{15}\frac{\partial f}{\partial x}(11, 5) + \frac{\partial f}{\partial y}(11, 5)$

(iii) $\frac{1}{15}\frac{\partial f}{\partial x}(\ln 15, 1) + \frac{\partial f}{\partial y}(\ln 15, 1)$

(iv) $\frac{2}{3}\frac{\partial f}{\partial x}(15, 1) + \frac{1}{2}\frac{\partial f}{\partial y}(15, 1)$

(v) $\frac{2}{3}\frac{\partial f}{\partial x}(11, 5) + \frac{1}{2}\frac{\partial f}{\partial y}(11, 5)$

(vi) $\frac{2}{3}\frac{\partial f}{\partial x}(\ln 15, 1) + \frac{1}{2}\frac{\partial f}{\partial y}(\ln 15, 1)$

Solution: When $(s, t) = (11, 5)$, the pair $(1, y)$ is equal to $(\ln 15, 1)$. The partial derivatives x_t and y_t are respectively $(2t)/(1 - s + t^2)$ and $1/(2\sqrt{t - 4})$; evaluating these at $(11, 5)$ gives $10/15 = 2/3$ and $1/2$ respectively. The formula

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}(x, y)\frac{\partial x}{\partial t}(s, t) + \frac{\partial f}{\partial y}(x, y)\frac{\partial y}{\partial t}(s, t)$$

then becomes

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}(\ln 15, 1) \cdot \frac{2}{3} + \frac{\partial f}{\partial y}(\ln 15, 1) \cdot \frac{1}{2}\,.$$

The correct answer is therefore (vi).

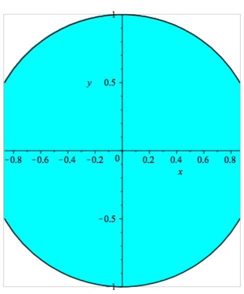
15. The region R inside the unit circle and above the line $y = -1/2$ can be described by which of these? Circle all that apply.

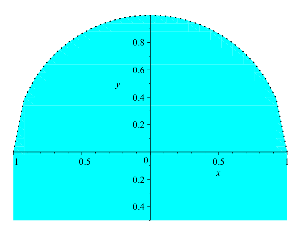
(i) $\left\{(x,y) : -\frac{1}{2} \le y \le 1 \text{ and } -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}\right\}$

(ii) $\left\{(x,y) : -\frac{\sqrt{3}}{2} \le x \le \frac{\sqrt{3}}{2} \text{ and } -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\right\}$

(iii) $\left\{(x,y) : -1 \le x \le 1 \text{ and } -\frac{\sqrt{3}}{2} \le y \le \sqrt{1-x^2}\right\}$

Solution: The correct answer is (i) only. In horizontal strips the least and greatest heights are respectively $-1/2$ and 1. Each horizontal strip runs from $x = -\sqrt{1-y^2}$ to $\sqrt{1-y^2}$. Therefore (i) is correct. In vertical strips there is a region in the middle where y goes from $-1/2$ to $\sqrt{1-x^2}$ but when x is between -1 and $-\sqrt{3}/2$ or between $\sqrt{3}/2$ and 1, then y goes from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$, not from $-1/2$ to $\sqrt{1-x^2}$. Therefore, neither (ii) nor (iii) is correct; the two figures show what is described by (ii) and (iii) respectively.





Logarithm Cheat Sheet

These values are accurate to within 1%:

$$
\begin{aligned}
e &\approx 2.7 \\
\ln(2) &\approx 0.7 \\
\ln(10) &\approx 2.3 \\
\log_{10}(2) &\approx 0.3 \\
\log_{10}(3) &\approx 0.48
\end{aligned}
$$

Some other useful quantities to with 1%:

$$
\begin{aligned}
\pi &\approx \frac{22}{7} \\
\sqrt{10} &\approx \pi \\
\sqrt{2} &\approx 1.4 \\
\sqrt{1/2} &\approx 0.7
\end{aligned}
$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k)$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \frac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a >$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x >$