# 8.8 Probability

The outcome of some events, such as a heavy rock falling from a great height, can be modeled so that we can predict with high accuracy what will happen. On the other hand, many events have more than one possible outcome and which one of them will occur is uncertain. If we toss a coin, a head or a tail will result with each outcome being equally likely, but we do not know in advance which one it will be. If we randomly select and then weigh a person from a large population, there are many possible weights the person might have, and it is not certain whether the weight will be between 180 and 190 lbs. We are told it is highly likely, but not known for sure, that an earthquake of magnitude 6.0 or greater on the Richter scale will occur near a major population area in California within the next one hundred years. Events having more than one possible outcome are *probabilistic* in nature, and when modeling them we assign a *probability* to the likelihood that a particular outcome may occur. In this section we show how calculus plays a central role in making predictions with probabilistic models.

## Random Variables

We begin our discussion with some familiar examples of uncertain events for which the collection of all possible outcomes is finite.

### EXAMPLE 1

**(a)** If we toss a coin once, there are two possible outcomes {H, T}, where H represents the coin landing head face up and T a tail landing face up. If we toss a coin three times, there are eight possible outcomes, taking into account the order in which a head or tail occurs. The set of outcomes is {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

**(b)** If we roll a six-sided die once, the set of possible outcomes is {1, 2, 3, 4, 5, 6} representing the six faces of the die.

**(c)** If we select at random two cards from a 52-card deck, there are 52 possible outcomes for the first card drawn and then 51 possibilities for the second card. Since the order of the cards does not matter, there are $(52 \cdot 51)/2 = 1{,}326$ possible outcomes altogether. ■

It is customary to refer to the set of all possible outcomes as the *sample space* for an event. With an uncertain event we are usually interested in which outcomes, if any, are more likely to occur than others, and to how large an extent. In tossing a coin three times, is it more likely that two heads or that one head will result? To answer such questions, we need a way to quantify the outcomes.

**DEFINITION** A **random variable** is a function $X$ that assigns a numerical value to each outcome in a sample space.

Random variables that have only finitely many values are called **discrete** random variables. A **continuous random variable** can take on values in an entire interval, and it is associated with a *distribution function*, which we explain later.

### EXAMPLE 2

**(a)** Suppose we toss a coin three times giving the possible outcomes {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}. Define the random variable $X$ to be the number of heads that appear. So $X(\text{HHT}) = 2$, $X(\text{THT}) = 1$, and so forth. Since $X$ can only assume the values 0, 1, 2, or 3, it is a discrete random variable.

**(b)** We spin an arrow anchored by a pin located at the origin. The arrow can wind up pointing in any possible direction and we define the random variable $X$ as the radian angle the arrow makes with the positive $x$-axis, measured counterclockwise. In this case, $X$ is a continuous random variable which can take on any value in the interval $[0, 2\pi)$.

**(c)** The weight of a randomly selected person in a given population is a continuous random variable $W$. The cholesterol level of a randomly chosen person, and the waiting time for service of a person in a queue at a bank, are also continuous random variables.

**(d)** The scores on the national ACT Examination for college admissions in a particular year are described by a discrete random variable $S$ taking on integer values between 1 and 36. If the number of outcomes is large, or for reasons involving statistical analysis, discrete random variables such as test scores are often modeled as continuous random variables (Example 13).

**(e)** We roll a pair of dice and define the random variable $X$ to be the sum of the numbers on the top faces. This sum can only assume the integer values from 2 through 12, so $X$ is a discrete random variable.

**(f)** A tire company produces tires for mid-sized sedans. The tires are guaranteed to last for 30,000 miles, but some will fail sooner and some will last many more miles beyond 30,000. The lifetime in miles of a tire is described by a continuous random variable $L$. ■

## Probability Distributions

A *probability distribution* describes the probabilistic behavior of a random variable. Our chief interest is in probability distributions associated with continuous random variables, but to gain some perspective we first consider a distribution for a discrete random variable.

Suppose we toss a coin three times, with each side H or T equally likely to occur on a given toss. We define the discrete random variable $X$ that assigns the number of heads appearing in each outcome, giving

$$\begin{array}{ccccccccc} & \{\text{HHH}, & \text{HHT}, & \text{HTH}, & \text{HTT}, & \text{THH}, & \text{THT}, & \text{TTH}, & \text{TTT}\} \\ X & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \end{array}$$

Next we count the *frequency* or number of times a specific value of $X$ occurs. Because each of the eight outcomes is equally likely to occur, we can calculate the probability of the random variable $X$ by dividing the frequency of each value by the total number of outcomes. We summarize our results as follows:

| **Value of $X$** | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| **Frequency** | 1 | 3 | 3 | 1 |
| **$P(X)$** | 1/8 | 3/8 | 3/8 | 1/8 |

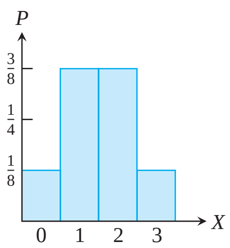


**FIGURE 8.21** Probability bar graph for the random variable $X$ when tossing a fair coin three times.

We display this information in a probability bar graph of the discrete random variable $X$, as shown in Figure 8.21. The values of $X$ are portrayed by intervals of length 1 on the $x$-axis so the area of each bar in the graph is the probability of the corresponding outcome. For instance, the probability that exactly two heads occurs in the three tosses of the coin is the area of the bar associated with the value $X = 2$, which is 3/8. Similarly, the probability that two or more heads occurs is the sum of areas of the bars associated with the values $X = 2$ and $X = 3$, or 4/8. The probability that either zero or three heads occurs is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$, and so forth. Note that the total area of all the bars in the graph is 1, which is the sum of all the probabilities for $X$.

