PSEUDOMODULAR FRICKE GROUPS

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ABSTRACT

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David Fithian

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The action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ by fractional linear transformations has exactly one orbit and encodes the classical Euclidean algorithm. Long and Reid sought discrete, nonarithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})$ that also act transitively on $\mathbb{P}^1(\mathbb{Q})$ through fractional linear transformations. By considering a particular two-parameter family Δ of discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$—more specifically, Fricke groups—they succeeded in their search. They dubbed the four groups they discovered "pseudomodular".

In this thesis we identify sufficient conditions for which a group in the family Δ has more than one orbit in its action on $\mathbb{P}^1(\mathbb{Q})$ and is thus *not* pseudomodular. These conditions are that the aforementioned two parameters, which are rational numbers, either have particular congruence properties or specific number-theoretical relations.

Our main technique is as follows. The groups in Δ are Fuchsian groups whose cusps form one orbit that is a subset of $\mathbb{P}^1(\mathbb{Q})$. We endow $\mathbb{P}^1(\mathbb{Q})$ with a topology that is given in a certain way by some p-adic fields and ask when the cusp sets of particular groups in Δ are dense in this topology. For each of our chosen topologies, if the set of cusps $\mathcal{C}$ of a particular group in Δ is not dense, then $\mathcal{C}$ cannot equal $\mathbb{P}^1(\mathbb{Q})$, so the group in question cannot be pseudomodular. Through this method, we find several families of groups in Δ that are not pseudomodular and, along the way, we show that cusp sets admit numerous types of behavior with respect to p-adic topologies.

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Chapter 1

Introduction

A prominent example of a discrete group of hyperbolic isometries is the modular group $\mathrm{SL}_2(\mathbb{Z})$, which acts on the hyperbolic plane $\mathcal{H}$ together with its boundary B at infinity. Under the action of this group, the rational points of B, identified with $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, form one orbit of cusps and are the vertices of the Farey tessellation of $\mathcal{H}$. We can use these facts to describe, for example, continued fractions [Ser1], the Rademacher function and Dedekind sums [KM].

Generally, for an algebraic number field K with ring of integers O_K, the action of $\mathrm{SL}_2(O_K)$ on $\mathbb{P}^1(K)$, induced as in the above case by multiplying column vectors on the left by entries of $\mathrm{SL}_2(O_K)$, has number of orbits equal to the class number of K. Motivated in part by this and other (e.g., [CLR]) connections between orbits of cusps and class numbers of number fields, Long and Reid sought examples of discrete groups of isometries of hyperbolic 2- and 3-space that had a *single* cusp orbit equal to a number field (together with infinity). This search turned out to be quite difficult in full generality, so they investigated Fricke groups [Abe], i.e., discrete groups G of isometries of $\mathcal{H}$ such that $G\backslash\mathcal{H}$ is a finite-area, once punctured torus with a complete hyperbolic metric. Among the Fricke groups with rational cusps, Long and Reid then found [LR1] finitely many nonarithmetic groups whose sets of cusps are exactly $\mathbb{Q} \cup \{\infty\}$. They dubbed these groups *pseudomodular* in analogy with the modular group. Their open questions include whether there are infinitely many commensurability classes of pseudomodular groups and whether their algorithmic methods can detect each pseudomodular Fricke group.

The work here extends that of the author in [Fit], in which we describe some obstructions to pseudomodularity of Fricke groups given by looking at how cusps behave with respect to "congruence data". More precisely, our work is motivated by the following question: if the set $\mathcal{C}$ of cusps of a Fricke group with only rational cusps is dense in some space X into which $\mathbb{P}^1(\mathbb{Q})$ densely includes, is $\mathcal{C}$ equal to $\mathbb{P}^1(\mathbb{Q})$? The spaces X that we use are defined in terms of p-adic numbers and adeles, and the resulting topologies on $\mathbb{P}^1(\mathbb{Q})$ constitute our "congruence data".

If we call a Fricke group whose cusps are contained in $\mathbb{P}^1(\mathbb{Q})$ *rational*, then our chief results are as follows.

THEOREM 4.2. A nonarithmetic, rational Fricke group having a cusp at infinity is pseudomodular if and only if its set of finite cusps is dense in the ring of finite adeles over $\mathbb{Q}$.

This theorem epitomizes the type of statements we seek and provides an "upper bound" for topologies of interest, as we discuss at the beginning of Chapter 5. Next, we consider topologies on $\mathbb{P}^1(\mathbb{Q})$ given by diagonal inclusions into products of the form $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$. We find several families of rational Fricke groups whose sets of cusps are not dense in such products and are thus proper subsets of $\mathbb{P}^1(\mathbb{Q})$. Effective results are given in the numerous propositions of Section 4.2 and are summarized variously by Corollary 4.17, Figures 4.4 and 4.5 and Table A.3.

These results should be of interest to any reader attempting to show a particular rational Fricke group Γ is not pseudomodular. As a side note, a Γ presented in terms of trace coordinates in the Teichmüller space of the once-punctured torus can be represented by the parameters u^2 and t we use throughout this work. This translation is made explicit early in the proof of Proposition 3.2. We must also note that our results do not identify all rational Fricke groups whose sets of cusps are not dense in some product $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$.

Moving on, we find, as a consequence of our p-adic considerations, the following theorem.

THEOREM 4.18. There are infinitely many commensurability classes of non-arith-

metic, non-pseudomodular Fricke groups with rational cusps.

Despite this statement, there are nonarithmetic, non-pseudomodular, rational Fricke groups that cannot be detected using products $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$, as stated in the following result.

THEOREM 4.27. Density of the set of cusps of a nonarithmetic, rational Fricke group in the product $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$, ranging over all primes ℓ, is not a sufficient condition for pseudomodularity.

Further refining our study, we construct in Chapter 5 a profinite topological space $\mathbb{K}$ containing $\mathbb{P}^1(\mathbb{Q})$ as a dense subset. This $\mathbb{K}$ induces a finer topology on $\mathbb{P}^1(\mathbb{Q})$ than $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$ does and provides strictly more information about pseudomodularity, showing that we can circumvent the result of Theorem 4.27. Propositions 5.9 and 5.11 identify explicitly some groups that are captured by the final theorem.

THEOREM 5.12. There exist rational Fricke groups whose sets of cusps are not dense in $\mathbb{K}$ but are nevertheless dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$.

In the last two sections of Chapter 5, we motivate and pose some open questions regarding density of cusp sets in topologies on $\mathbb{P}^1(\mathbb{Q})$ induced by $\mathbb{K}$ and other profinite spaces.

Chapter 2

Preliminaries

2.1 Generalities and Notation

We let $\mathbb{Z}_p$ be the ring of p-adic integers, $\mathbb{Q}_p$ its quotient field, i.e., the p-adic completion of $\mathbb{Q}$, $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and $\mathbb{A}_{\mathbb{Q},f}$ the ring of finite adeles of $\mathbb{Q}$. Details about these rings are recalled when necessary.

For a commutative unital ring R, we define

$$\mathbb{P}^1(R) := \{(a,b) \in R \times R : aR + bR = R\}/R^\times$$

where the quotient is by diagonal scalar multiplication. Then we have identifications $\mathbb{P}^1(\mathbb{Z}) = \mathbb{P}^1(\mathbb{Q})$, $\mathbb{P}^1(\mathbb{Z}_p) = \mathbb{P}^1(\mathbb{Q}_p)$ and $\mathbb{P}^1(\widehat{\mathbb{Z}}) = \mathbb{P}^1(\mathbb{A}_{\mathbb{Q},f})$ each induced by coordinate-wise inclusion. If F is a field, then $\mathbb{P}^1(F)$ is the union of F—whose elements are represented by the classes of $(a,1)$—and the class of $(1,0)$, which we denote by ∞ or $b/0$ for $b \neq 0$ in F.

Say a fraction $a/b \in \mathbb{P}^1(\mathbb{Q})$ is in *lowest terms* if a and b are integers whose GCD is 1. Here, b may be zero. We shall need this notion frequently later, so we adopt the following convention and notation: when we write $a/b \in \mathbb{P}^1(\mathbb{Z})$ or $x \doteq a/b$, we assume that a/b is in lowest terms. All other fractions, including those written as $a/b \in \mathbb{P}^1(\mathbb{Q})$, are not assumed to be in lowest terms unless explicitly noted.

The GCD of a set of integers is always denoted by $\gcd(-)$. If an integer a divides, resp. does not divide, an integer b then we write $a \mid b$, resp. $a \nmid b$.

We shall also use these notions for residues modulo N. If N is a residue and a and b are residues mod N then $\gcd(a,b,N)$ is the GCD of N and any lifts a' and b'

of a and b. If a is a divisor of N and b is a residue mod N then a divides b if and only if $b \equiv 0 \pmod{a}$.

For the purposes of calling a group action continuous, we assume that all groups carry the discrete topology. Thus a *continuous group action* of G on a topological space X is a group action such that each mapping $X \to X : x \mapsto g \cdot x$, for $g \in G$, is continuous.

Finally, we denote by $\mathcal{C}(G)$ the set of cusps of a Fuchsian group G. These terms will be defined in the next section.

2.2   Fuchsian groups

The material here can be found collectively in [Bea] Ch. 7, [Shi] Ch. 1 and [MR] Ch. 1.

A *Fuchsian group* is a discrete subgroup Γ of the orientation-preserving isometry group of the hyperbolic plane $\mathcal{H}$. This isometry group is identified with $\mathrm{PSL}_2(\mathbb{R})$ and its left action is described by the *fractional linear* (or *Möbius*) transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

where z is a point of $\mathcal{H}$ taken in the complex upper half-plane model.

For Fuchsian groups Γ, the quotient $\Gamma \backslash \mathcal{H}$ is a surface with a complex structure induced by that of $\mathcal{H}$, even if Γ has torsion elements. We say that Γ *uniformizes* its corresponding quotient surface. If Γ uniformizes a surface of finite hyperbolic area, then Γ has *finite coarea*.

The Möbius action above extends to $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ through the same formula. An element g of $\mathrm{PSL}_2(\mathbb{R})$ with exactly one fixed point, resp. two fixed points, in $\mathbb{P}^1(\mathbb{R})$ is called *parabolic*, resp. *hyperbolic*. These conditions are characterized by the trace of g: if g is nontrivial and $|\operatorname{tr} g| = 2$ then g is parabolic; if $|\operatorname{tr} g| > 2$, then g is hyperbolic.

A *cusp* of a Fuchsian group Γ is a point of $\mathbb{P}^1(\mathbb{R})$ that is fixed by a parabolic element of Γ. The action of Γ on $\mathbb{P}^1(\mathbb{R})$ sends cusps to cusps. A Fuchsian group is *zonal* if ∞ is among its cusps.

For a fixed Fuchsian group Γ we can append to $\mathcal{H}$ the cusps of Γ to form a space $\mathcal{H}^*$ with a topology such that $\Gamma \backslash \mathcal{H}^*$ has the structure of a Riemann surface. When

$\Gamma \backslash \mathcal{H}^*$ is compact, we say Γ is of *the first kind.* All Fuchsian groups that we consider in this work are of the first kind. For such groups, the complement of $\Gamma \backslash \mathcal{H}$ in $\Gamma \backslash \mathcal{H}^*$ is a finite set of points, each of which is represented by a Γ-orbit of cusps. We also refer to these points or the corresponding punctures of $\Gamma \backslash \mathcal{H}$ as *cusps.*

The quotient $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ of the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ is arguably the most ubiquitous finite coarea Fuchsian group; we also call it the modular group. Its cusps form the set $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ in one orbit under the group action.

Two subgroups of a group G are *(directly) commensurable* if their intersection is of finite index in each of them; they are *commensurable (in the wide sense)* if one is directly commensurable with some conjugate of the other in G. Both types of commensurability are equivalence relations. Hereafter, the term "commensurable" is taken to mean commensurable in the wide sense unless otherwise noted. A Fuchsian group having cusps is *arithmetic* if it is commensurable with the modular group. (Under the broader definition of arithmeticity, which is not needed here, this follows from Theorem 8.2.7 of [MR].) The set of cusps of a Fuchsian group is invariant under direct commensurability.

We will find it convenient at times to use non-unimodular matrices. Accordingly, we note that $\mathrm{GL}_2^+(\mathbb{R})$, the group of invertible 2×2 matrices having positive determinant, has a fractional linear action on $\mathcal{H}$ that extends the action of $\mathrm{SL}_2(\mathbb{R})$ and has as kernel the scalar matrices. We implicitly identify $\mathrm{PGL}_2^+(\mathbb{R})$ with $\mathrm{PSL}_2(\mathbb{R})$ so that we can consider, e.g., $\mathrm{PGL}_2^+(\mathbb{Q})$, as a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Also, we note that conjugation of a Fuchsian group in $\mathrm{PGL}_2(\mathbb{R})$ gives another Fuchsian group since any conjugation preserves discreteness and the sign of the determinant.

Elements of any group $\mathrm{PSL}_2(-)$ or $\mathrm{PGL}_2(-)$ will always be denoted by representative matrices. We will explicitly distinguish between a matrix and its equivalence class when necessary.

2.3 Fricke groups and the family Δ

As in [LR1], we restrict our study to Fuchsian groups uniformizing hyperbolic tori with one cusp.

DEFINITION 2.1 ([Abe] Def. 1.1). A Fuchsian group generated freely by A and B is a *Fricke group* if A and B are hyperbolic and the trace of $B^{-1}A^{-1}BA$ is -2.

The trace condition is well-defined since it is invariant under negating either A or B and under exchanging A and B. Every once punctured torus with a complete hyperbolic metric is uniformized by some Fricke group.

