Name: ______________________________________

Final Exam for Math 110, Spring 2015

May 11, 2015

Problem	Points	Score
1	12	
2	12	
3	12	
4	6	
5	6	
6	8	
7	12	
8	20	
9	12	
10	12	
11	12	
12	12	
13	12	
14	6	
15	6	
Total	160	

- You have two hours for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. A cheat sheet of any length is allowed, provided it is freshly handwritten by you.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. Let $f(x, y) = x^2y - y^2$.

   (a) Compute ∇f.

   (b) From the point $(1, 1)$, which direction should you move in order to increase f the fastest? Give a unit vector in this direction.

   (c) How fast does f increase per unit movement in this direction?

   (d) At what rate does f increase per unit moved in a direction that differs from the previous direction by $45°$?

2. Find the location and the value of the maximum of the function

$$f(x, y) = 3x^2 + y^2$$

over the triangle $x \geq 0, y \geq 0, x + 2y \leq 13$. You need to justify why this, and not some other point, is the maximum.

3. Survey data shows that satisfaction for purchasers of jetskis can be modeled by $u(x, y) = x - y^2/x$ where x is the maximum speed attainable in MPH and y is the number of accidents and breakdowns reported per year per 1,000 owners.

   (a) What is the gradient of the utility function $u(x, y)$ at the point $(60, 90)$?

   (b) Compute the slope of the level curve through of $u(x, y)$ through the point $(60, 90)$ and give a sketch showing the point $(60, 90)$, the gradient there, and the level curve as it passes through $(60, 90)$.

   (c) How many more MPH must next year's model go for each accident or breakdown per thousand increase over this year's model if it is to maintain the same satisfaction and this year's model goes 60 MPH with 90 reported accidents or breakdowns per 1,000 owners?

4. Graph the function $y = \dfrac{x}{1+x^2}$. Please give a scale, choosing one that allows you to show important features. Please any maxima, minima, asymptotes or discontinuities.

5. Circle the best approximation to 1.03^{90}.

(i) 2.7

(ii) 3.7

(iii) 15

(iv) 27

(v) 37

(vi) $1,000,000$

(vii) $1,000,000,000,000,000,000,000,000,000,000$

6. True or false?

(a) $\ln x \ll x^{1/8}$ as $x \to \infty$

(b) $x^{1/2} = o(x^{1/3})$ as $x \to 0^+$

(c) $\frac{x^2}{\sqrt{1+x^3}} \ll x$ as $x \to \infty$

(d) $\sqrt{1+x^4} \sim x^2$ as $x \to \infty$

7. Write the sum $\frac{1}{6} - \frac{1}{12} + \frac{1}{24} - \frac{1}{48} + \cdots$ (don't overlook the negative signs) in Sigma notation and evaluate it.

8. Uncle Sam has a debt of 18 trillion dollars (as of January 1, 2015). Senator Paul decides he needs to pay it off, and proposes an installment scheme, under which Uncle Sam will make payments of one trillion dollars on December 31, 2015 and every December 31 thereafter. Unfortunately, the debt increases by 5% during the course of every year.

   (a) Write expressions for the amount owed by Uncle Sam on January 1 of 2015, 2016 and 2017.

   (b) Write an expression, which will involve the Sigma notation, for how much Uncle Sam owes on January 1 of year n, counting 2015 as year 0.

(c) Evaluate the sum to get an algebraic expression.

(d) Solve for the number of years, n, in which the debt will be paid off. Please leave this as an exact expression (it is OK if it leads to a value which is not a whole number).

(e) Estimate the numerical value of n to the nearest whole number.

9. (a) Write the improper integral $\int_0^\infty xe^{-x}\,dx$ as a limit and determine whether it converges.

(b) For what value of C is Cxe^{-x} a probability density on $[0,\infty)$?

(c) What is the mean of this probability density?