With a continuous random variable, even when the outcomes are equally likely, we cannot simply count the number of outcomes in the sample space or the frequencies of outcomes that lead to a specific value of $X$. In fact, the probability that $X$ takes on any particular one of its values is zero. What *is* meaningful to ask is how probable it is that the random variable takes on a value within some specified *interval* of values.

We capture the information we need about the probabilities of $X$ in a function whose graph behaves much like the bar graph in Figure 8.21. That is, we take a nonnegative function $f$ defined over the range of the random variable with the property that the total area beneath the graph of $f$ is 1. The probability that a value of the random variable $X$ lies within some specified interval $[c, d]$ is then the area under the graph of $f$ over that interval. The following definition assumes the range of the continuous random variable $X$ is any real value, but the definition is general enough to account for random variables having a range of finite length.

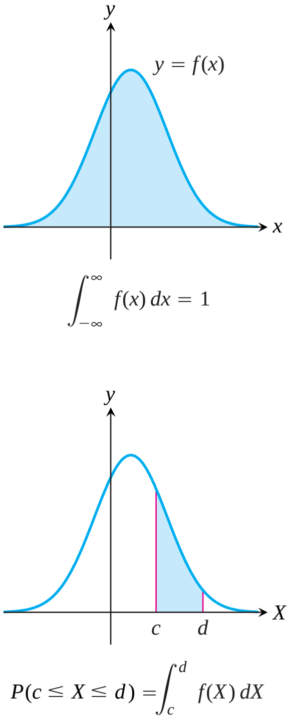


**FIGURE 8.22** A probability density function for the continuous random variable $X$.

**DEFINITIONS** A **probability density function** for a continuous random variable is a function $f$ defined over $(-\infty, \infty)$ and having the following properties:

**1.** $f$ is continuous, except possibly at a finite number of points.

**2.** $f$ is nonnegative, so $f \geq 0$.

**3.** $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

If $X$ is a continuous random variable with probability density function $f$, the **probability** that $X$ assumes a value in the interval between $X = c$ and $X = d$ is the area integral

$$P(c \leq X \leq d) = \int_c^d f(X)\,dX.$$

We note that the probability a continuous random variable $X$ assumes a particular real value $c$ is $P(X = c) = \int_c^c f(X)\,dX = 0$, consistent with our previous assertion. Since the area under the graph of $f$ over the interval $[c, d]$ is only a portion of the total area beneath the graph, the probability $P(c \leq X \leq d)$ is always a number between zero and one. Figure 8.22 illustrates a probability density function.

A probability density function for a random variable $X$ resembles the density function for a wire of varying density. To obtain the mass of a segment of the wire, we integrate the density of the wire over an interval. To obtain the probability that a random variable has values in a particular interval, we integrate the probability density function over that interval.

**EXAMPLE 3** Let $f(x) = 2e^{-2x}$ if $0 \leq x < \infty$ and $f(x) = 0$ for all negative values of $x$.

**(a)** Verify that $f$ is a probability density function.

**(b)** The time $T$ in hours until a car passes a spot on a remote road is described by the probability density function $f$. Find the probability $P(T \leq 1)$ that a hitchhiker at that spot will see a car within one hour.

**(c)** Find the probability $P(T = 1)$ that a car passes by the spot after precisely one hour.

**Solution**

**(a)** The function $f$ is continuous except at $x = 0$, and is everywhere nonnegative. Moreover,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx = \lim_{b\to\infty} \int_0^b 2e^{-2x}\,dx = \lim_{b\to\infty} \left(1 - e^{-2b}\right) = 1.$$

So all of the conditions are satisfied and we have shown that $f$ is a probability density function.

**(b)** The probability that a car comes after a time lapse between zero and one hour is given by integrating the probability density function over the interval [0, 1]. So

$$P(T \leq 1) = \int_0^1 2e^{-2T}\,dT = -e^{-2T}\Big]_0^1 = 1 - e^{-2} \approx 0.865.$$

This result can be interpreted to mean that if 100 people were to hitchhike at that spot, about 87 of them can expect to see a car within one hour.

**(c)** This probability is the integral $\int_1^1 f(T)\,dT$ which equals zero. We interpret this to mean that a sufficiently accurate measurement of the time until a car comes by the spot would have no possibility of being precisely equal to one hour. It might be very close, perhaps, but it would not be exactly one hour. ■

We can extend the definition to finite intervals. If $f$ is a nonnegative function with at most finitely many discontinuities over the interval $[a, b]$, and its extension $F$ to $(-\infty, \infty)$, obtained by defining $F$ to be 0 outside of $[a, b]$, satisfies the definition for a probability density function, then $f$ is a **probability density function for [*a*, *b*]**. This means that $\int_a^b f(x)\,dx = 1$. Similar definitions can be made for the intervals $(a, b)$, $(a, b]$, and $[a, b)$.

**EXAMPLE 4** Show that $f(x) = \frac{4}{27}x^2(3 - x)$ is a probability density function over the interval [0, 3].

**Solution** The function $f$ is continuous and nonnegative over [0, 3]. Also,

$$\int_0^3 \frac{4}{27}x^2(3 - x)\,dx = \frac{4}{27}\left[x^3 - \frac{1}{4}x^4\right]_0^3 = \frac{4}{27}\left(27 - \frac{81}{4}\right) = 1.$$

We conclude that $f$ is a probability density function over [0, 3]. ■

## Exponentially Decreasing Distributions

The distribution in Example 3 is called an *exponentially decreasing probability density function*. These probability density functions always take on the form

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$$

(Exercise 23). Exponential density functions can provide models for describing random variables such as the lifetimes of light bulbs, radioactive particles, tooth crowns, and many kinds of electronic components. They also model the amount of time until some specific event occurs, such as the time until a pollinator arrives at a flower, the arrival times of a bus at a stop, the time between individuals joining a queue, the waiting time between phone calls at a help desk, and even the lengths of the phone calls themselves. A graph of an exponential density function is shown in Figure 8.23.