The pair of elements freely generating a Fricke group is not unique. Should we need to specify generators A and B satisfying the conditions of the definition above, we call these generators *marked*. Marked generators will always implicitly form an ordered pair (A, B) and not just a set.

Again following [LR1], we focus on the specific two-parameter family of Fricke groups defined as follows. For u^2 and t real numbers with $0 < u^2 < t - 1$, the group $\Delta(u^2, 2t)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the hyperbolic elements

$$g_1 = \frac{1}{\sqrt{-1 + t - u^2}} \begin{pmatrix} t - 1 & u^2 \\ 1 & 1 \end{pmatrix}, \quad g_2 = \frac{1}{u\sqrt{-1 + t - u^2}} \begin{pmatrix} u^2 & u^2 \\ 1 & t - u^2 \end{pmatrix}. \quad (*)$$

A fundamental domain for the action of $\Delta(u^2, 2t)$ on $\mathcal{H}$ is the ideal polygon with vertices ∞, -1, 0 and u^2, as in Figure 2.1. The element g_1 maps the oriented geodesic from -1 to 0 to that from ∞ to u^2; likewise, g_2 maps the oriented geodesic from -1 to ∞ to that from 0 to u^2. Each $\Delta(u^2, 2t)$ is a Fricke group freely generated by g_1 and g_2 and thus has a single orbit of cusps, which corresponds to the puncture on the quotient surface $\Delta(u^2, 2t)\backslash\mathcal{H}$. The commutator $g_1 g_2^{-1} g_1^{-1} g_2$ is a parabolic element generating the stabilizer of ∞ in $\Delta(u^2, 2t)$, so each $\Delta(u^2, 2t)$ is zonal. Since each matrix in $\Delta(u^2, 2t)$ is represented up to a scalar by a matrix with coordinates in $\mathbb{Q}(u^2, t)$, the cusps of $\Delta(u^2, 2t)$ are contained in $\mathbb{P}^1(\mathbb{Q}(u^2, t))$.

Hereafter we unconditionally assume for each given pair of parameters $(u^2, 2t)$ that $0 < u^2 < t - 1$. Where we need the value u itself, we select it to be positive.

2.4   Pseudomodular groups

Long and Reid initiate the study of pseudomodular groups in [LR1]. In this section, we describe some of their results and methods.

DEFINITION 2.2. A Fuchsian group Γ is *pseudomodular* if it is nonarithmetic, has finite coarea and its set of cusps is $\mathbb{P}^1(\mathbb{Q})$.
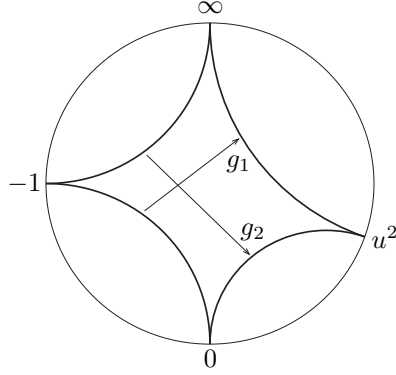
Figure 2.1: A fundamental domain for $\Delta(u^2, 2t)$ in $\mathcal{H}$

As noted above, the set of cusps of $\Delta(u^2, 2t)$ lies in $\mathbb{P}^1(\mathbb{Q}(u^2, t))$. In studying the sets of cusps of potentially pseudomodular groups, we restrict hereafter to the case where both u^2 and t are rational.

Since each $\Delta(u^2, 2t)$ has one orbit of cusps and that orbit contains ∞, one can show that $\Delta(u^2, 2t)$ is pseudomodular by showing that for each $x \in \mathbb{Q}$, there is a $g \in \Delta(u^2, 2t)$ such that $gx = \infty$.

In the case of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, such a g can be found through the Euclidean algorithm, as follows. The two matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathrm{PSL}_2(\mathbb{Z})$ (see [DS] Exercise 1.1.1) and in particular

$$ ST^m \cdot \frac{r}{s} = \frac{-s}{r + ms} $$

for r/s in $\mathbb{P}^1(\mathbb{Q})$ and m an integer. If r and s are coprime integers, then so too are $-s$ and $r + ms$. Assuming s is nonzero, we can choose m so that $0 \leq r + ms \leq s - 1$. By replacing r/s with $(-s)/(r + ms)$ and iterating this process, we recover the Euclidean algorithm. At each step in the iteration, we decrease the denominator of a rational number by applying to it an element of the modular group. In particular, starting with any r/s with r and s coprime integers, we ultimately arrive at $\infty = 1/0$ in $\mathbb{P}^1(\mathbb{Q})$.

Long and Reid apply the same principle in [LR1] using elements of the groups $\Delta(u^2, 2t)$. If $g \in \Delta(u^2, 2t)$ does not fix ∞, then g is represented by some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of integers with $c \neq 0$. For $r/s \in \mathbb{Q}$ with r and s coprime integers, we have

$g \cdot (r/s) = (ar + bs)/(cr + ds)$, whose denominator, when written in lowest terms, is at most $cr + ds$. Therefore, if $|cr + ds| < |s|$, or equivalently, if $|x + d/c| < 1/|c|$ for $x = r/s$, then $g \cdot (r/s)$ has a smaller denominator than r/s. The open interval of reals defined by this last inequality is the *killer interval* of g.

When killer intervals for elements of a fixed, nonarithmetic $\Delta(u^2, 2t)$ cover the real line, they collectively produce a "pseudo-Euclidean" algorithm analogous to the Euclidean algorithm described above: starting with any rational number, one can iteratively map it by elements of $\Delta(u^2, 2t)$ to rationals of strictly decreasing denominators, until eventually ∞ is hit. This establishes that $\Delta(u^2, 2t)$ is pseudomodular. Using this technique, Long and Reid show:

THEOREM 2.3 ([LR1] Theorem 1.2). *The groups $\Delta(5/7, 6)$, $\Delta(2/5, 4)$, $\Delta(3/7, 4)$ and $\Delta(3/11, 4)$ are pseudomodular and pairwise noncommensurable. Therefore pseudomodular groups exist.*

They also produce finitely many groups $\Delta(u^2, 2t)$ that are neither pseudomodular nor arithmetic by exhibiting, for each, a hyperbolic element of $\Delta(u^2, 2t)$ that fixes a rational number x. They call such an element and the rational numbers it fixes *special*. Special points in $\mathbb{P}^1(\mathbb{Q})$ cannot also be cusps as $\Delta(u^2, 2t)$ is discrete; see the proof of Theorem 8.3.1 in [Bea]. It is yet unclear whether special points exist for all non-pseudomodular, nonarithmetic $\Delta(u^2, 2t)$.

In the appendix, we reproduce Tables 5.1 and 5.2 of [LR1], which list some cusp data found using the above ideas.

The open problems stated in [LR1] include determining whether there are infinitely many commensurability classes of pseudomodular groups and finding the pairs $(u^2, 2t)$ such that $\Delta(u^2, 2t)$ is pseudomodular. In this work, we approach the latter question by providing sufficient conditions for $\Delta(u^2, 2t)$ to have more than one orbit and so to *not* be pseudomodular. These are given by requiring the parameters u^2 and t to satisfy some congruence or number-theoretical relations. Notably, our proofs neither require nor produce special points.

We do not yet have a method for producing more pseudomodular groups. One of our overarching goals is to find necessary, sufficient and effective conditions for pseudomodularity that extend the results presented herein.

Chapter 3

Equivalences for Δ

In this chapter we use trace coordinates of Fricke groups to construct some results exhibiting equivalences between groups $\Delta(u^2, 2t)$ with different parameters. These results will later be indispensable in extending certain results to wider classes of Fricke groups.

3.1 Trace coordinates

Suppose Γ is a Fricke group with marked generators A and B. Lift these to respective representatives $\widetilde{A}$ and $\widetilde{B}$ in $\mathrm{SL}_2(\mathbb{R})$ each with positive trace. If we set

$$(x, y, z) = (\operatorname{tr} \widetilde{A}, \operatorname{tr} \widetilde{B}, \operatorname{tr} \widetilde{A}\widetilde{B}) \tag{3.1.1}$$

then the conditions of Definition 2.1 are characterized by

$$x^2 + y^2 + z^2 = xyz, \qquad x, y, z > 2. \tag{3.1.2}$$

See [Abe] or [Wol] and the trace relations in §3.4 of [MR]. Call the 3-tuple of (3.1.1) the *trace coordinates* of Γ. This is well-defined since marked generators are ordered. The next lemma justifies our interest in trace coordinates.

LEMMA 3.1 (Equivalence Lemma). *If Γ and Σ are Fricke groups with the same trace coordinates then they are conjugate in* $\mathrm{PGL}_2(\mathbb{R})$.

Proof. We shall employ numerous results from [CS] on varieties of group representations. The approach herein is suggested by Long and Reid in the proof of Lemma 2.1 of [LR1].

Let the marked generators A and B of Γ and the marked generators P and Q of Σ be lifted to elements $\widetilde{A}, \widetilde{B}, \widetilde{P}, \widetilde{Q}$ of $\mathrm{SL}_2(\mathbb{R})$ each with positive trace. Define $\widetilde{\Gamma}$, resp. $\widetilde{\Sigma}$, to be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated (necessarily freely) by $\widetilde{A}$ and $\widetilde{B}$, resp. by $\widetilde{P}$ and $\widetilde{Q}$.

Fix two generators h_1 and h_2 of $\mathbb{Z} * \mathbb{Z}$ and form the representations

$$\rho_1 : \mathbb{Z} * \mathbb{Z} \longrightarrow \mathrm{SL}_2(\mathbb{R}), \qquad\qquad \rho_2 : \mathbb{Z} * \mathbb{Z} \longrightarrow \mathrm{SL}_2(\mathbb{R})$$
$$h_1 \longmapsto \widetilde{A} \qquad\qquad\qquad\qquad h_2 \longmapsto \widetilde{P}$$
$$h_2 \longmapsto \widetilde{B} \qquad\qquad\qquad\qquad h_2 \longmapsto \widetilde{Q},$$

which map onto $\widetilde{\Gamma}$ and $\widetilde{\Sigma}$, respectively. Through the definition of Fricke group we find that

$$\mathrm{tr}\, \rho_1(h_1 h_2^{-1} h_1^{-1} h_2) = \mathrm{tr}\, \rho_2(h_1 h_2^{-1} h_1^{-1} h_2) = -2.$$

The presence of a commutator of trace -2 implies that ρ_1 and ρ_2 are irreducible representations, according to [CS] Lemma 1.2.1.

As in the proof of [CS] Proposition 1.5.5, we now invoke [CS] Proposition 1.4.1 to say that the character of a representation in $\mathrm{SL}_2(\mathbb{C})$ of a free group generated by h_1 and h_2 is determined its values on h_1, h_2 and $h_1 h_2$. By construction, then, the characters of ρ_1 and ρ_2 are equal, whence, by irreducibility and [CS] Proposition 1.5.2, ρ_1 and ρ_2 are equivalent as representations into $\mathrm{GL}_2(\mathbb{C})$; i.e., $\widetilde{\Gamma}$ and $\widetilde{\Sigma}$ are conjugate in $\mathrm{GL}_2(\mathbb{C})$.

In their fractional linear actions on $\mathbb{P}^1(\mathbb{C})$, both $\widetilde{\Gamma}$ and $\widetilde{\Sigma}$ fix $\mathbb{P}^1(\mathbb{R})$ setwise, so they in fact conjugate in $\mathrm{GL}_2(\mathbb{R})$. This completes the proof. $\boxtimes$

3.2 Equivalent parameters

Below, we will use the Equivalence Lemma to exhibit instances of distinct groups $\Delta(u^2, 2t)$ that are conjugate in $\mathrm{PGL}_2(\mathbb{Q})$. First, we show that, up to conjugation, the family Δ represents all Fricke groups whose set of cusps is in $\mathbb{P}^1(\mathbb{Q})$.

PROPOSITION 3.2. *Every Fricke group whose cusps are contained in $\mathbb{P}^1(\mathbb{Q})$ is conjugate in $\mathrm{PGL}_2(\mathbb{Q})$ to some $\Delta(u^2, 2t)$ with u^2 and t in $\mathbb{Q}$.*

Proof. Say Γ is a Fricke group whose cusps are in $\mathbb{P}^1(\mathbb{Q})$ and let (x, y, z) be its trace coordinates. Define

$$u = \frac{x}{y}, \qquad t = \frac{x(x^2 + y^2)}{yz}.$$

Then, using (3.1.2),

$$-1 + t - u^2 = -1 + \frac{x(x^2 + y^2)}{yz} - \frac{x^2}{y^2} = \frac{(x^2 + y^2)(xyz - z^2)}{y^2 z^2} = \frac{(x^2 + y^2)^2}{y^2 z^2},$$

which is positive, so that $0 < u^2 < t - 1$ and $\Delta(u^2, 2t)$ is defined.

Choose g_1 and g_2 as marked generators of $\Delta(u^2, 2t)$. Then, using the lifts to $\mathrm{SL}_2(\mathbb{R})$ given by $(*)$, we compute

$$\mathrm{tr}\, g_1 = \frac{t}{\sqrt{-1 + t - u^2}} = \frac{x(x^2 + y^2)}{yz} \cdot \frac{yz}{x^2 + y^2} = x$$

and

$$\mathrm{tr}\, g_2 = \frac{t}{u\sqrt{-1 + t - u^2}} = \frac{x}{u} = y.$$

Moreover,

$$\begin{aligned}
\mathrm{tr}\, g_1 g_2 &= \frac{-t(1 + u^2)}{u - tu + u^3} \\
&= \frac{-x((x^2 + y^2)/yz)(1 + x^2/y^2)}{(x/y)(1 - x(x^2 + y^2)/yz + x^2/y^2)} \\
&= \frac{-x(x^2 + y^2)(x^2 + y^2)}{x(y^2 z - xy(x^2 + y^2) + x^2 z)} \\
&= \frac{x^2 + y^2}{xy - z} = \frac{xyz - z^2}{xy - z} = z.
\end{aligned}$$

Therefore Γ and $\Delta(u^2, 2t)$ share the same trace coordinates and by the Equivalence Lemma are conjugate in $\mathrm{PGL}_2(\mathbb{R})$.