10. Let $f(x) = \int_1^x \frac{e^x}{x^2}\,dx$.

(a) Compute the linear and quadratic Taylor polynomials for f about the point $x = 1$.

$$\begin{aligned} L(x) &= \\ P_2(x) &= \end{aligned}$$

(b) Use these to give two estimates of $f(3/2)$. Leave as exact expresssions; do not evaluate numerically.

(c) Now estimate the same integral by a trapezoidal approximation with just one trapezoid. Again, leave as an exact expression.

11. Sketch the region and evaluate the integral.

$$\int_{4}^{9} \int_{0}^{\sqrt{x}} e^{y/\sqrt{x}}\, dy\, dx$$

12. Five million barrels of oil are spilled in the Gulf of Mexico. Natural forces break the oil down continuously at a rate of 8% of the present amount per year. Unfortunately, the broken rig was not properly capped and continues to leak $100,000$ barrels per year.

    (a) Write an initial value problem for this.

    (b) Solve the initial value problem.

(c) How much oil is present in the gulf after ten years? Please give an exact answer.

(d) Give a numerical approximation to this exact value.

13. Let $f(x) = \sqrt[3]{1+x}$.

(a) Compute the quadratic MacLaurin polynomial $P_2(x)$.

(b) Evaluate this at $x = 1/2$, giving both an exact expression and a numerical estimate.

(c) Use Taylor's Remainder Theorem to give bounds on the difference between $f(1/2)$ and $P_2(1/2)$.

14. Suppose that $f(x, y)$ is a function and that x and y depend on parameters s and t by the formulas $x = \ln(1 - s + t^2)$ and $y = \sqrt{t - 4}$. Which of the following expressions correctly describes the rate of change of f with respect to t when (s, t) starts at the value $(11, 5)$ and then t is varied while s is held constant? You need only circle the correct number from (i) to (v).

(i) $\frac{1}{15}\frac{\partial f}{\partial x}(15, 1) + \frac{\partial f}{\partial y}(15, 1)$

(ii) $\frac{1}{15}\frac{\partial f}{\partial x}(11, 5) + \frac{\partial f}{\partial y}(11, 5)$

(iii) $\frac{1}{15}\frac{\partial f}{\partial x}(\ln 15, 1) + \frac{\partial f}{\partial y}(\ln 15, 1)$

(iv) $\frac{2}{3}\frac{\partial f}{\partial x}(15, 1) + \frac{1}{2}\frac{\partial f}{\partial y}(15, 1)$

(v) $\frac{2}{3}\frac{\partial f}{\partial x}(11, 5) + \frac{1}{2}\frac{\partial f}{\partial y}(11, 5)$

(vi) $\frac{2}{3}\frac{\partial f}{\partial x}(\ln 15, 1) + \frac{1}{2}\frac{\partial f}{\partial y}(\ln 15, 1)$

15. The region R inside the unit circle and above the line $y = -1/2$ can be described by which of these? Circle all that apply.

(i) $\left\{(x, y) : -\frac{1}{2} \leq y \leq 1 \text{ and } -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}\right\}$

(ii) $\left\{(x, y) : -\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2} \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\right\}$

(iii) $\left\{(x, y) : -1 \leq x \leq 1 \text{ and } -\frac{1}{2} \leq y \leq \sqrt{1-x^2}\right\}$

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$
\begin{aligned}
e &\approx 2.7 \\
\ln(2) &\approx 0.7 \\
\ln(10) &\approx 2.3 \\
\log_{10}(2) &\approx 0.3 \\
\log_{10}(3) &\approx 0.48
\end{aligned}
$$

Some other useful quantities to with 1%:

$$
\begin{aligned}
\pi &\approx \frac{22}{7} \\
\sqrt{10} &\approx \pi \\
\sqrt{2} &\approx 1.4 \\
\sqrt{1/2} &\approx 0.7
\end{aligned}
$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$

3. $\int \frac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a >$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x >$