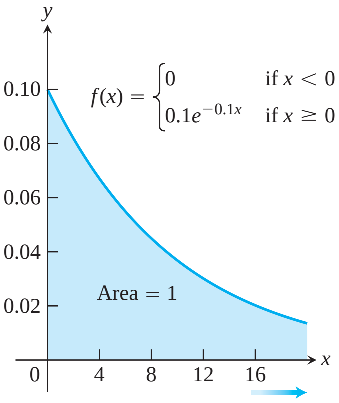


**FIGURE 8.23** An exponentially decreasing probability density function.

Random variables with exponential distributions are *memoryless*. If we think of $X$ as describing the lifetime of some object, then the probability that the object survives for at least $s + t$ hours, given that it has survived $t$ hours, is the same as the initial probability that it survives for at least $s$ hours. For instance, the current age $t$ of a radioactive particle does not change the probability that it will survive for at least another time period of length $s$. Sometimes the exponential distribution is used as a model when the memoryless principle is violated, because it provides reasonable approximations that are good enough for their intended use. For instance, this might be the case when predicting the lifetime of an artificial hip replacement or heart valve for a particular individual. Here is an application illustrating the exponential distribution.

**EXAMPLE 5** An electronics company models the lifetime $T$ in years of a chip they manufacture with the exponential density function

$$f(T) = \begin{cases} 0 & \text{if } T < 0 \\ 0.1e^{-0.1T} & \text{if } T \geq 0 \end{cases}$$

Using this model,

**(a)** Find the probability $P(T > 2)$ that a chip will last for more than two years.

**(b)** Find the probability $P(4 \leq T \leq 5)$ that a chip will fail in the fifth year.

**(c)** If 1000 chips are shipped to a customer, how many can be expected to fail within three years?

**Solution**

**(a)** The probability that a chip lasts at least two years is

$$P(T > 2) = \int_2^\infty 0.1e^{-0.1T}\,dT = \lim_{b\to\infty}\int_2^b 0.1e^{-0.1T}\,dT$$
$$= \lim_{b\to\infty}\left[e^{-0.2} - e^{-0.1b}\right] = e^{-0.2} \approx 0.819.$$

That is, about 82% of the chips last more than two years.

**(b)** The probability is

$$P(4 \leq T \leq 5) = \int_4^5 0.1e^{-0.1T}\,dT = -e^{-0.1T}\Big]_4^5 = e^{-0.4} - e^{-0.5} \approx 0.064$$

which means that about 6% of the chips fail during the fifth year.

**(c)** We want the probability

$$P(0 \leq T \leq 3) = \int_0^3 0.1e^{-0.1T}\,dT = -e^{-0.1T}\Big]_0^3 = 1 - e^{-0.3} \approx 0.259.$$

We can expect that about 259 of the 1000 chips will fail within three years. ■

## Expected Values, Means, and Medians

Suppose the weight in lbs of a steer raised on a cattle ranch is a continuous random variable $W$ with probability density function $f(W)$ and that the rancher can sell a steer for $g(W)$ dollars. How much can the rancher expect to earn for a randomly chosen steer on the ranch?

To answer this question, we consider a small interval $[W_i, W_{i+1}]$ of width $\Delta W_i$ and note that the probability a steer has weight in this interval is

$$\int_{W_i}^{W_{i+1}} f(W)\,dW \approx f(W_i)\,\Delta W_i\,.$$

The earning on a steer in this interval is approximately $g(W_i)$. The Riemann sum

$$\sum g(W_i) f(W_i)\,\Delta W_i$$

then approximates the amount the rancher would receive for a steer. We assume that steers have a maximum weight, so $f$ is zero outside some finite interval $[0, b]$. Then taking the limit of the Riemann sum as the width of each interval approaches zero gives the integral

$$\int_{-\infty}^{\infty} g(W) f(W)\,dW.$$

This integral estimates how much the rancher can expect to earn for a typical steer on the ranch and is the *expected value of the function g*.

The expected values of certain functions of a random variable $X$ have particular importance in probability and statistics. One of the most important of these functions is the expected value of the function $g(X) = X$.

**DEFINITION** The **expected value** or **mean** of a continuous random variable $X$ with probability density function $f$ is the number

$$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} Xf(X)\,dX.$$

The expected value E($X$) can be thought of as a weighted average of the random variable $X$, where each value of $X$ is weighted by $f(X)$. The mean can also be interpreted as the long-run average value of the random variable $X$, and it is one measure of the centrality of the random variable $X$.

**EXAMPLE 6** Find the mean of the random variable $X$ with exponential probability density function

$$f(X) = \begin{cases} 0 & \text{if } X < 0 \\ ce^{-cX} & \text{if } X \geq 0 \end{cases}$$

**Solution** From the definition we have

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} Xf(X)\,dX = \int_{0}^{\infty} Xce^{-cX}\,dX \\ &= \lim_{b\to\infty} \int_{0}^{b} Xce^{-cX}\,dX = \lim_{b\to\infty} \left( -Xe^{-cX} \Big]_0^b + \int_0^b e^{-cX}\,dX \right) \\ &= \lim_{b\to\infty} \left( -be^{-cb} - \frac{1}{c}e^{-cb} + \frac{1}{c} \right) = \frac{1}{c}. \qquad \text{l'Hôpital's rule on 1st term} \end{aligned}$$

Therefore, the mean is $\mu = 1/c$. ■

From the result in Example 6, knowing the mean or expected value $\mu$ of a random variable $X$ having an exponential density function allows us to write its entire formula.

**Exponential density function for a random variable $X$ with mean $\mu$**

$$f(X) = \begin{cases} 0 & \text{if } X < 0 \\ \mu^{-1}e^{-X/\mu} & \text{if } X \geq 0 \end{cases}$$

**EXAMPLE 7** Suppose the time $T$ before a chip fails in Example 5 is modeled instead by the exponential density function with a mean of eight years. Find the probability that a chip will fail within five years.