Since the cusps of $\Delta(u^2, 2t)$ include ∞, 0 and -1 and the cusps of Γ are all in $\mathbb{P}^1(\mathbb{Q})$, and because fractional linear transformations are determined by images of three points, $\Delta(u^2, 2t)$ and Γ are in fact conjugate in $\mathrm{PGL}_2(\mathbb{Q})$.

This implies that the cusps of $\Delta(u^2, 2t)$ lie in $\mathbb{P}^1(\mathbb{Q})$. The values $u^2 = g_2 \cdot \infty$ and $t - 1 = g_1 \cdot \infty$ are evidently cusps of $\Delta(u^2, 2t)$, so u^2 and t lie in $\mathbb{Q}$. ☐

The preceding proposition—together with the fact that the property of pseudo-modularity is invariant under conjugation in $\mathrm{PGL}_2(\mathbb{Q})$—implies that we can study

the pseudomodularity of all Fricke groups with only rational cusps by restricting attention to those groups $\Delta(u^2, 2t)$ with u^2 and t rational (as already assumed above in Section 2.4).

Having proved this, we now turn our attention to equivalences between groups in the family $\Delta(u^2, 2t)$.

DEFINITION 3.3. Two groups $\Delta(u_1^2, 2t_1)$ and $\Delta(u_2^2, 2t_2)$ are *equivalent* if they are conjugate in $\mathrm{PGL}_2(\mathbb{Q})$.

This notion of equivalence preserves the property of whether or not the set of cusps is equal to $\mathbb{P}^1(\mathbb{Q})$. Of course, it also defines an equivalence relation. We next exhibit some explicit equivalences. In the course of the upcoming proof, we repeatedly and implicitly use that the trace coordinates of $\Delta(u^2, 2t)$, with the marked generators g_1 and g_2, are

$$\left(\frac{t}{\sqrt{-1+t-u^2}}, \frac{t}{u\sqrt{-1+t-u^2}}, \frac{t(u^2+1)}{u(-1+t-u^2)} \right)$$

as functions of u and t.

PROPOSITION 3.4. *The following groups are equivalent to $\Delta(u^2, 2t)$:*

(a) $\Delta(u^{-2}, 2tu^{-2})$,

(b) $\Delta\left(u^2, \dfrac{2t(u^2+1)}{-1+t-u^2} \right)$,

(c) $\Delta\left(\dfrac{(u^2+1)^2}{-1+t-u^2}, \dfrac{2t(u^2+1)}{-1+t-u^2} \right)$,

(d) $\Delta\left(-1+t-u^2, \dfrac{2t(t-u^2)}{u^2} \right)$.

Proof. Fix the parameters u^2 and t. In this proof, g_1 and g_2 always denote the usual marked generators of the fixed group $\Delta(u^2, 2t)$.

For each of the other given groups $\Delta(v^2, 2s)$, we have to show that $-1+s-v^2 > 0$ to ensure that it is well-defined. (Already, in each case, v^2 is visibly positive since $-1+t-u^2 > 0$.) Having done this, we will give the trace coordinates of each group and show that there are generators of $\Delta(u^2, 2t)$ that yield the same traces.

Case (a): Set $v = u^{-1}$ and $s = tu^{-2}$. The given group $\Delta(v^2, 2s)$ is well-defined since

$$-1 + s - v^2 = u^{-2}(-1 + t - u^2) > 0.$$

Its trace coordinates are

$$\left(\frac{t}{u\sqrt{-1+t-u^2}}, \frac{t}{\sqrt{-1+t-u^2}}, \frac{t(u^2+1)}{u(-1+t-u^2)} \right).$$

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These coordinates are also given by $(\operatorname{tr} g_2, \operatorname{tr} g_1, \operatorname{tr} g_2 g_1)$. According to the remarks at the beginning of Section 3.1, this implies that (g_2, g_1) can be taken as a pair of marked generators for the Fricke group $\Delta(u^2, 2t)$. Using the Equivalence Lemma and the fact that the cusps of $\Delta(u^2, 2t)$ and of $\Delta(v^2, 2s)$ lie in $\mathbb{P}^1(\mathbb{Q})$, we find that $\Delta(u^2, 2t)$ and $\Delta(v^2, 2s)$ are equivalent as claimed.

Case (b): Set $v = u$ and $s = t(u^2 + 1)/(-1 + t - u^2)$. The group in question is well-defined since

$$-1 + s - v^2 = \frac{(u^2 + 1)^2}{-1 + t - u^2} > 0.$$

Its trace coordinates are

$$\left(\frac{t}{\sqrt{-1 + t - u^2}}, \ \frac{t}{u\sqrt{-1 + t - u^2}}, \ \frac{t}{u}\right) = (\operatorname{tr} g_1^{-1}, \operatorname{tr} g_2, \operatorname{tr} g_1^{-1} g_2).$$

Thus g_1^{-1} and g_2 serve as marked generators for $\Delta(u^2, 2t)$, which is then equivalent to $\Delta(v^2, 2s)$.

Case (c): Set $v = (u^2 + 1)/\sqrt{-1 + t - u^2}$ and $s = t(u^2 + 1)/(-1 + t - u^2)$. Then

$$-1 + s - v^2 = \frac{1 - t + u^2 + t(u^2 + 1) - (u^2 + 1)^2}{-1 + t - u^2} = u^2 > 0,$$

whence the given group $\Delta(v^2, 2s)$ is well-defined. The trace coordinates of $\Delta(v^2, 2s)$ are

$$\left(\frac{t(u^2 + 1)}{u(-1 + t - u^2)}, \ \frac{t}{u\sqrt{-1 + t - u^2}}, \ \frac{t(u^4 + u^2 + t)}{u^2(-1 + t - u^2)^{3/2}}\right) = (\operatorname{tr} g_1 g_2, \operatorname{tr} g_2, \operatorname{tr} g_1 g_2^2),$$

so $g_1 g_2$ and g_2 serve as marked generators for $\Delta(u^2, 2t)$. The claimed equivalence is again true by the Equivalence Lemma.

Case (d): Set $v = \sqrt{-1 + t - u^2}$ and $s = t(t - u^2)/u^2$. Then

$$-1 + s - v^2 = \frac{t(t - u^2) - tu^2 + u^4}{u^2} = \frac{(u^2 - t)^2}{u^2} > 0,$$

so the group given in part (d) is well-defined. We find that this group's trace coordinates are

$$\left(\frac{t}{u}, \ \frac{t}{u\sqrt{-1 + t - u^2}}, \ \frac{t}{\sqrt{-1 + t - u^2}}\right) = (\operatorname{tr} g_1 g_2^{-1}, \operatorname{tr} g_2, \operatorname{tr} g_1).$$

Consequently $g_1 g_2^{-1}$ and g_2 can be taken as a pair of marked generators for $\Delta(u^2, 2t)$. As above we have that $\Delta(u^2, 2t)$ and $\Delta(v^2, 2s)$ are equivalent. $\boxtimes$

The next definition is motivated by the "changes of parameters" given by each of the four equivalences just established.

DEFINITION 3.5. Let

$$\mathcal{P} := \{(u^2, 2t) \in \mathbb{R}^2 : 0 < u^2 < t - 1\}$$

be the space of parameters $(u^2, 2t)$ for which $\Delta(u^2, 2t)$ is defined. Label the following functions $\mathcal{P} \to \mathcal{P}$:

$$\beta : (u^2, 2t) \longmapsto \left(\frac{1}{u^2}, \frac{2t}{u^2} \right),$$

$$\sigma : (u^2, 2t) \longmapsto \left(u^2, \frac{2t(u^2 + 1)}{-1 + t - u^2} \right),$$

$$\psi : (u^2, 2t) \longmapsto \left(\frac{(u^2 + 1)^2}{-1 + t - u^2}, \frac{2t(u^2 + 1)}{-1 + t - u^2} \right),$$

$$\psi^{-1} : (u^2, 2t) \longmapsto \left(-1 + t - u^2, \frac{2t(t - u^2)}{u^2} \right).$$

We claim that P and σ are involutions and that the map labeled ψ^{-1} is in fact the inverse of ψ, so that β, σ and ψ are bijections of $\mathcal{P}$. Denote by $\mathcal{G}$ the group of bijections of $\mathcal{P}$ generated by β, σ and ψ. For a function $f \in \mathcal{G}$, define ${}^f\Delta(u^2, 2t)$ to be $\Delta(v^2, 2s)$ where $(v^2, 2s) = f(u^2, 2t)$.

Now according to Proposition 3.4 above, ${}^f\Delta(u^2, 2t)$ and $\Delta(u^2, 2t)$ are equivalent for each choice of $f \in \mathcal{G}$ and of $(u^2, 2t) \in \mathcal{P}$. Using $f = \psi\beta\sigma\beta$, we recover Lemma 2.1 of [LR1].

PROPOSITION 3.6. *The groups $\Delta(u^2, 2t)$ and $\Delta(-1 + t - u^2, 2t)$ are equivalent.*

Equivalences will see use in Sections 4.2 and 4.3.

Remark. Although we do not need this fact for our work, the elements of the above discussion are related to the Teichmüller space of the once-punctured torus and to the automorphism group of $\mathbb{Z} * \mathbb{Z}$. Indeed, we claim without proof that $\mathcal{G}$ is isomorphic to the opposite group of $\mathrm{Aut}(\mathbb{Z} * \mathbb{Z})$. The reader interested on following up on this material should consult [Abe].

Chapter 4

Density obstructions

Clearly, if the group $\Delta(u^2, 2t)$ is arithmetic or pseudomodular, then its set of cusps, respectively finite cusps, is dense in any space into which $\mathbb{P}^1(\mathbb{Q})$, respectively $\mathbb{Q}$, has a dense inclusion. In this chapter we use the contrapositive of this statement to provide some obstructions to pseudomodularity, using number-theoretically defined topologies on $Y = \mathbb{Q}$ or $Y = \mathbb{P}^1(\mathbb{Q})$. Each of the topologies on Y we consider in this chapter is given by a dense inclusion of Y into some topological space X. A subset of Y is dense in Y in the topology induced by said inclusion if and only if it is dense in X. We will use this implication without further comment.

We call any result in which we have the set of cusps of a (zonal) Fuchsian group not dense in a specified X a *density obstruction* to pseudomodularity, since pseudomodularity implies having the set of (finite) cusps dense in X.

4.1   Adelic characterization

We first give a condition characterizing pseudomodularity using the finite-adelic topology on $\mathbb{Q}$. This will not be useful in producing effective results but gives an upper bound on topologies of interest, as we shall discuss at the beginning of Chapter 5.

Recall that the finite adele ring $\mathbb{A}_{\mathbb{Q},f}$ of $\mathbb{Q}$ is the restricted direct product of the non-archimedean completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the compact subrings $\mathbb{Z}_p$ of p-adic integers; see Chapter VII of [Lan]. Alternatively, we may write $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$,

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. The topology induced on $\mathbb{A}_{\mathbb{Q},f}$ by the profinite topology on $\widehat{\mathbb{Z}}$ is the same as that given by the restricted direct product. The inclusion of $\mathbb{Q}$ into $\mathbb{A}_{\mathbb{Q},f}$ is dense by the Strong Approximation Theorem [Cas]. Therefore it is of interest to see when the set of finite cusps of a $\Delta(u^2, 2t)$ is dense in $\mathbb{Q}$.

LEMMA 4.1. *Let S be a nonempty subset of $\mathbb{Q}$ such that $S = S + \alpha$ for some nonzero $\alpha \in \mathbb{Q}$. Then $\mathbb{Q} - S$ is not dense in $\mathbb{A}_{\mathbb{Q},f}$.*

Proof. It suffices to find a nonempty open set U in $\mathbb{A}_{\mathbb{Q},f}$ such that $\mathbb{Q} \cap U \subseteq S$. (Since $\mathbb{Q}$ is dense in $\mathbb{A}_{\mathbb{Q},f}$, the intersection $\mathbb{Q} \cap U$ will be nonempty.)

Fix an $s_0 \in S$. Write $\alpha = m/n$ for integers m and n and let the prime factorization of m be $p_1^{e_1} \cdots p_r^{e_r}$. Define the open neighborhood U_0 of 0 in $\mathbb{A}_{\mathbb{Q},f}$ by

$$U_0 = p_1^{e_1}\mathbb{Z}_{p_1} \times \cdots \times p_r^{e_r}\mathbb{Z}_{p_r} \times \prod_{p \nmid m} \mathbb{Z}_p$$

and set $U = s_0 + U_0$. Suppose $x \in \mathbb{Q} \cap U$. Then $x - s_0 \in U_0 \cap \mathbb{Q}$, but this set is exactly the ideal $p_1^{e_1} \cdots p_r^{e_r}\mathbb{Z} = m\mathbb{Z}$. It follows that for some integer b, $x = s_0 + mb = s_0 + nb\alpha$, which lands in S by hypothesis. Thus $\mathbb{Q} \cap U \subseteq S$. ☑

Remark. The proof holds if we choose U_0 to take the denominator of α into account too; that is, we can choose $U_0 = (m/n) \prod \mathbb{Z}_p$ instead of $U_0 = m \prod \mathbb{Z}_p$. Moreover, we can take the union of the U as $s_0 \in S$ varies to get an open set in $\mathbb{A}_{\mathbb{Q},f}$ whose intersection with $\mathbb{Q}$ is exactly S, so such S are themselves open in $\mathbb{Q}$ in the finite-adelic topology.

