Final exam

Math 103
5/5/2016

Name: ______________________________

ID: ______________________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this exam:"

Signature: ______________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all 10 pages are present.
- The exam questions are not in a particular order. If you get stuck, move on to the next problem.
- One 8.5×11 sheet of handwritten notes (front and back) is allowed.
- You will be provided with scratch paper; you must turn it in with your exam.
- You MUST show work to receive credit.
- Good luck and thanks for a great semester!

1	20		6	20	
2	20		7	20	
3	30		8	30	
4	20		9	20	
5	20				
Total	200				

1. Compute each limit or explain why it does not exist:

1. $\lim_{x \to -\infty} \arctan x$

2. $\lim_{x \to 0^+} e^{-1/x} \ln(x)$

3. $\lim_{x \to -\infty} \dfrac{x^2 - 7x}{x + 1}$

4. $\lim_{x \to \infty} \dfrac{1}{\sqrt{x}} \int_1^x \dfrac{dt}{\sqrt{t}}$ (Hint: L'Hopital's Rule applies....)

2. The curves $y = \frac{2}{3\pi}x$ and $y = -\sin x$ bound a finite region in the plane.

1. Carefully sketch the region.
2. Represent the area using integrals with the variable of integration being x.
3. Compute the area.

3.

1. Find the derivative of each function below:

   (a) $\int_{x^2}^{e^x} t \ln(t)\, dt$

   (b) $\arcsin\left(\sqrt{1-x^2}\right)$

   (c) $\frac{2^x}{\tanh(x)}$

2. Find the following antiderivatives (remember you can check yourself by differentiating a proposed answer):

   (a) $\displaystyle\int \sec(x)\tan(x) + e^{3x} + \frac{1}{\sqrt{x}}\, dx$

   (b) $\displaystyle\int \frac{\ln\left(\sqrt{x}\right)}{x}\, dx$

   (c) $\displaystyle\int \frac{dx}{x\sqrt{1+(\ln(x))^2}}$

4. Consider a circular cylindrical tank of height 12 feet and radius 6 feet. Starting from nothing, water begins flowing into the empty tank. The flow accelerates constantly at a rate of 2 cubic feet per minute per minute (i.e., the flow rate is getting bigger with time). How long will it take to fill the tank?

5. For each of the following statements, say whether it is necessarily true or could be false. If it could be false, explain why (for instance, by drawing a picture or giving a counterexample).

a. Continuous functions defined over a closed interval always attain a global maximum.

b. For all integrable functions f, g, if $f(x) \geq g(x)$ for all $x \in [0,1]$, then $\int_0^1 f(x)\,dx \geq \int_0^1 g(x)\,dx$.

c. Suppose $\lim_{x\to\infty} f'(x) > 0$. Then $\lim_{x\to\infty} f(x) = \infty$.

d. All integrable functions are differentiable.

6. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by the function

$$N(x) = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

7. Consider the definite integral $\int_0^2 5x\,dx$.

1. Write this integral as a limit of a Riemann sum.
2. Use the rules of sums/Sigma notation to compute the limit of the sum you wrote in the first part.

8. If a circle and a curve meet at a given point and both the circle and curve have the same first derivative and same second derivative at the point, we call the circle an "osculating circle". Do the following to find values h, k, and a such that $(x-h)^2 + (y-k)^2 = a^2$ is an osculating circle to the curve $y = x^2 + 1$ at the point $(1, 2)$:

1. Sketch a picture illustrating the situation. (You don't know h, k, a yet, but you can draw what an osculating circle to $y = x^2 + 1$ looks like.)

2. Given that the desired circle passes through $(1, 2)$, write a single equation relating h, k, and a.

3. Find an equation giving the slope $\frac{dy}{dx}$ of a tangent line to the circle. (Hint: Use implicit differentiation.)

4. Plug something into the preceding equation to get a single equation just relating h and k.

5. Find an equation giving the second derivative $\frac{d^2y}{dx^2}$ of the circle.

6. Plug something into the preceding equation to solve for k.

7. Solve for h and a. (To help you check your work: h will be a negative integer.)

(Write clearly and explain how you go from step to step. If we understand what you are doing, you will get credit for correct moves in later steps even if you made an error earlier.)

9. Consider the function $f(x) = x + \sin x$.

1. What is the largest value attained by the derivative of f?
2. Give the location of any points of inflection of f on the interval $[0, 2\pi]$.
3. Does f have an inverse? Explain why you say "yes" or "no".