**Solution** The exponential density function with mean $\mu = 8$ is

$$f(T) = \begin{cases} 0 & \text{if } T < 0 \\ \frac{1}{8}e^{-T/8} & \text{if } T \geq 0 \end{cases}$$

Then the probability a chip will fail within five years is the definite integral

$$P(0 \leq T \leq 5) = \int_0^5 0.125e^{-0.125T}\,dT = -e^{-0.125T}\Big]_0^5 = 1 - e^{-0.625} \approx 0.465$$

so about 47% of the chips can be expected to fail within five years. ■

**EXAMPLE 8** Find the expected value for the random variable $X$ with probability density function given by Example 4.

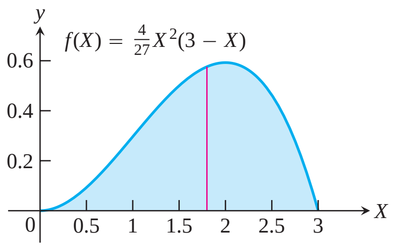


**FIGURE 8.24** The expected value of a random variable with this probability density function is $\mu = 1.8$ (Example 8).

**Solution** The expected value is

$$\mu = \mathrm{E}(X) = \int_0^3 \frac{4}{27}X^3(3 - X)\,dX = \frac{4}{27}\left[\frac{3}{4}X^4 - \frac{1}{5}X^5\right]_0^3$$

$$= \frac{4}{27}\left(\frac{243}{4} - \frac{243}{5}\right) = 1.8$$

From Figure 8.24, you can see that this expected value is reasonable because the region beneath the probability density function appears to be balanced about the vertical line $X = 1.8$. That is, the horizontal coordinate of the centroid of a plate described by the region is $\overline{X} = 1.8$. ■

There are other ways to measure the centrality of a random variable with a given probability density function.

> **DEFINITION** The **median** of a continuous random variable $X$ with probability density function $f$ is the number $m$ for which
>
> $$\int_{-\infty}^{m} f(X)\,dX = \frac{1}{2} \quad \text{and} \quad \int_{m}^{\infty} f(X)\,dX = \frac{1}{2}.$$

The definition of the median means that there is an equal likelihood that the random variable $X$ will be smaller than $m$ or larger than $m$.

**EXAMPLE 9** Find the median of a random variable $X$ with exponential probability density function

$$f(X) = \begin{cases} 0 & \text{if } X < 0 \\ ce^{-cX} & \text{if } X \geq 0 \end{cases}$$

**Solution** The median $m$ must satisfy

$$\frac{1}{2} = \int_0^m ce^{-cX}\,dX = -e^{-cX}\Big]_0^m = 1 - e^{-cm}.$$

It follows that

$$e^{-cm} = \frac{1}{2} \quad \text{or} \quad m = \frac{1}{c}\ln 2.$$

Also,

$$\frac{1}{2} = \lim_{b\to\infty}\int_m^b ce^{-cX}\,dX = \lim_{b\to\infty}\left(-e^{-cX}\Big]_m^b\right) = \lim_{b\to\infty}\left(e^{-cm} - e^{-cb}\right) = e^{-cm}$$

giving the same value for $m$. Since $1/c$ is the mean $\mu$ of $X$ with an exponential distribution, we conclude that the median is $m = \mu \ln 2$. The mean and median differ because the probability density function is skewed and spreads towards the right. ■

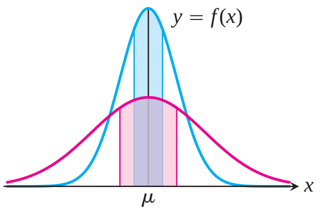


**FIGURE 8.25** Probability density functions with the same mean can have different spreads in relation to the mean. The blue and red regions under the curves have equal area.

## Variance and Standard Deviation

Random variables with exactly the same mean $\mu$ but different distributions can behave very differently (see Figure 8.25). The *variance* of a random variable $X$ measures how spread out the values of $X$ are in relation to the mean, and we measure this dispersion by the expected value of $(X - \mu)^2$. Since the variance measures the expected square of the difference from the mean, we often work instead with its square root.

**DEFINITIONS** The **variance** of a random variable $X$ with probability density function $f$ is the expected value of $(X - \mu)^2$:

$$\operatorname{Var}(X) = \int_{-\infty}^{\infty} (X - \mu)^2 f(X)\, dX$$

The **standard deviation** of $X$ is

$$\sigma_X = \sqrt{\operatorname{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (X - \mu)^2 f(X)\, dX}.$$

**EXAMPLE 10** Find the standard deviation of the random variable $T$ in Example 5, and find the probability that $T$ lies within one standard deviation of the mean.

**Solution** The probability density function is the exponential density function with mean $\mu = 10$ by Example 6. To find the standard deviation we first calculate the variance integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} (T - \mu)^2 f(T)\, dT &= \int_{0}^{\infty} (T - 10)^2 \left(0.1e^{-0.1T}\right) dT \\
&= \lim_{b\to\infty} \int_{0}^{b} (T - 10)^2 \left(0.1e^{-0.1T}\right) dT \\
&= \lim_{b\to\infty} \left[\left(-(T - 10)^2 - 20(T - 10)\right)e^{-0.1T}\right]_0^b \\
&\quad + \lim_{b\to\infty} \int_0^b 20e^{-0.1T}\, dT \qquad \text{Integrating by parts} \\
&= [0 + (-10)^2 + 20(-10)] - 20 \lim_{b\to\infty} \left(10e^{-0.1T}\right]_0^b \\
&= -100 - 200 \lim_{b\to\infty} \left(e^{-0.1b} - 1\right) = 100.
\end{aligned}$$

The standard deviation is the square root of the variance, so $\sigma = 10.0$.