THEOREM 4.2. *The group $\Delta(u^2, 2t)$ is pseudomodular or arithmetic if and only if its set of finite cusps is dense in the ring $\mathbb{A}_{\mathbb{Q},f}$ of finite adeles over $\mathbb{Q}$.*

Proof. The set of points of $\mathbb{P}^1(\mathbb{Q})$ that are not cusps of $\Delta(u^2, 2t)$ is either empty or is a nonempty subset S of $\mathbb{Q}$ such that $S + 2t = S$. Using that t is nonzero, we apply the preceding lemma. ☑

Through this proof, we see that this theorem in fact applies to any Fuchsian subgroup that is represented by elements of $\mathrm{GL}_2^+(\mathbb{Q})$ and that has a cusp at infinity.

We now make a few remarks related to the implicit choice of the cusp ∞, to which the result of Theorem 4.2 is sensitive.

If we have a Fuchsian group Γ whose cusps are in $\mathbb{P}^1(\mathbb{Q})$ and x_0 is one of the cusps, then we can get a statement that is analogous to Theorem 4.2 and related to x_0 as follows. Endow $\mathbb{Q}$ with the topology inherited from $\mathbb{A}_{\mathbb{Q},f}$, let τ be a fractional linear transformation mapping $\mathbb{P}^1(\mathbb{Q})$ onto itself (setwise) and mapping x_0 to ∞, and then give $\mathbb{P}^1(\mathbb{Q}) \backslash \{x_0\}$ the topology induced by the bijection $\tau|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{P}^1(\mathbb{Q}) \backslash \{x_0\}$. If τ_0 is another such τ, then $\tau \circ \tau_0^{-1}$ fixes ∞ and is thus some affine transformation $x \mapsto ax + b$ with a and b rational and a nonzero. Such a transformation is a homeomorphism of $\mathbb{Q}$ with the finite-adelic topology, so the topology on $\mathbb{P}^1(\mathbb{Q}) \backslash \{x_0\}$ is well-defined. Now Γ is pseudomodular or arithmetic if and only if its set of cusps excluding x_0 is dense in $\mathbb{P}^1(\mathbb{Q}) \backslash \{x_0\}$.

In this sense, ∞ is not a special value for the result of Theorem 4.2. However, for the family of groups $\Delta(u^2, 2t)$, the most useful choice of cusp is among those common to all groups in the family, which are (at least) 0, -1 and ∞. We select ∞ since it makes the statement of Theorem 4.2 most natural: we can simply discuss $\mathbb{Q}$ without cumbersome references to fractional linear transformations.

Below we will be stating results and questions in terms of topologies $\mathcal{T}$ on $\mathbb{P}^1(\mathbb{Q})$. Although this is most natural, it is not strictly necessary. Each $\mathcal{T}$ considered below, in this chapter and the next, is Hausdorff and thus has $\{\infty\}$ as a closed subset. Therefore a set $X \subset \mathbb{Q}$ is dense in $\mathbb{Q}$ with the subspace topology induced by a fixed $\mathcal{T}$ if and only if $X \cup \{\infty\}$ is dense in $\mathbb{P}^1(\mathbb{Q})$ with the same topology $\mathcal{T}$. All results below can thus be restated in terms of density of the set of finite cusps in $\mathbb{Q}$ with the appropriate (subspace) topology. Hence they are comparable with Theorem 4.2 above despite their initial appearance. We shall say more later about comparisons between various results.

Finally, through the homeomorphisms

$$\mathbb{P}^1(\mathbb{A}_{\mathbb{Q},f}) = \mathbb{P}^1(\widehat{\mathbb{Z}}) \cong \prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Z}_\ell) = \prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell),$$

we show in Theorem 4.27 below that density of the cusp set in $\mathbb{P}^1(\mathbb{A}_{\mathbb{Q},f})$ does not ensure that the cusp set is $\mathbb{P}^1(\mathbb{Q})$. Thus, the analogue of Theorem 4.2 using the inclusion of the set of cusps into $\mathbb{P}^1(\mathbb{A}_{\mathbb{Q},f})$ is false.

4.2 Some p-adic obstructions

In this section, all topologies on $\mathbb{P}^1(\mathbb{Q})$ arise from the diagonal embedding ι_S of $\mathbb{P}^1(\mathbb{Q})$ into a product of distinct spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$ over a set S of primes ℓ. Using the product topology and Theorem 1 of §II.1 of [Lan] we see that the diagonal embedding of $\mathbb{Q}$ into any product $\prod_\ell \mathbb{Q}_\ell$ is dense. Since also each inclusion of $\prod_\ell \mathbb{Q}_\ell$ into $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$ is dense, each inclusion ι_S is dense. Hence we are justified in investigating the density of the cusp sets of $\Delta(u^2, 2t)$ in the aforementioned products.

For a rational prime p, we let $v_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ denote the standard logarithmic valuation. Extend v_p to $\mathbb{P}^1(\mathbb{Q}_p)$ by setting $v_p(0) = \infty$ and $v_p(\infty) = -\infty$ in the affinely extended real numbers. Note that v_p is defined on $\mathbb{P}^1(\mathbb{Q})$ for each prime p. Recall that the set of cusps of a Fuchsian group G is denoted by $\mathcal{C}(G)$.

4.2.1 One-prime obstructions

Our first proposition says that if the valuation of u^2 with respect to p is sufficiently low, then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$. The lemma preceding this proposition demonstrates the general technique we use to show that $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in some product of spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$: we exhibit a $\Delta(u^2, 2t)$-invariant set of rationals that is nonempty, proper and open in the induced topology on $\mathbb{P}^1(\mathbb{Q})$.

LEMMA 4.3. *Suppose that $u^2 = m/p^a$ and $t = n/p^b$, where m is p-adically a unit, n is p-adically integral, a is positive and b is nonnegative. If $b - a < -b - 1$, then $\Delta(u^2, 2t)$ fixes the set of rationals $U = \{x \in \mathbb{Q} : b - a + 1 \leq v_p(x) \leq -b - 1\}$.*

Proof. Suppose $x \in U$. Write x as a quotient r/s of coprime integers and assign $e = v_p(s) = -v_p(x)$. Then $p \nmid r$, $p^e \mid s$ and $s_0 := s/p^e$ is a p-adic unit. We note that, by hypothesis,

$$b + 1 \leq e \leq a - b - 1. \tag{4.2.1}$$

We compute the valuations of images of x under the generators of $\Delta(u^2, 2t)$ and their inverses. First, represent the generators and their inverses in $\mathrm{PGL}_2(\mathbb{R})$ by

matrices whose entries are p-adically integral:

$$g_1 = \begin{pmatrix} (n - p^b)p^a & mp^b \\ p^{a+b} & p^{a+b} \end{pmatrix}, \qquad g_1^{-1} = \begin{pmatrix} p^{a+b} & -mp^b \\ -p^{a+b} & (n - p^b)p^a \end{pmatrix},$$

$$g_2 = \begin{pmatrix} mp^b & mp^b \\ p^{a+b} & np^a - mp^b \end{pmatrix}, \qquad g_2^{-1} = \begin{pmatrix} np^a - mp^b & -mp^b \\ -p^{a+b} & mp^b \end{pmatrix}.$$

Now use these to compute:

$$
\begin{aligned}
v_p(g_1 x) &= v_p((n - p^b)p^a r + mp^b s) - v_p(p^{a+b}r + p^{a+b}s) \\
&= v_p((n - p^b)p^a r + mp^{b+e}s_0) - (a + b) \\
&= b + e - (a + b) \hspace{4cm} (b + e < a \text{ by } 4.2.1) \\
&= e - a \\
v_p(g_1^{-1}x) &= v_p(p^{a+b}r - mp^b s) - v_p(-p^{a+b}r + (n - p^b)p^a s) \\
&= v_p(p^{a+b}r - mp^{b+e}s_0) - v_p(-p^{a+b}r + (n - p^b)p^{a+e}s_0) \\
&= b + e - (a + b) \hspace{4cm} (b < e < a \text{ by } 4.2.1) \\
&= e - a \\
v_p(g_2 x) &= v_p(mp^b r + mp^b s) - v_p(p^{a+b}r + (np^a - mp^b)s) \\
&= b - b - v_p(p^a r + (np^{a-b} - m)p^e s_0) \\
&= -e \hspace{5.5cm} (e < a) \\
v_p(g_2^{-1}x) &= v_p((np^a - mp^b)r - mp^b s) - v_p(-p^{a+b}r + mp^b s) \\
&= b + v_p((np^{a-b} - m)r - mp^e s_0) - b - v_p(-p^a r + mp^e s_0) \\
&= -e. \hspace{5.5cm} (e < a)
\end{aligned}
$$

We immediately see that $g_2 x$ and $g_2^{-1}x$ are in U, since their p-valuations equal that of x. The inequality 4.2.1 implies

$$b - a + 1 \le e - a \le -b - 1,$$

whence $g_1 x$ and $g_1^{-1}x$ are also in U. Since g_1 and g_2 generate $\Delta(u^2, 2t)$, we conclude that U is $\Delta(u^2, 2t)$-invariant. ☑

PROPOSITION 4.4 (One-prime p-adic Obstruction I). *Let p be prime. If* (a) $v_p(t) \ge 0$ *and* $v_p(u^2) \le -2$, *or if* (b) $v_p(t) < 0$ *and* $v_p(u^2) \le 2v_p(t) - 2$, *then* $\mathcal{C}(\Delta(u^2, 2t))$ *is not dense in* $\mathbb{P}^1(\mathbb{Q}_p)$, *and so* $\Delta(u^2, 2t)$ *is neither pseudomodular nor arithmetic.*

Proof. Write $u^2 = m/p^a$ and $t = n/p^b$ as in Lemma 4.3. Hypothesis (a) is equivalent to setting $b = 0$ and $a \geq 2$, while hypothesis (b) is equivalent to assuming that $b > 0$ and $-a \leq -2b - 2$ and also that n is a p-adic unit. The inequality $b - a < -b - 1$ that is a hypothesis for the above lemma is satisfied by the conditions of (a) and is equivalent to the inequality condition of (b). Therefore, in either case, Lemma 4.3 applies and there is a nonempty $\Delta(u^2, 2t)$-invariant set $U \subset \mathbb{P}^1(\mathbb{Q})$ that is relatively open in $\mathbb{P}^1(\mathbb{Q}_p)$ and does not contain ∞. Since $\Delta(u^2, 2t)$ has only one orbit of cusps and ∞ is a cusp, U contains no cusps. The proposition follows. $\boxtimes$

Remark. This proposition generalizes the phenomenon witnessed by Long and Reid for $\Delta(3/4, 4)$ in Theorem 2.9 of [LR1], though in contrast to their example, it does not identify any rational hyperbolic fixed points. The proof of Lemma 4.3 does, however, give a lower bound for the number of orbits of $\Delta(u^2, 2t)$ on $\mathbb{P}^1(\mathbb{Q})$. Indeed, those x with $v_p(x) \in \{-e, e - a\}$ constitute a $\Delta(u^2, 2t)$-invariant set for each e satisfying condition 4.2.1. This fact partitions the set of valuations in the interval $[b + 1, a - b - 1]$ into pairs, except for the value $a/2$ (corresponding to the case $-e = e - a$) if it is an integer. We so obtain $\lfloor (a - 2b)/2 \rfloor$ sets of rationals that are not cusps, each invariant under $\Delta(u^2, 2t)$, thus giving a lower bound of $\lfloor (a - 2b)/2 \rfloor + 1$ for the number of orbits on $\mathbb{P}^1(\mathbb{Q})$.

The computations in the proof of Lemma 4.3 showed that, for a, b, e fixed and as given there, if we define U_1 and U_2 to be the sets of rationals with p-valuations, respectively, $e - a$ and $-e$, then $g_1^{\pm 1}$ exchanges U_1 and U_2, while $g_2^{\pm 1}$ fixes U_1 and U_2 setwise. (If $a = 2e$, then $U_1 = U_2$.) This shows that $U_1 \cup U_2$ is $\Delta(u^2, 2t)$-invariant. Since this phenomenon occurs numerous times below, we make a definition.

DEFINITION 4.5. Fix a $\Delta(u^2, 2t)$. Let U_1 and U_2 be nonempty, proper subsets of $\mathbb{P}^1(\mathbb{Q})$ such that g_1 and g_1^{-1} exchange U_1 and U_2 and g_2 and g_2^{-1} fix U_1 and U_2 setwise. Then (U_1, U_2) is a *cooperating pair* for the action of $\Delta(u^2, 2t)$; we also say U_1 and U_2 *cooperate for* $\Delta(u^2, 2t)$. When the context is clear, we will omit the reference to the group.

The chosen order of the generators in this definition is deliberate and will be an important detail in Section 4.2.4 below.

As a general principle, if U_1 and U_2 cooperate for $\Delta(u^2, 2t)$ and their union is not all of $\mathbb{P}^1(\mathbb{Q})$, then $\Delta(u^2, 2t)$ can be neither pseudomodular nor arithmetic.