To find the probability that $T$ lies within one standard deviation of the mean, we find the probability $P(\mu - \sigma \le T \le \mu + \sigma)$. For this example we have,

$$P(10 - 10 \le T \le 10 + 10) = \int_0^{20} 0.1e^{-0.1T}\, dT = -e^{-0.1T}\Big]_0^{20} = 1 - e^{-2} \approx 0.865$$

This means that about 87% of the chips will fail within twenty years. ■

## Uniform Distributions

The **uniform distribution** is very simple, but it occurs commonly in applications. The probability density function for this distribution on the interval $[a, b]$ is

$$f(x) = \frac{1}{b - a}, \quad a \le x \le b.$$

If each outcome in the sample space is equally likely to occur, then the random variable $X$ has a uniform distribution. Since $f$ is constant on $[a, b]$, a random variable with a uniform distribution is just as likely to be in one subinterval of a fixed length as in any other of the same length. The probability that $X$ assumes a value in a subinterval of $[a, b]$ is the length of that subinterval divided by $(b - a)$.

**EXAMPLE 11** An anchored arrow is spun around the origin and the random variable $X$ is the radian angle the arrow makes with the positive $x$-axis, measured within the interval $[0, 2\pi)$. Assuming there is equal probability for the arrow pointing in any direction, find the probability density function and the probability that the arrow ends up pointing between North and East.

**Solution** We model the probability density function with the uniform distribution $f(x) = 1/2\pi$, $0 \le x < 2\pi$, and $f(x) = 0$ elsewhere.

The probability that the arrow ends up pointing between North and East is given by

$$P\left(0 \le X \le \frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{1}{2\pi}\, dx = \frac{1}{4}. \quad \blacksquare$$

## Normal Distributions

Numerous applications use the **normal distribution,** which is defined by the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

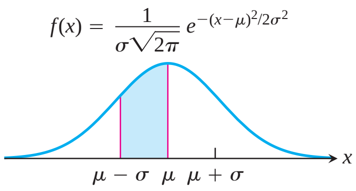


**FIGURE 8.26** The normal probability density function with mean $\mu$ and standard deviation $\sigma$.

It can be shown that the mean of a random variable $X$ with this probability density function is $\mu$ and its standard deviation is $\sigma$. The values of $\mu$ and $\sigma$ are often estimated using large sets of data. The function is graphed in Figure 8.26, and the graph is sometimes called a *bell curve* because of its shape. Since the curve is symmetric about the mean, the median for $X$ is the same as its mean. It is often observed in practice that many random variables have approximately a normal distribution. Some examples illustrating this phenomenon are the height of a man, the annual rainfall in a certain region, an individual's blood pressure, the serum cholesterol level in the blood, the brain weights in a certain population of adults, and the amount of growth in a given period for a population of sunflower seeds.

The normal probability density function does not have an antiderivative expressible in terms of familiar functions. Once $\mu$ and $\sigma$ are fixed, however, an integral involving the normal probability density function can be computed using numerical integration methods. Usually we use the numerical integration capability of a computer or calculator to estimate the values of these integrals. Such computations show that for any normal distribution, we get the following values for the probability that the random variable $X$ lies within $k = 1, 2, 3,$ or $4$ standard deviations of the mean:

$$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68269$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95450$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.99730$$
$$P(\mu - 4\sigma < X < \mu + 4\sigma) \approx 0.99994$$

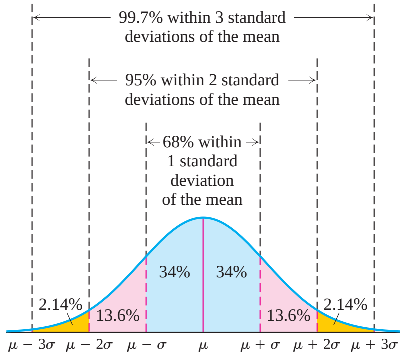


**FIGURE 8.27** Probabilities of the normal distribution within its standard deviation bands.

This means, for instance, that the random variable $X$ will take on a value within two standard deviations of the mean about 95% of the time. About 68% of the time, $X$ will lie within one standard deviation of the mean (see Figure 8.27).

**EXAMPLE 12** An individual's blood pressure is an important indicator of overall health. A medical study of healthy individuals between 14 and 70 years of age modeled their systolic blood pressure using a normal distribution with mean 119.7 mm Hg and standard deviation 10.9 mm Hg.

**(a)** Using this model, what percentage of the population has a systolic blood pressure between 140 and 160 mm Hg, the levels set by the American Heart Association for Stage 1 hypertension?

**(b)** What percentage has a blood pressure between 160 and 180 mm Hg, the levels set by the American Heart Association for Stage 2 hypertension?

**(c)** What percentage has a blood pressure in the normal range of 90–120, as set by the American Heart Association?

**Solution**

**(a)** Since we cannot find an antiderivative, we use a computer to evaluate the probability integral of the normal probability density function with $\mu = 119.7$ and $\sigma = 10.9$:

$$P(140 \leq X \leq 160) = \int_{140}^{160} \frac{1}{10.9\sqrt{2\pi}} e^{-(X-119.7)^2/2(10.9)^2}\, dX \approx 0.03117.$$

This means that about 3% of the population in the studied age range have Stage 1 hypertension.

**(b)** Again we use a computer to calculate the probability that the blood pressure is between 160 and 180 mm Hg:

$$P(160 \leq X \leq 180) = \int_{160}^{180} \frac{1}{10.9\sqrt{2\pi}} e^{-(X-119.7)^2/2(10.9)^2}\, dX \approx 0.00011.$$

We conclude that about 0.011% of the population has Stage 2 hypertension.

**(c)** The probability that the blood pressure falls in the normal range is

$$P(90 \leq X \leq 120) = \int_{90}^{120} \frac{1}{10.9\sqrt{2\pi}} e^{-(X-119.7)^2/2(10.9)^2}\, dX \approx 0.50776.$$

That is, about 51% of the population has a normal systolic blood pressure. ■

Many national tests are standardized using the normal distribution. The following example illustrates modeling the discrete random variable for scores on a test using the normal distribution function for a continuous random variable.

**EXAMPLE 13** The ACT is a standardized test taken by high school students seeking admission to many colleges and universities. The test measures knowledge skills and proficiency in the areas of English, math, and science with scores ranging over the interval [1, 36]. Nearly 1.5 million high school students took the test in 2009, and the composite mean score across the academic areas was $\mu = 21.1$ with standard deviation $\sigma = 5.1$.

**(a)** What percentage of the population had an ACT score between 18 and 24?

**(b)** What is the ranking of a student who scored 27 on the test?

**(c)** What is the minimal integer score a student needed to get in order to be in the top 8% of the scoring population?

**Solution**

**(a)** We use a computer to evaluate the probability integral of the normal probability density function with $\mu = 21.1$ and $\sigma = 5.1$:

$$P(18 \leq X \leq 24) = \int_{18}^{24} \frac{1}{5.1\sqrt{2\pi}} e^{-(X-21.1)^2/2(5.1)^2}\, dX \approx 0.44355.$$

This means that about 44% of the students had an ACT score between 18 and 24.

**(b)** Again we use a computer to calculate the probability of a student getting a score lower than 27 on the test:

$$P(1 \le X < 27) = \int_{1}^{27} \frac{1}{5.1\sqrt{2\pi}} e^{-(X-21.1)^2/2(5.1)^2}\, dX \approx 0.87630.$$

We conclude that about 88% of the students scored below a score of 27, so the student ranked in the top 12% of the population.

**(c)** We look at how many students had a mark higher than 28:

$$P(28 < X \le 36) = \int_{28}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(X-21.1)^2/2(5.1)^2}\, dX \approx 0.0863.$$

Since this number gives more than 8% of the students, we look at the next higher integer score:

$$P(29 < X \le 36) = \int_{29}^{36} \frac{1}{5.1\sqrt{2\pi}} e^{-(X-21.1)^2/2(5.1)^2}\, dX \approx 0.0595.$$

Therefore, 29 is the lowest integer score a student could get in order to score in the top 8% of the population (and actually scoring here in the top 6%). ■

The simplest form for a normal distribution of $X$ occurs when its mean is zero and its standard deviation is one. The *standard normal probability density function f* giving mean $\mu = 0$ and standard deviation $\sigma = 1$ is

$$f(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2}.$$

Note that the substitution $z = (X - \mu)/\sigma$ gives the equivalent integrals

$$\int_{a}^{b} \frac{1}{\sigma\sqrt{2\pi}} e^{-((X-\mu)/\sigma)^2/2}\, dX = \int_{\alpha}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz,$$

where $\alpha = (a - \mu)/\sigma$ and $\beta = (b - \mu)/\sigma$. So we can convert random variable values to the "$z$-values" to standardize a normal distribution, and then use the integral on the right-hand side of the last equation to calculate probabilities for the original random variable normal distribution with mean $\mu$ and standard deviation $\sigma$. In a normal distribution, we know that 95.5% of the population lies within two standard deviations of the mean, so a random variable $X$ converted to a $z$-value has more than a 95% chance of occurring in the interval $[-2, 2]$.

## Exercises 8.8

### Probability Density Functions

In Exercises 1–8, determine which are probability density functions and justify your answer.

1. $f(x) = \frac{1}{18}x$ over $[4, 8]$

2. $f(x) = \frac{1}{2}(2 - x)$ over $[0, 2]$

3. $f(x) = 2^x$ over $\left[0, \frac{\ln(1 + \ln 2)}{\ln 2}\right]$

4. $f(x) = x - 1$ over $\left[0, 1 + \sqrt{3}\right]$

5. $f(x) = \begin{cases} \frac{1}{x^2} & x \ge 1 \\ 0 & x < 1 \end{cases}$

6. $f(x) = \begin{cases} \frac{8}{\pi(4 + x^2)} & x \ge 0 \\ 0 & x < 0 \end{cases}$

7. $f(x) = 2\cos 2x$ over $\left[0, \frac{\pi}{4}\right]$

8. $f(x) = \frac{1}{x}$ over $(0, e)$

**9.** Let $f$ be the probability density function for the random variable $L$ in Example 2f. Explain the meaning of each integral.

**a.** $\int_{25,000}^{32,000} f(L)\, dL$ **b.** $\int_{30,000}^{\infty} f(L)\, dL$

**c.** $\int_{0}^{20,000} f(L)\, dL$ **d.** $\int_{-\infty}^{15,000} f(L)\, dL$

**10.** Let $f(X)$ be the uniform distribution for the random variable $X$ in Example 11. Express the following probabilities as integrals.

**a.** The probability that the arrow points either between South and West or between North and West.

**b.** The probability that the arrow makes an angle of at least 2 radians.

Verify that the functions in Exercises 11–16 are probability density functions for a continuous random variable $X$ over the given interval. Determine the specified probability.

**11.** $f(x) = xe^{-x}$ over $[0, \infty)$, $P(1 \le X \le 3)$

T **12.** $f(x) = \dfrac{\ln x}{x^2}$ over $[1, \infty)$, $P(2 < X < 15)$

**13.** $f(x) = \dfrac{3}{2}x(2 - x)$ over $[0, 1]$, $P(0.5 > X)$

T **14.** $f(x) = \dfrac{\sin^2 \pi x}{\pi x^2}$ over $\left[\dfrac{200}{1059}, \infty\right)$, $P(X < \pi/6)$

**15.** $f(x) = \begin{cases} \dfrac{2}{x^3} & x > 1 \\ 0 & x \le 1 \end{cases}$ over $(-\infty, \infty)$, $P(4 \le X < 9)$

**16.** $f(x) = \sin x$ over $[0, \pi/2]$, $P\left(\dfrac{\pi}{6} < X \le \dfrac{\pi}{4}\right)$

In Exercises 17–20, find the value of the constant $c$ so that the given function is a probability density function for a random variable over the specified interval.