This is because we can then form two disjoint, nonempty $\Delta(u^2, 2t)$-invariant sets, namely $U_1 \cup U_2$ and its complement in $\mathbb{P}^1(\mathbb{Q})$. Whichever one of these two sets does not contain ∞ also contains no other cusps since $\mathcal{C}(\Delta(u^2, 2t))$ is exactly the $\Delta(u^2, 2t)$-orbit of ∞.

Armed with this principle, we next strengthen part (a) of Proposition 4.4 by allowing the p-valuation of u^2 to be higher, assuming mild additional conditions on p and t.

PROPOSITION 4.6 (One-prime p-adic Obstruction II). *Let p be an odd prime. If $v_p(t) > 0$ and $v_p(u^2) = -1$ then the following pair of subsets of $\mathbb{P}^1(\mathbb{Q})$ is cooperative for $\Delta(u^2, 2t)$:*

$$U_1 = \{x : v_p(x) = 0 \text{ and } x \not\equiv -1 \ (\mathrm{mod}\, p)\},$$
$$U_2 = \{x : v_p(x) = -1 \text{ and } px \not\equiv pu^2 \ (\mathrm{mod}\, p)\}.$$

Consequently $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$ and $\Delta(u^2, 2t)$ is neither pseudo-modular nor arithmetic.

Proof. We mimic the proofs of the above lemmas. Write $t = ap$ and $u^2 = m/p$ where a is p-adically integral and m is p-adically a unit. The generators of $\Delta(u^2, 2t)$ and their inverses are represented in $\mathrm{PGL}_2(\mathbb{R})$ by

$$g_1 = \begin{pmatrix} p(ap-1) & m \\ p & p \end{pmatrix}, \qquad g_1^{-1} = \begin{pmatrix} p & -m \\ -p & p(ap-1) \end{pmatrix},$$
$$g_2 = \begin{pmatrix} m & m \\ p & ap^2 - m \end{pmatrix}, \qquad g_2^{-1} = \begin{pmatrix} ap^2 - m & -m \\ -p & m \end{pmatrix}.$$

Say $x \in U_1$. Accordingly, we write $x = r/s$ for integers r and s with $p \nmid r$, $p \nmid s$ and $v_p(r + s) = 0$ (i.e., $r \not\equiv -s \bmod p$). Compute:

$$p \cdot g_1 x = \frac{p(ap-1)r + ms}{r + s} \equiv \frac{ms}{r + s} \ (\mathrm{mod}\, p),$$
$$p \cdot g_1^{-1} x = \frac{pr - ms}{-r + (ap-1)s} \equiv \frac{-ms}{-r - s} \equiv \frac{ms}{r + s} \ (\mathrm{mod}\, p),$$
$$g_2 x = \frac{mr + ms}{pr + (ap^2 - m)s} \equiv \frac{m(r + s)}{-ms} \equiv \frac{r + s}{-s} \ (\mathrm{mod}\, p),$$
$$g_2^{-1} x = \frac{(ap^2 - m)r - ms}{-pr + ms} \equiv \frac{-m(r + s)}{ms} \equiv \frac{r + s}{-s} \ (\mathrm{mod}\, p).$$

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Since p does not divide r, we have that $ms/(r+s) \not\equiv m = pu^2$ and $(r+s)/(-s) \not\equiv -1$ modulo p, so $g_1^{\pm 1} x \in U_2$ and $g_2^{\pm 1} x \in U_1$.

If $x \in U_2$, then we write $x = r/ps$ with r and s integers not divisible by p and $v_p(r - ms) = 0$ (using that $m = pu^2$). Then:

$$g_1 x = \frac{p(ap-1)r + mps}{pr + p^2 s} = \frac{(ap-1)r + ms}{r + ps} \equiv \frac{-r + ms}{r} \pmod{p},$$

$$g_1^{-1} x = \frac{pr - mps}{-pr + p(ap-1)ps} = \frac{r - ms}{-r + p(ap-1)s} \equiv \frac{-r + ms}{r} \pmod{p},$$

$$p \cdot g_2 x = \frac{mr + mps}{r + (ap^2 - m)s} \equiv \frac{mr}{r - ms} \pmod{p},$$

$$p \cdot g_2^{-1} x = \frac{(ap^2 - m)r - mps}{-r + ms} \equiv \frac{mr}{r - ms} \pmod{p}.$$

Because p does not divide ms, we have modulo p that $mr/(r - ms) \not\equiv m$ and $(-r + ms)/r \not\equiv -1$. Thus $g_1^{\pm 1} x \in U_1$ and $g_2^{\pm 1} x \in U_2$.

It follows that U_1 and U_2 cooperate for $\Delta(u^2, 2t)$, so $U_1 \cup U_2$, a nonempty open set in the p-adic topology on $\mathbb{P}^1(\mathbb{Q})$, is $\Delta(u^2, 2t)$-invariant. Since $U_1 \cup U_2$ does not contain ∞, it contains no cusps and the proof is complete. ☑

Again weakening the valuation condition on u^2 and also assuming an additional congruence condition on u^2, we can still produce a cooperating pair of sets. This result is stated in a standalone lemma here, since we will use it below.

LEMMA 4.7. *Let p be an odd prime. If $v_p(t) > 0$ and u^2 is a p-adic unit not congruent to -1 modulo p, then the following pair of subsets of $\mathbb{P}^1(\mathbb{Q})$ is cooperative for $\Delta(u^2, 2t)$:*

$$U_1 = \{x : v_p(x) > 0 \text{ or } (v_p(x) = 0 \text{ and } x \equiv -1 \pmod{p})\},$$
$$U_2 = \{x : v_p(x) < 0 \text{ or } (v_p(x) = 0 \text{ and } x \equiv u^2 \pmod{p})\}.$$

Proof. Write $t = ap$ with $v_p(a) \geq 0$. Represent $g_1^{\pm 1}$ and $g_2^{\pm 1}$ in $\mathrm{PGL}_2(\mathbb{R})$ by:

$$g_1 = \begin{pmatrix} ap - 1 & u^2 \\ 1 & 1 \end{pmatrix}, \qquad\qquad g_1^{-1} = \begin{pmatrix} 1 & -u^2 \\ -1 & ap - 1 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} u^2 & u^2 \\ 1 & ap - u^2 \end{pmatrix}, \qquad\qquad g_2^{-1} = \begin{pmatrix} ap - u^2 & -u^2 \\ -1 & u^2 \end{pmatrix}.$$

Suppose $x \in \mathbb{P}^1(\mathbb{Q})$ and $v_p(x) > 0$. Write $x \doteq r/s$ with p dividing r. Then

$$g_1 x = \frac{(ap-1)r + u^2 s}{r+s} \equiv u^2 \equiv \frac{r - u^2 s}{-r + (ap-1)s} = g_1^{-1} x$$

and

$$g_2 x = \frac{u^2 r + u^2 s}{r + (ap - u^2)s} \equiv -1 \equiv \frac{(ap - u^2)r - u^2 s}{-r + u^2 s} = g_2^{-1} x$$

where the quotients are p-adically integral and equivalences are taken modulo p.

Next assume $v_p(x) = 0$ and $x \equiv -1 \pmod{p}$. Then

$$g_1 x = \frac{(ap-1)x + u^2}{x+1} \quad \text{and} \quad g_1^{-1} x = \frac{x - u^2}{-x + ap - 1}$$

have negative p-valuation and

$$g_2 x = \frac{u^2(x+1)}{x + ap - u^2} \quad \text{and} \quad g_2^{-1} x = \frac{(ap - u^2)x - u^2}{-x + u^2}$$

have positive p-valuation since $u^2 \not\equiv -1 \pmod{p}$. Taken with the above computations, this says that $g_1^{\pm 1} U_1 \subset U_2$ and $g_2^{\pm 1} U_1 \subset U_1$.

Now let x have negative p-valuation; write $x = r/s$ as above but with p dividing s. Using the above expressions in r and s, we find that

$$g_1^{\pm 1} x \equiv -1 \quad \text{and} \quad g_2^{\pm 1} x \equiv u^2 \pmod{p}.$$

Finally, select x with $v_p(x) = 0$ and $x \equiv u^2 \pmod{p}$. Using the expressions in x above, we determine that $v_p(g_1^{\pm 1} x) > 0$ and $v_p(g_2^{\pm 1} x) < 0$, again using that $u^2 \not\equiv -1 \pmod{p}$. Therefore $g_1^{\pm 1} U_2 \subset U_1$ and $g_2^{\pm 1} U_2 \subset U_2$. ☑

The lemma gives another one-prime obstruction.

PROPOSITION 4.8 (One-prime p-adic Obstruction III). *Let p be a prime greater than 3. If $v_p(t) > 0$ and u^2 is a p-adic unit not congruent to -1 modulo p, then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$, whence $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic.*

Proof. Let U_1 and U_2 be as in the preceding lemma. Their union and thus the complement of their union,

$$U := \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) = 0, \ x \not\equiv -1, \ x \not\equiv u^2 \pmod{p}\}$$

is invariant under the action of $\Delta(u^2, 2t)$. This U is nonempty since $p > 3$. It is also p-adically open and contains no cusps since it misses ∞. ☑

4.2.2 Two-prime obstructions

Notice that if we take p to be 3 in the statement of Proposition 4.8 then we get a cooperating pair of complementary sets in $\mathbb{P}^1(\mathbb{Q})$. We cannot produce with this pair and the methods above a $\Delta(u^2, 2t)$-invariant, 3-adically open, nonempty, proper subset of $\mathbb{P}^1(\mathbb{Q})$. However, this circumstance still allows us to produce some density obstructions and so motivates the next definition.

DEFINITION 4.9. For a prime p, a pair of parameters $(u^2, 2t)$ is *borderline with respect to p* (or *p-borderline*) if there is a cooperating pair of p-adically open, disjoint sets for $\Delta(u^2, 2t)$ whose union is $\mathbb{P}^1(\mathbb{Q})$.

Remark. Having $(u^2, 2t)$ be borderline with respect to only one prime is not sufficient to produce a density obstruction. Indeed, $\Delta(5/7, 6)$ is pseudomodular [LR1] although $(u^2, 2t) = (5/7, 6)$ is borderline with respect to 7.

Our interest in borderline cases is justified by the following statement.

PROPOSITION 4.10. *If $(u^2, 2t)$ is borderline with respect to two distinct primes p and q then the set of cusps of $\Delta(u^2, 2t)$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$. In particular, $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic.*

Proof. Our idea is to produce a cooperating pair by intersecting sets from the two implicitly given pairs.

Suppose (U_1, U_2) and (V_1, V_2) are cooperating pairs for $\Delta(u^2, 2t)$ that demonstrate, respectively, that $(u^2, 2t)$ is p-borderline and q-borderline. Define the following two subsets of $\mathbb{P}^1(\mathbb{Q})$:

$$X_1 = (U_1 \cap V_1) \cup (U_2 \cap V_2), \quad X_2 = (U_1 \cap V_2) \cup (U_2 \cap V_1).$$

Since the U_i are p-adically open and the V_j are q-adically open, X_1 and X_2 are open in $\mathbb{P}^1(\mathbb{Q})$ in the topology induced by diagonal inclusion into $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$. Because $U_1 \cap U_2$ and $V_1 \cap V_2$ are each empty, X_1 and X_2 do not intersect. Moreover, each X_k is nonempty by the density of $\mathbb{P}^1(\mathbb{Q})$ in $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$. Finally, $X_1 \cup X_2$ is of course $\mathbb{P}^1(\mathbb{Q})$ by definition of the U_i and V_j.

Armed with all of this, we assume without loss of generality (relabeling the V_j if necessary) that ∞ is not in X_1. By construction, $U_1 \cap V_1$ and $U_2 \cap V_2$ cooperate

for $\Delta(u^2, 2t)$. Therefore, X_1 is $\Delta(u^2, 2t)$-invariant. Now X_1 contains no cusps as it does not contain ∞ and as $\Delta(u^2, 2t)$ has only one orbit of cusps. The fact that X_1 is nonempty and open in the topology induced by $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$ completes the proof. $\qquad\qquad\boxtimes$
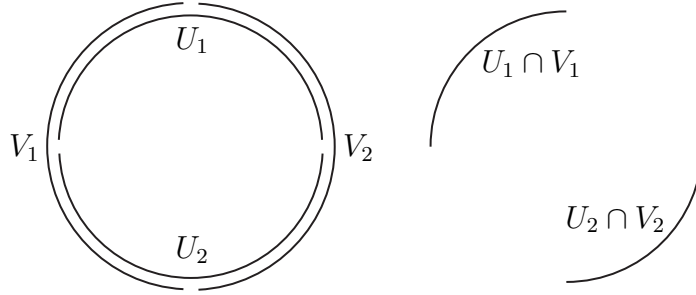

Figure 4.1: Produce a third cooperating pair using intersections

Now we can produce effective two-prime obstructions simply by finding numerous (effective) sets of hypotheses under which $(u^2, 2t)$ is p-borderline and then pairing these together for different primes p.

First we state the above-noted variation of Proposition 4.8 with $p = 3$.

LEMMA 4.11 (Borderline Lemma I). *If $v_3(t) > 0$ and u^2 is a 3-adic unit congruent to 1 modulo 3, then the following pair of subsets of $\mathbb{P}^1(\mathbb{Q})$ is cooperative for $\Delta(u^2, 2t)$:*

$$U_1 = \{x : v_3(x) > 0 \ or \ (v_3(x) = 0 \ and \ x \equiv -1 \ (\mathrm{mod}\, 3))\},$$
$$U_2 = \{x : v_3(x) < 0 \ or \ (v_3(x) = 0 \ and \ x \equiv 1 \ (\mathrm{mod}\, 3))\}.$$

In particular $(u^2, 2t)$ is 3-borderline.