**17.** $f(x) = \dfrac{1}{6}x$ over $[3, c]$ **18.** $f(x) = \dfrac{1}{x}$ over $[c, c + 1]$

**19.** $f(x) = 4e^{-2x}$ over $[0, c]$ **20.** $f(x) = cx\sqrt{25 - x^2}$ over $[0, 5]$

**21.** Let $f(x) = \dfrac{c}{1 + x^2}$. Find the value of $c$ so that $f$ is a probability density function. If $f$ is a probability density function for the random variable $X$, find the probability $P(1 \le X < 2)$.

**22.** Find the value of $c$ so that $f(x) = c\sqrt{x}(1 - x)$ is a probability density function for the random variable $X$ over $[0, 1]$, and find the probability $P(0.25 \le X \le 0.5)$.

**23.** Show that if the exponentially decreasing function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ Ae^{-cx} & \text{if } x \ge 0 \end{cases}$$

is a probability density function, then $A = c$.

**24.** Suppose $f$ is a probability density function for the random variable $X$ with mean $\mu$. Show that its variance satisfies

$$\text{Var}(X) = \int_{-\infty}^{\infty} X^2 f(X)\, dX - \mu^2.$$

Compute the mean and median for a random variable with the probability density functions in Exercises 25–28.

**25.** $f(x) = \dfrac{1}{8}x$ over $[0, 4]$ **26.** $f(x) = \dfrac{1}{9}x^2$ over $[0, 3]$

**27.** $f(x) = \begin{cases} \dfrac{2}{x^3} & x \ge 1 \\ 0 & x < 1 \end{cases}$ **28.** $f(x) = \begin{cases} \dfrac{1}{x} & 1 \le x \le e \\ 0 & \text{Otherwise} \end{cases}$

### Exponential Distributions

**29. Digestion time** The digestion time in hours of a fixed amount of food is exponentially distributed with a mean of 1 hour. What is the probability that the food is digested in less than 30 minutes?

**30. Pollinating flowers** A biologist models the time in minutes until a bee arrives at a flowering plant with an exponential distribution having a mean of 4 minutes. If 1000 flowers are in a field, how many can be expected to be pollinated within 5 minutes?

**31. Lifetime of light bulbs** A manufacturer of light bulbs finds that the mean lifetime of a bulb is 1200 hours. Assume the life of a bulb is exponentially distributed.

**a.** Find the probability that a bulb will last less than its guaranteed lifetime of 1000 hours.

**b.** In a batch of light bulbs, what is the expected time until half the light bulbs in the batch fail?

**32. Lifetime of an electronic component** The life expectancy in years of a component in a microcomputer is exponentially distributed, and 1/3 of the components fail in the first 3 years. The company that manufactures the component offers a 1 year warranty. What is the probability that a component will fail during the warranty period?

**33. Lifetime of an organism** A *hydra* is a small fresh-water animal, and studies have shown that its probability of dying does not increase with the passage of time. The lack of influence of age on mortality rates for this species indicates that an exponential distribution is an appropriate model for the mortality of hydra. A biologist studies a population of 500 hydra and observes that 200 of them die within the first 2 years. How many of the hydra would you expect to die within the first six months?

**34. Car accidents** The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. Based on historical data, an insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of the calendar year. In a group of 100 high-risk drivers, how many do you expect to be involved in an accident during the first 80 days of the calendar year?

**35. Customer service time** The mean waiting time to get served after walking into a bakery is 30 seconds. Assume that an exponential density function describes the waiting times.

**a.** What is the probability a customer waits 15 seconds or less?

**b.** What is the probability a customer waits longer than one minute?

**c.** What is the probability a customer waits exactly 5 minutes?

**d.** If 200 customers come to the bakery in a day, how many are likely to be served within three minutes?

**36. Airport waiting time** According to the U.S. Customs and Border Protection Agency, the average airport wait time at Chicago's O'Hare International airport is 16 minutes for a traveler arriving during the hours 7–8 A.M., and 32 minutes for arrival during the hours 4–5 P.M. The wait time is defined as the total processing time from arrival at the airport until the completion of a passenger's security screening. Assume the wait time is exponentially distributed.

**a.** What is the probability of waiting between 10 and 30 minutes for a traveler arriving during the 7–8 A.M. hour?

**b.** What is the probability of waiting more than 25 minutes for a traveler arriving during the 7–8 A.M. hour?

**c.** What is the probability of waiting between 35 and 50 minutes for a traveler arriving during the 4–5 P.M. hour?

**d.** What is the probability of waiting less than 20 minutes for a traveler arriving during the 4–5 P.M. hour?

**37. Printer lifetime** The lifetime of a $200 printer is exponentially distributed with a mean of 2 years. The manufacturer agrees to pay a full refund to a buyer if the printer fails during the first year following its purchase, and a one-half refund if it fails during the second year. If the manufacturer sells 100 printers, how much should it expect to pay in refunds?

**38. Failure time** The time between failures of a photo copier is exponentially distributed. Half of the copiers at a university require service during the first 2 years of operations. If the university purchased 150 copiers, how many do you expect to require service during the first year of their operation?

## T Normal distributions

**39. Cholesterol levels** The serum cholesterol levels of children aged 12 to 14 years follows a normal distribution with mean $\mu = 162$ mg/dl and standard deviation $\sigma = 28$ mg/dl. In a population of 1000 of these children, how many would you expect to have serum cholesterol levels between 165 and 193? between 148 and 167?