The next two lemmas, which also exhibit borderline sets of parameters, are given by weakening the inequalities on $v_p(u^2)$ in each of the parts of Proposition 4.4. (This also justifies the term "borderline", as can be seen in, e.g., Figure 4.3 below.)

LEMMA 4.12 (Borderline Lemma II). *Let p be prime, and assume that $v_p(t) \geq 0$ and $v_p(u^2) = -1$. Define $U_1 = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) \geq 0\}$ and $U_2 = \mathbb{P}^1(\mathbb{Q}) \setminus U_1$. Then U_1 and U_2 cooperate for the action of $\Delta(u^2, 2t)$ on $\mathbb{P}^1(\mathbb{Q})$, so $(u^2, 2t)$ is p-borderline.*

Proof. Write $u^2 = m/p$ for m a p-adic unit. Fix an $x \in \mathbb{P}^1(\mathbb{Q})$ and write $x \doteq r/s$. Represent g_1 and g_2 in $\mathrm{PGL}_2(\mathbb{R})$ by matrices whose entries are p-adically integral:

$$g_1 = \begin{pmatrix} (t-1)p & m \\ p & p \end{pmatrix}, \qquad\qquad g_2 = \begin{pmatrix} m & m \\ p & tp - m \end{pmatrix}.$$

Then

$$v_p(g_1 x) = v_p((t-1)pr + ms) - v_p(r+s) - 1,$$
$$v_p(g_2 x) = v_p(r+s) - v_p(pr + (tp - m)s).$$

If $x \in U_1$, then p does not divide s, so

$$v_p(g_1 x) = -v_p(r+s) - 1 < 0 \quad \text{and} \quad v_p(g_2 x) = v_p(r+s) \geq 0,$$

whence $g_1 x \in U_2$ and $g_2 x \in U_1$. On the other hand, if $x \in U_2$, then p divides s and does not divide r, so

$$v_p(g_1 x) = 1 + v_p((t-1)r + ms/p) - 1 \geq 0, \quad \text{and}$$
$$v_p(g_2 x) = -1 - v_p(r + (tp - m)s/p) < 0.$$

Thus $g_1 x \in U_1$ and $g_2 x \in U_2$. It follows that g_1 maps U_1 into U_2 and maps U_2 into U_1, and that g_2 maps each U_i into itself. Since U_1 and U_2 are complements of each other in $\mathbb{P}^1(\mathbb{Q})$, U_1 and U_2 cooperate. ☐

Remark. If p is odd and $v_p(t) > 0$ then the above lemma is superseded by Proposition 4.6. For the sake of brevity, we will omit this note in our statements of two-prime obstructions below.

LEMMA 4.13 (Borderline Lemma III). *Let p be prime, and suppose that $v_p(t) < 0$ and that $v_p(u^2) = 2v_p(t) - 1$. Define $U_1 = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) \geq v_p(t)\}$ and $U_2 = \mathbb{P}^1(\mathbb{Q}) \setminus U_1$. Then U_1 and U_2 cooperate for the action of $\Delta(u^2, 2t)$ on $\mathbb{P}^1(\mathbb{Q})$, so $(u^2, 2t)$ is p-borderline.*

Proof. Write $t = n/p^a$ and $u^2 = m/p^{2a+1}$ with n and m p-adic units and a a positive integer. Fix an $x \in \mathbb{P}^1(\mathbb{Q})$ and write $x \doteq r/s$. Represent g_1 and g_2 in $\mathrm{PGL}_2(\mathbb{R})$ by matrices whose entries are p-adically integral:

$$g_1 = \begin{pmatrix} np^{a+1} - p^{2a+1} & m \\ p^{2a+1} & p^{2a+1} \end{pmatrix}, \qquad g_2 = \begin{pmatrix} m & m \\ p^{2a+1} & np^{a+1} - m \end{pmatrix}.$$

Compute:

$$v_p(g_1 x) = v_p(p^{a+1}(n - p^a)r + ms) - v_p(r + s) - (2a + 1), \qquad (4.2.2)$$

$$v_p(g_2 x) = v_p(r + s) - v_p(p^{2a+1}r + (np^{a+1} - m)s). \qquad (4.2.3)$$

First we see how equation 4.2.2 responds to various values of x. If $v_p(x) \geq 0$, then $p \nmid s$ and so

$$v_p(g_1 x) = 0 - v_p(r + s) - (2a + 1) < -a = v_p(t).$$

If $v_p(t) \leq v_p(x) < 0$, then $p \nmid r$, $p \mid s$ and $p^{a+1} \nmid s$. Thus the first term on the right hand side of equation 4.2.2 is between 1 and a inclusive. Since $v_p(r + s) = 0$, it follows that $v_p(g_1 x) \leq a - (2a + 1) = -a - 1 < v_p(t)$. These two computations collectively exhaust all $x \in U_1$, so g_1 maps U_1 to U_2. For $x \in U_2$, i.e. with $p \nmid r$ and $p^{a+1} \mid s$, we see

$$v_p(g_1 x) = a + 1 + v_p((n - p^a)r + ms/p^{a+1}) - (2a + 1) \geq -a = v_p(t).$$

Therefore $g_1 x \in U_1$. Since the sets U_1 and U_2 are complementary in $\mathbb{P}^1(\mathbb{Q})$, g_1 and g_1^{-1} exchange them.

Similarly considering equation 4.2.3, we have the cases

$$0 \leq v_p(x): \qquad v_p(g_2 x) = v_p(r + s) \geq 0 > v_p(t);$$

$$v_p(t) \leq v_p(x) < 0: \quad v_p(g_2 x) = 0 - v_p(s) = v_p(x) \geq v_p(t);$$

$$v_p(x) < v_p(t): \qquad v_p(g_2 x) = -v_p(p^{2a+1}r + (np^{a+1} - m)s) \leq -(a + 1) < v_p(t).$$

Consequently, g_2 maps each of the sets U_1 and U_2 into itself. Again we use that U_1 and U_2 are complements of each other to say that both g_2 and g_2^{-1} fix U_1 and U_2 setwise.

We conclude that U_1 and U_2 cooperate for $\Delta(u^2, 2t)$. ☐

Now, as promised, we can produce explicit two-prime obstructions using the preceding lemmas.

PROPOSITION 4.14 (Two-prime p-adic Obstructions I). *Fix $(u^2, 2t)$. Assume that $v_3(t)$ is positive and that u^2 is a 3-adic unit congruent to 1 modulo 3. Let p be a prime distinct from 3. If* (a) $v_p(t) \geq 0$ *and* $v_p(u^2) = -1$ *or if* (b) $v_p(t) < 0$ *and* $v_p(u^2) = 2v_p(t) - 1$, *then* $\mathcal{C}(\Delta(u^2, 2t))$ *is not dense in* $\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_p)$ *and* $\Delta(u^2, 2t)$ *is neither pseudomodular nor arithmetic.*

Proof. For part (a), select Lemmas 4.11 and 4.12 and invoke Proposition 4.10. In this case, one subset of $\mathbb{P}^1(\mathbb{Q})$ containing no cusps of $\Delta(u^2, 2t)$ is

$$\left\{x : v_p(x) < 0 \text{ and } \left(v_3(x) > 0 \text{ or } (v_3(x) = 0 \text{ and } x \equiv -1 \ (3))\right)\right\}$$
$$\cup \left\{x : v_p(x) \geq 0 \text{ and } \left(v_3(x) < 0 \text{ or } (v_3(x) = 0 \text{ and } x \equiv 1 \ (3))\right)\right\}.$$

Do the same for part (b), except selecting Lemma 4.13 in place of 4.12. Here, a set that misses $\mathcal{C}(\Delta(u^2, 2t))$ is

$$\left\{x : v_p(x) < v_p(t) \text{ and } \left(v_3(x) > 0 \text{ or } (v_3(x) = 0 \text{ and } x \equiv -1 \ (3))\right)\right\}$$
$$\cup \left\{x : v_p(x) \geq v_p(t) \text{ and } \left(v_3(x) < 0 \text{ or } (v_3(x) = 0 \text{ and } x \equiv 1 \ (3))\right)\right\}.$$

$$\boxtimes$$

PROPOSITION 4.15 (Two-prime p-adic Obstructions II). *Fix $(u^2, 2t)$ and let p and q be distinct primes. If:*

(a) $v_p(t) \geq 0$, $v_q(t) \geq 0$ *and* $v_p(u^2) = v_q(u^2) = -1$;

(b) $v_p(t) \geq 0$, $v_q(t) < 0$, $v_p(u^2) = -1$ *and* $v_q(u^2) = 2v_q(t) - 1$; *or*

(c) $v_p(t) < 0$, $v_q(t) < 0$, $v_p(u^2) = 2v_p(t) - 1$ *and* $v_q(u^2) = 2v_q(t) - 1$,

then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$ and $\Delta(u^2, 2t)$ is neither pseudo-modular nor arithmetic.

Proof. In each case we invoke Proposition 4.10 as in the proof of the previous proposition.

Case (a): Select Lemma 4.12 for each of the primes p and q. A set that misses $\mathcal{C}(\Delta(u^2, 2t))$ is

$$\left\{x : (v_p(x) \geq 0 \text{ and } v_q(x) < 0) \text{ or } (v_p(x) < 0 \text{ and } v_q(x) \geq 0)\right\}.$$

Case (b): Select Lemma 4.12 using the prime p and Lemma 4.13 using the prime q. A set that misses $\mathcal{C}(\Delta(u^2, 2t))$ is

$$\left\{x : (v_p(x) \geq 0 \text{ and } v_q(x) < v_q(t)) \text{ or } (v_p(x) < 0 \text{ and } v_q(x) \geq v_q(t))\right\}.$$

Case (c): Select Lemma 4.13 for each of the primes p and q. A set that misses $\mathcal{C}(\Delta(u^2, 2t))$ is

$$\{x : (v_p(x) \geq v_p(t) \text{ and } v_q(x) < v_q(t)) \text{ or } (v_p(x) < v_p(t) \text{ and } v_q(x) \geq v_q(t))\}.$$

$\square$

Were we to be given a triple of pairs of subsets of $\mathbb{P}^1(\mathbb{Q})$ that are borderline with respect to mutually distinct primes, we could pair any two of them to obtain a two-prime obstruction. In this sense, taking more than two borderline pairs does not produce a density obstruction in cases where we could not otherwise find one.

A conceivably more fruitful type of generalization would be to iterate the process that we now describe. For a pair of subsets U_1 and U_2 of $\mathbb{P}^1(\mathbb{Q})$, let condition (Q_1) on this pair be the condition that they cooperate. Let condition (Q_2) be some condition on such a pair such that if we take another pair (V_1, V_2) satisfying (Q_2), then each of $(U_1 \cap V_1, U_2 \cap V_2)$ and $(U_1 \cap V_2, U_2 \cap V_1)$ satisfies (Q_1). For example, we may take (Q_2) to be the condition of being borderline with respect to a prime p. (Accordingly, we may need to add some parameters to the definitions of our conditions, such as the prime with respect to which a pair is borderline.) Iteratively, for j at least 3, let (Q_j) be a condition on a pair such as above so that, given two pairs as above, the noted pairs of intersections each satisfy condition (Q_{j-1}). Then, with sufficiently many pairs of subsets of $\mathbb{P}^1(\mathbb{Q})$ satisfying a fixed (Q_j)—seemingly, rather a lot as j increases—one could produce more obstructions.

We have yet to investigate the possibilities so described, but pairs of sets satisfying, e.g., some condition (Q_3), may be easier to find than their borderline or cooperating counterparts. This section and the previous one of course address the first and second steps of this iteration.

4.2.3 Corollaries

The one- and two-prime obstructions given above are summarized by Figures 4.2 and 4.3. Table A.3 in the appendix also collects the above results.

Now we give some consequences of the above obstructions that are comparable with the tables of [LR1], which are reproduced (and augmented) in the appendix.
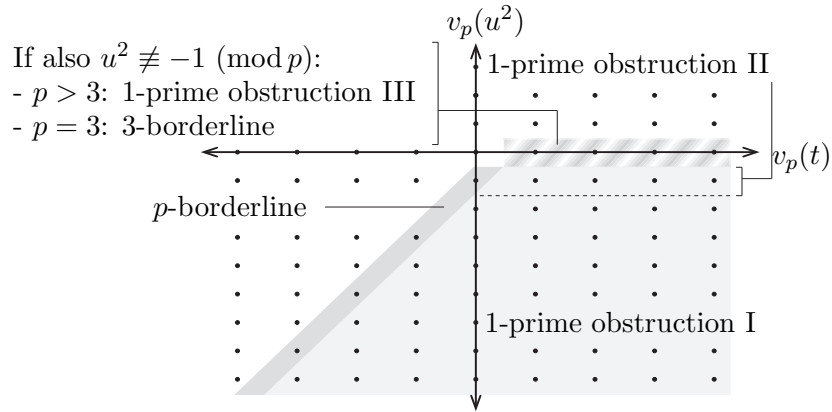
Figure 4.2: Plot of obstructions for $p \geq 3$ (1×1 grid)
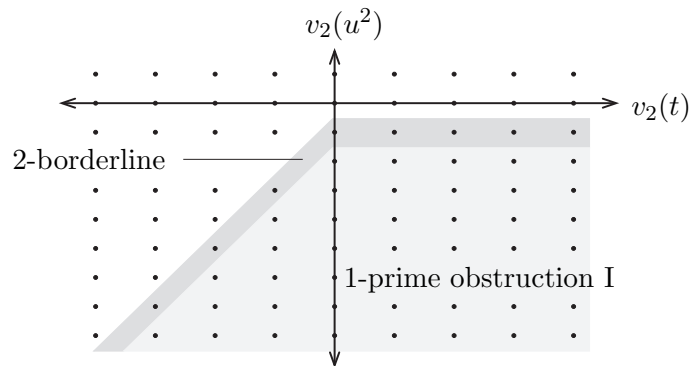

Figure 4.3: Plot of obstructions for $p = 2$ (1×1 grid)

The next result, in particular, applies to all of the visibly nonarithmetic, non-pseudomodular examples in Table A.1, with $t = 2$, and many of those in Table A.2, where $t = 3$.

COROLLARY 4.16. *If t is an integer (necessarily at least 2) and u^2 has composite denominator, then $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic.*

Proof. If the denominator of u^2 is not square-free, then there is a prime p such that $v_p(u^2) < -1$. The result then follows from part (a) of Proposition 4.4. Otherwise, there are two distinct primes p and q such that $v_p(u^2) = v_q(u^2) = -1$ and Proposition 4.15(a) affirms the desired result. ☑

More generally, we have:

COROLLARY 4.17. *Let t be an integer and suppose $\Delta(u^2, 2t)$ is pseudomodular or arithmetic. Then*

(a) *u^2 has prime or unit denominator, say p,*

(b) *if this p is an odd prime, then p does not divide t, and*

(c) *if q is a prime dividing t and $q > 3$, then u^2 (which is, by part (b), q-adically integral) is congruent to 0 or -1 modulo q; and*

(d) *if 3 divides t and $p \neq 1$, then u^2 is congruent to 0 or -1 modulo 3.*

Proof. Part (a) is a reformulation of the preceding corollary. Part (b) follows immediately from Proposition 4.6. Part (c) follows from Proposition 4.8 and part (d) is a consequence Proposition 4.14(a). ☑

A particular implication of this last result is that if $\Delta(u^2, 2t)$ is pseudomodular and t is an integer divisible by large primes, then each such prime enforces a strong congruence condition on u^2.

In Table A.2, where $t = 3$, the corollary ensures that those groups $\Delta(u^2, 2t)$ with u^2 among the set

$$\{1/3,\ 2/3,\ 2/5,\ 1/7,\ 4/7,\ 2/11,\ 5/11,\ 8/11\},$$

in addition to those with composite denominator, are not pseudomodular. Some groups in this table whose cusp sets are known *a priori* to be proper subsets of $\mathbb{P}^1(\mathbb{Q})$ are not eliminated by the above propositions. We will show in Section 4.3 that among these groups are some whose sets of cusps are in fact dense in *all* finite products of distinct spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$.

Obviously, now, infinitely many nonarithmetic $\Delta(u^2, 2t)$ are not pseudomodular. A strictly stronger statement is true. For the sake of its proof, we briefly identify certain subgroups of the groups $\Delta(u^2, 2t)$. For each $(u^2, 2t)$, let $\Lambda(u^2, 2t)$ be the kernel of the group homomorphism from $\Delta(u^2, 2t)$ to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $g_1 \mapsto (1, 0)$ and $g_2 \mapsto (0, 1)$. More explicit information about the groups $\Lambda(u^2, 2t)$ is given in Chapter 5.

THEOREM 4.18. *There are infinitely many commensurability classes of non-arithmetic, non-pseudomodular Fricke groups with rational cusps.*

Proof. We shall establish that the family of groups $\Lambda(1/4, 2t)$, with t ranging over integers at least 2, spans infinitely many commensurability classes. This suffices since $\Lambda(1/4, 2t)$ is always of finite index in $\Delta(1/4, 2t)$.

First, we claim that $\Lambda(1/4, 2t)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z}[(4t - 5)^{-1}])$. The computations justifying this are in Section 5.2. Using Dirichlet's Theorem, we select a sequence of distinct integers $\{t_i\}$ with $t_i \geq 2$ (ensuring $\Delta(1/4, 2t_i)$ is defined) such that $q_i := 4t_i - 5$ is prime for each i.

Fix index i. Since $\Lambda_i := \Lambda(1/4, 2t_i) \subset \mathrm{PSL}_2(\mathbb{Z}[q_i^{-1}])$, the traces of elements in Λ_i are p-adically integral for $p \neq q_i$. Suppose now that $G \subset \Lambda_i$ is a subgroup of finite index. The Fuchsian group G has cusps and is of finite coarea, so by Theorem 1 of [Tak], if G has integral traces (i.e., all traces of elements in G are integers) then G is arithmetic. But we have already established in Corollary 4.16 that the cusp set of Λ_i is not $\mathbb{P}^1(\mathbb{Q})$, so Λ_i, and therefore G, is not arithmetic. It follows that G cannot have integral traces. Consequently, since $G \subset \mathrm{PSL}_2(\mathbb{Z}[q_i^{-1}])$, there is an element of G with trace not q_i-adically integral.

Now consider Λ_i and Λ_j for $i \neq j$. Let H_i (resp. H_j) be finite index in Λ_i (resp. H_j). Then, for $k \in \{i, j\}$, the traces of H_k are p-adically integral except for $p = q_k$ and H_k has a trace not integral at q_k. Since $q_i \neq q_j$ and trace is invariant under conjugation, H_i and H_j are not conjugate subgroups of $\mathrm{PSL}_2(\mathbb{R})$. This holds

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regardless of choice of H_i and H_j, so Λ_i and Λ_j are not commensurable. It follows that $\{\Delta(1/4, 2t_i)\}_i$ is a family of nonarithmetic, non-pseudomodular groups with rational cusps, spanning infinitely many commensurability classes. $\boxed{}$

4.2.4 Using equivalences

Here we employ the equivalences of Section 3.2 to extend our one- and two-prime obstructions. The results so obtained are as effective as those above in terms of hypotheses on the parameters u and t but, in contrast, do not explicitly describe $\Delta(u^2, 2t)$-invariant sets of rationals.

We must of course note that equivalence of groups $\Delta(u^2, 2t)$ preserves topological properties of sets of cusps with respect to the topologies induced by products of spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$. This is because $\mathrm{GL}_2(\mathbb{Q})$ acts continuously on $\mathbb{P}^1(\mathbb{Q})$ with respect to all such topologies.

If U_1 and U_2 form a cooperating pair for $\Delta(u^2, 2t)$ and we have through an equivalence that $\Delta(v^2, 2s) = A\Delta(u^2, 2t)A^{-1}$ for an $A \in \mathrm{PGL}_2(\mathbb{Q})$, AU_1 and AU_2 do *not* necessarily form a cooperating pair for $\Delta(v^2, 2s)$. The reason for this is that the definition of cooperation is not symmetric in the marked generators of $\Delta(u^2, 2t)$. For example, by the proof of Proposition 3.4(a), we have that the marked generators of $^\beta\Delta(u^2, 2t)$ are Ag_2A^{-1} and Ag_1A^{-1}, where g_1 and g_2 are the marked generators of $\Delta(u^2, 2t)$. Therefore, if h_1 and h_2 are the marked generators of $^\beta\Delta(u^2, 2t)$, then h_1 and its inverse fix the sets AU_i setwise while the elements $h_2^{\pm 1}$ exchange them. Note that this phenomenon does not occur when we use σ, ψ or ψ^{-1} in place of β.

This motivates a new definition in which we must also take into account that the borderline properties are defined in terms of cooperation.

DEFINITION 4.19. Fix a $\Delta(u^2, 2t)$. Let U_1 and U_2 be nonempty, proper subsets of $\mathbb{P}^1(\mathbb{Q})$ such that g_1 and g_1^{-1} fix U_1 and U_2 setwise and g_2 and g_2^{-1} exchange U_1 and U_2. Then (U_1, U_2) is a *flip-cooperating pair* for the action of $\Delta(u^2, 2t)$. For a prime p, a pair of parameters $(u^2, 2t)$ is *flip-borderline with respect to p* (or *p-flip-borderline*) if there is a flip-cooperating pair of p-adically open, disjoint sets for $\Delta(u^2, 2t)$ whose union is $\mathbb{P}^1(\mathbb{Q})$.

The distinction made in this definition is necessary because we cannot produce two-prime obstructions by taking a pair $(u^2, 2t)$ that is for distinct primes p and

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q both p-borderline and q-flip-borderline. Pairs of flip-borderline conditions still produce two-prime obstructions as above; likewise, flip-cooperating sets still yield one-prime obstructions.

We now let $\mathcal{P}_{\mathbb{Q}} \subset \mathcal{P}$ comprise those pairs $(u^2, 2t)$ such that u^2 and t are rational. With $\mathcal{G}$ defined as in Section 3.2, $\mathcal{P}_{\mathbb{Q}}$ is $\mathcal{G}$-invariant. The action of $\mathcal{G}$ on $\mathcal{P}_{\mathbb{Q}}$ will now be used to extend our obstructions above.

PROPOSITION 4.20. *Let p be prime. If $2 \leq v_p(u^2) \leq 2v_p(t) - 2$ then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$.*

Proof. Define

$$\mathcal{S}_1 = \{(u^2, 2t) \in \mathcal{P}_{\mathbb{Q}} : v_p(t) \geq 0, \ v_p(u^2) \leq -2\},$$
$$\mathcal{S}_2 = \{(u^2, 2t) \in \mathcal{P}_{\mathbb{Q}} : v_p(t) < 0, \ v_p(u^2) \leq 2v_p(t) - 2\}.$$

For $(u^2, 2t)$ in either of these sets, $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$ according to Proposition 4.4.

We now compute the images $\beta \mathcal{S}_i$ for $i = 1, 2$. Assign $x = v_p(t)$ and $y = v_p(u^2)$. If $(v^2, 2s) = (u^{-2}, 2tu^{-2}) = \beta(u^2, 2t)$, then $v_p(s) = x - y$ and $v_p(v^2) = -y$.

For $(u^2, 2t) \in \mathcal{S}_1$, we have that $x \geq 0$ and $y \leq -2$. Therefore $v_p(s) \geq v_p(v^2) \geq 2$. Conversely, if this condition on s and v^2 holds, then $x - y \geq -y \geq 2$ and so x is necessarily nonnegative and y is at most -2. Therefore

$$\beta \mathcal{S}_1 = \{(u^2, 2t) \in \mathcal{P}_{\mathbb{Q}} : v_p(t) \geq v_p(u^2) \geq 2\}.$$

If $(u^2, 2t) \in \mathcal{S}_2$, then $x \leq -1$ and $y \leq 2x - 2$. Note that $x \leq -1$ if and only if $v_p(s) = x - y < -y = v_p(v^2)$. Also, $y \leq 2x - 2$ exactly when

$$v_p(v^2) = -y \leq 2(x - y) - 2 = 2v_p(s) - 2.$$

Hence

$$\beta \mathcal{S}_2 = \{(u^2, 2t) \in \mathcal{P}_{\mathbb{Q}} : v_p(t) < v_p(u^2) \leq 2v_p(t) - 2\}.$$

The union of $\beta \mathcal{S}_1$ and $\beta \mathcal{S}_2$ is exactly the subset of $\mathcal{P}_{\mathbb{Q}}$ described by the hypotheses of this proposition. Since equivalence preserves topological properties of cusp sets, this completes the proof. ◻

PROPOSITION 4.21. *Let p be an odd prime. If $v_p(t) \geq 2$ and $v_p(u^2) = 1$ then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$.*

Proof. Set $\mathcal{S} = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) > 0, \ v_p(u^2) = -1\}$. Proposition 4.6 says that for all $(u^2, 2t) \in \mathcal{S}$, the set of cusps of $\Delta(u^2, 2t)$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$—this statement uses that p is an odd prime.

We show that the given hypotheses describe the set $\beta\mathcal{S}$. Suppose x, y, v and s are as in the proof of the preceding result. If $(u^2, 2t) \in \mathcal{S}$ then $x > 0$ and $y = -1$. Thus $v_p(s) \geq 2$ and $v_p(v^2) = 1$. Conversely, if $v_p(s) \geq 2$ and $v_p(v^2) = 1$, then $x - y \geq 2$ and $-y = 1$, which together imply that $v_p(u^2) = y$ is -1 and that $v_p(t) = x$ is positive. We conclude that

$$\beta\mathcal{S} = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) \geq 2, \ v_p(u^2) = 1\},$$

which completes the proof. $\boxdot$

The previous two propositions are derived from the first and second One-prime Obstructions. Curiously, the third One-prime Obstruction (Proposition 4.8) does not similarly extend because the subset of $\mathcal{P}_\mathbb{Q}$ described by its hypotheses is $\mathcal{G}$-invariant. We record this now.

PROPOSITION 4.22. *Let p be a prime. Then the set*

$$\mathcal{S} = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) \geq 1, \ v_p(u^2) = 0, \ u^2 \not\equiv -1 \ (\mathrm{mod}\, p)\}$$

is $\mathcal{G}$-invariant.

Proof. When p is 2, the set $\mathcal{S}$ is empty and this is obviously true, so assume hereafter that $p > 2$. It suffices to show that β, σ, ψ and ψ^{-1} map $(u^2, 2t)$ into $\mathcal{S}$ for each $(u^2, 2t)$ in $\mathcal{S}$. Fix a pair $(u^2, 2t)$ in $\mathcal{S}$.

If $(v^2, 2s) = \beta(u^2, 2t)$ then $v_p(v^2) = v_p(u^{-2}) = 0$ and $v^2 = (u^2)^{-1}$ is not equivalent to -1 mod p. Also, $v_p(s) = v_p(tu^{-2}) = v_p(t) \geq 1$. Thus $\beta\mathcal{S} \subset \mathcal{S}$.