**40. Annual rainfall** The annual rainfall in inches for San Francisco, California is approximately a normal random variable with mean 20.11 in. and standard deviation 4.7 in. What is the probability that next year's rainfall will exceed 17 inches?

**41. Manufacturing time** The assembly time in minutes for a component at an electronic manufacturing plant is normally distributed with a mean of $\mu = 55$ and standard deviation $\sigma = 4$. What is the probability that a component will be made in less than one hour?

**42. Lifetime of a tire** Assume the random variable $L$ in Example 2f is normally distributed with mean $\mu = 22{,}000$ miles and $\sigma = 4{,}000$ miles.

**a.** In a batch of 4000 tires, how many can be expected to last for at least 18,000 miles?

**b.** What is the minimum number of miles you would expect to find as the lifetime for 90% of the tires?

**43. Height** The average height of American females aged 18–24 is normally distributed with mean $\mu = 65.5$ inches and $\sigma = 2.5$ inches.

**a.** What percentage of females are taller than 68 inches?

**b.** What is the probability a female is between 5′1″ and 5′4″ tall?

**44. Life expectancy** At birth, a French citizen has an average life expectancy of 81 years with a standard deviation of 7 years. If 100 newly born French babies are selected at random, how many would you expect to live between 75 and 85 years? Assume life expectancy is normally distributed.

**45. Length of pregnancy** A team of medical practitioners determines that in a population of 1000 females with ages ranging from 20 to 35 years, the length of pregnancy from conception to birth is approximately normally distributed with a mean of 266 days and a standard deviation of 16 days. How many of these females would you expect to have a pregnancy lasting from 36 weeks to 40 weeks?

**46. Brain weights** In a population of 500 adult Swedish males, medical researchers find their brain weights to be approximately normally distributed with mean $\mu = 1400$ gm and standard deviation $\sigma = 100$ gm.

**a.** What percentage of brain weights are between 1325 and 1450 gm?

**b.** How many males in the population would you except to have a brain weight exceeding 1480 gm?

**47. Blood pressure** Diastolic blood pressure in adults is normally distributed with $\mu = 80$ mm Hg and $\sigma = 12$ mm Hg. In a random sample of 300 adults, how many would be expected to have a diastolic blood pressure below 70 mm Hg?

**48. Albumin levels** Serum albumin in healthy 20 year old males is normally distributed with $\mu = 4.4$ and $\sigma = 0.2$. How likely is it for a healthy 20 year old male to have a level in the range 4.3 to 4.45?

**49. Quality control** A manufacturer of generator shafts finds that it needs to add additional weight to its shafts in order to achieve proper static and dynamic balance. Based on experimental tests, the average weight it needs to add is $\mu = 35$ gms with $\sigma = 9$ gms. Assuming a normal distribution, from 1000 randomly selected shafts, how many would be expected to need an added weight in excess of 40 gms?

**50. Miles driven** A taxicab company in New York City analyzed the daily number of miles driven by each of their drivers. It found the average distance was 200 mi with a standard deviation of 30 mi. Assuming a normal distribution, what prediction can we make about the percentage of drivers who will log in either more than 260 mi or less than 170 mi?

**51. Germination of sunflower seeds** The germination rate of a particular seed is the percentage of seeds in the batch which successfully emerge as plants. Assume that the germination rate for a batch of sunflower seeds is 80%, and that among a large population of $n$ seeds the number of successful germinations is normally distributed with mean $\mu = 0.8n$ and $\sigma = 0.4\sqrt{n}$.

**a.** In a batch of $n = 2500$ seeds, what is the probability that at least 1960 will successfully germinate?

**b.** In a batch of $n = 2500$ seeds, what is the probability that at most 1980 will successfully germinate?

**c.** In a batch of $n = 2500$ seeds, what is the probability that between 1940 and 2020 will successfully germinate?

**52.** Suppose you toss a fair coin $n$ times and record the number of heads that land. Assume that $n$ is large and approximate the discrete random variable $X$ with a continuous random variable that is normally distributed with $\mu = n/2$ and $\sigma = \sqrt{n}/2$. If $n = 400$, find the given probabilities.

**a.** $P(190 \leq X < 210)$ **b.** $P(X < 170)$

**c.** $P(X > 220)$ **d.** $P(X = 300)$

## Discrete Random Variables

**53.** A fair coin is tossed four times and the random variable $X$ assigns the number of tails that appear in each outcome.

**a.** Determine the set of possible outcomes.

**b.** Find the value of $X$ for each outcome.

**c.** Create a probability bar graph for $X$, as in Figure 8.21. What is the probability that at least two heads appear in the four tosses of the coin?

**54.** You roll a pair of six-sided dice and the random variable $X$ assigns to each outcome the sum of the number of dots showing on each face, as in Example 2e.

**a.** Find the set of possible outcomes.

**b.** Create a probability bar graph for $X$.

**c.** What is the probability that $X = 8$?

**d.** What is the probability that $X \leq 5$? $X > 9$?

**55.** Three people are asked their opinion in a poll about a particular brand of a common product found in grocery stores. They can answer in one of three ways: "Like the product brand" (L), "Dislike the product brand" (D), or "Undecided" (U). For each outcome, the random variable $X$ assigns the number of L's that appear.

**a.** Find the set of possible outcomes and the range of $X$.

**b.** Create a probability bar graph for $X$.

**c.** What is the probability that at least two people like the product brand?

**d.** What is the probability that no more than one person dislikes the product brand?

**56. Spacecraft components** A component of a spacecraft has both a main system and a backup system. The probability that both systems perform satisfactorily throughout the duration of a flight is 0.5596, and that both systems fail is 0.0148. Assuming that each system separately has the same success rate, what is the probability that the main system fails during the flight?