Next, if $(v^2, 2s) = \sigma(u^2, 2t)$ then since $u^2 + 1$ and $-1 + t - u^2$ are p-adic units, $s = t(u^2 + 1)/(-1 + t - u^2)$ has the same (positive) valuation as t. Additionally, $v^2 = u^2$ so v^2 is a p-adic unit not congruent to -1 mod p, whence $\sigma\mathcal{S} \subset \mathcal{S}$.

Thirdly, if $(v^2, 2s) = \psi(u^2, 2t)$, then $v^2 = (u^2 + 1)^2/(-1 + t - u^2)$ is a quotient of p-adic units. Modulo p, v^2 is congruent to $-(u^2 + 1)$, which is not congruent to -1 by hypothesis. Since here s is again $t(u^2 + 1)/(-1 + t - u^2)$, we find that $\psi\mathcal{S} \subset \mathcal{S}$.

Finally, setting $(v^2, 2s) = \psi^{-1}(u^2, 2t)$, we have that $v^2 = -1 + t - u^2$ is a p-adic unit congruent to $-(u^2 + 1)$ mod p and so not congruent to -1 by hypothesis on u^2. Here $s = t(t - u^2)/u^2$ is a product of t and a p-adic unit, so its valuation is positive. Therefore $\psi^{-1}\mathcal{S} \subset \mathcal{S}$. ☒

Now we translate our Borderline Lemmas above into flip-borderline conditions using the same techniques.

PROPOSITION 4.23. *Let p be prime. If* (a) $v_p(t) \geq 1$ *and* $v_p(u^2) = 1$ *or* (b) $v_p(t) \geq 2$ *and* $v_p(u^2) = 2v_p(t) - 1$, *then* $(u^2, 2t)$ *is flip-borderline with respect to p.*

Proof. Define
$$\mathcal{S}_1 = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) \geq 0, \ v_p(u^2) = -1\},$$
$$\mathcal{S}_2 = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) < 0, \ v_p(u^2) = 2v_p(t) - 1\}.$$

Pairs $(u^2, 2t)$ in either of these sets are p-borderline by Lemmas 4.12 and 4.13.

We will identify the images $\beta\mathcal{S}_i$ as we did in Proposition 4.20. The pairs in the sets $\beta\mathcal{S}_i$ will be p-flip-borderline since the equivalence β is given by exchanging marked generators and then conjugating. Again set $x = v_p(t)$ and $y = v_p(u^2)$ and recall that if $(v^2, 2s) = \beta(u^2, 2t)$, then $v_p(s) = x - y$ and $v_p(v^2) = -y$.

For part (a), we have
$$(u^2, 2t) \in \mathcal{S}_1 \iff x \geq 0 \text{ and } y = -1 \iff v_p(s) \geq 1 \text{ and } v_p(v^2) = 1,$$
so
$$\beta\mathcal{S}_1 = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) \geq 1, \ v_p(u^2) = 1\}.$$

As to part (b), note first that $y = v_p(u^2) = 2v_p(t) - 1 = 2x - 1$ if and only if $v_p(v^2) = -y = 2(x - y) - 1 = 2v_p(s) - 1$. Given that $y = 2x - 1$, we also have that $v_p(s) = x - y \geq 2$ exactly when $x - (2x - 1) \geq 2$, which holds if and only if $v_p(t) = x \leq -1$. Thus
$$\beta\mathcal{S}_2 = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_p(t) \geq 2, \ v_p(u^2) = 2v_p(t) - 1\}.$$

☒

The final proposition of this section invokes Borderline Lemma I.

PROPOSITION 4.24. *If $v_3(t) > 0$ and u^2 is a 3-adic unit congruent to 1 modulo 3, then $(u^2, 2t)$ is 3-flip-borderline.*

Proof. Let

$$\mathcal{S} = \{(u^2, 2t) \in \mathcal{P}_\mathbb{Q} : v_3(t) > 0, \ v_3(u^2) = 0, \ u^2 \equiv 1 \ (\mathrm{mod}\, 3)\}.$$

Points in $\mathcal{S}$ are 3-borderline by Lemma 4.11. Invoke Proposition 4.22 with $p = 3$ to see that $\mathcal{S}$ is $\mathcal{G}$-invariant. In particular $\beta\mathcal{S} = \mathcal{S}$. ☒

The results from this section together with the original one-prime obstructions and borderline results we found are plotted together in Figures 4.4 and 4.5.
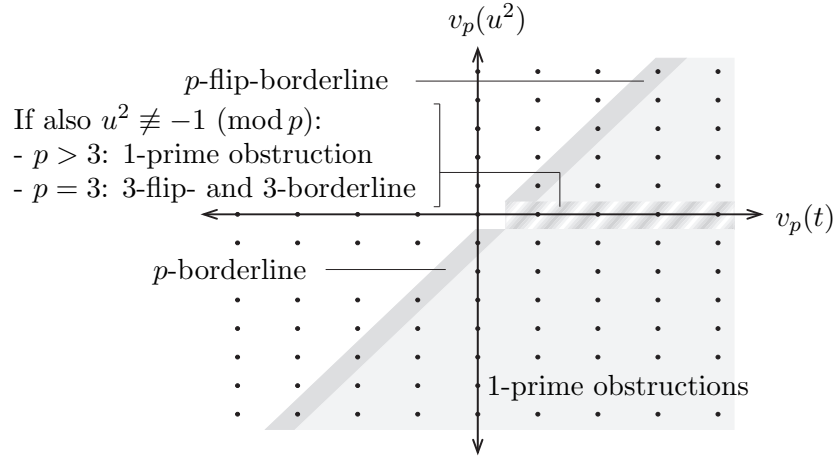
Figure 4.4: Plot of extended obstructions for $p \geq 3$ (1×1 grid)

Though we do not state any more two-prime obstructions explicitly, we note that we can pair hypotheses from the previous two propositions just as we did with the Borderline Lemmas of Section 4.2.2.

We can continue producing obstructions by applying generators of $\mathcal{G}$ to the sets of hypotheses of results in this section. These hypotheses seem to become increasingly complicated as we repeatedly apply generators of $\mathcal{G}$. For example, by applying ψ to the set $\beta\mathcal{S}$ in the proof of Proposition 4.21, we can show (we claim) that if p is an odd prime, $v_p(t) \geq 2$, $v_p(u^2) = 0$ and $v_p(u^2 + 1) = 1$ then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$. Because of this increased complexity, we have
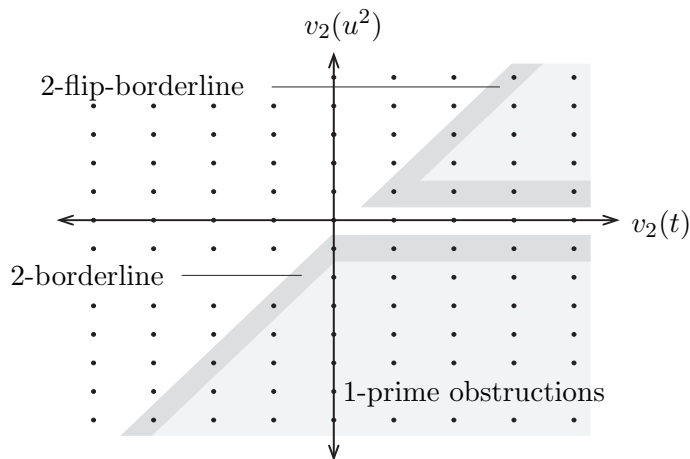
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Figure 4.5: Plot of extended obstructions for $p = 2$ (1×1 grid)

yet to investigate the full extent to which we can so generate more obstructions. Of course, this means that we cannot claim that our obstructions are exhaustive.

Much of the foreseeable work extending results of this section is to describe the $\mathcal{G}$-orbits of subsets of $\mathcal{P}_{\mathbb{Q}}$ that are described by our results' hypotheses. It is yet unclear whether these could collectively identify all pairs $(u^2, 2t)$ such that $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$.

4.3   Insufficiency of p-adic obstructions

We next show that we cannot promote the results of the previous section to a collective, complete obstruction to pseudomodularity. More precisely, we show that there are non-pseudomodular, nonarithmetic groups whose sets of cusps are dense in all (finite or infinite) products of the form $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$.

The key ideas of our proof are as follows.

(1) In the topology of a product of the form just given, there is a neighborhood $U \subset \mathbb{P}^1(\mathbb{Q})$ of some fixed cusp such that the cusps in U form a dense subset of U. This holds for all $\Delta(u^2, 2t)$.

(2) For a fixed $\Delta(u^2, 2t)$ and a U as above, the union of all gU as g ranges over $\Delta(u^2, 2t)$ is $\mathbb{P}^1(\mathbb{Q})$. This is not always the case, as per the obstructions above. Below, we actually only use a small set of elements g.

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(3) There exist groups $\Delta(u^2, 2t)$ that satisfy the condition of (2) and are neither pseudomodular nor arithmetic, either by an obstruction of Chapter 5 or by prior knowledge (i.e., the tables of [LR1]).

The first two ideas are implemented in the proposition that follows the next lemma.

LEMMA 4.25. *Endow $\mathbb{P}^1(\mathbb{Q})$ with the subspace topology from the diagonal inclusion into a finite product $H := \prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$. Let Δ be a fixed $\Delta(u^2, 2t)$ and fix x in $\mathbb{P}^1(\mathbb{Q})$. If for every neighborhood U of 0 in H, there is a $g \in \Delta$ such that $gx \in U$, then x is in the closure of $\mathcal{C}(\Delta(u^2, 2t))$ inside H.*

Proof. The generators of Δ are represented by matrices in $\mathrm{PGL}_2(\mathbb{Q})$, so Δ acts continuously by Möbius transformations on $\mathbb{P}^1(\mathbb{Q}_\ell)$ for each ℓ, in each case extending its action on $\mathbb{P}^1(\mathbb{Q})$. Thus the component-wise action of Δ on H, which extends its action on the diagonal $\mathbb{P}^1(\mathbb{Q})$, is continuous.

Since the translation $z \mapsto z + 2t$ is in Δ and 0 is a cusp of Δ, the set $2t\mathbb{Z}$ is contained in $\mathcal{C}(\Delta)$. Let $U = 2t \prod_\ell \mathbb{Z}_\ell$ in H. Then U is open and closed and $U \cap \mathbb{P}^1(\mathbb{Q}) = 2t\mathbb{Z}$ is a subset of $\mathcal{C}(\Delta)$. It follows that U is in the closure of $\mathcal{C}(\Delta)$ in H. Now, by hypothesis, there is a $g \in \Delta$ such that $gx \in U$, i.e., such that $x \in g^{-1}U$. Since g fixes $\mathcal{C}(\Delta)$ setwise and acts by homeomorphism on H, $g^{-1}U$ is in the closure of $\mathcal{C}(\Delta)$ in H. Therefore x is in the closure of $\mathcal{C}(\Delta)$. ☒

We shall show that, in certain situations, this lemma applies to all x in $\mathbb{P}^1(\mathbb{Q})$, thus establishing that the cusp set of the $\Delta(u^2, 2t)$ in question is dense in a specified product H as defined above. Of course, it is equivalent to show that each x in $\mathbb{P}^1(\mathbb{Q})$ can be moved by an element of $\Delta(u^2, 2t)$ to the U defined in the preceding proof. Indeed, any element of $2t\mathbb{Z}$ can be moved arbitrarily close to 0 in H by applying some power of the matrix $\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$, which lies in $\Delta(u^2, 2t)$.

For the proof of the upcoming proposition, we introduce a family of groups Δ_0 related to the family Δ. By Lemma 2.6 of [LR1], there is for each $(u^2, 2t)$ a Fuchsian group $\Delta_0(u^2, 2t)$ containing $\Delta(u^2, 2t)$ as a subgroup of index 2. This $\Delta_0(u^2, 2t)$ is isomorphic to the 3-fold free product of $\mathbb{Z}/2$ and is generated by the order-2 elements

$$S := \begin{pmatrix} 0 & -u \\ u^{-1} & 0 \end{pmatrix}, \quad A := Sg_1, \quad B := Sg_2.$$

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The stabilizer of ∞ is generated by $T := SAB = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. (See the beginning of Section 2 of [LR2].)
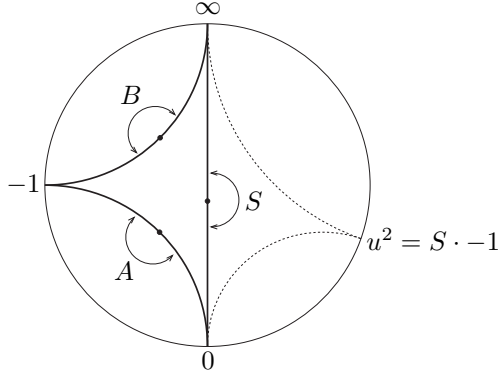

Figure 4.6: A fundamental domain for $\Delta_0(u^2, 2t)$ in $\mathcal{H}$

PROPOSITION 4.26. *If t equals a prime q, p is 1 or a prime distinct from q and $u^2 = m/p$ for some integer m coprime to p and divisible by q, then the set of cusps of $\Delta := \Delta(u^2, 2t)$ is dense in all finite products of $\mathbb{P}^1(\mathbb{Q}_\ell)$'s.*

Proof. We establish the proposition using $\Delta_0(u^2, 2t)$ in place of $\Delta(u^2, 2t)$, for our technique uses elements from the group $\Delta_0(u^2, 2t)$. This suffices since cusp sets are invariant under direct commensurability and $\Delta(u^2, 2t)$ is a finite-index subgroup of $\Delta_0(u^2, 2t)$.

Assume that p is not 1. The case in which p is 1 will be handled at the end of the proof.

Let Z be a fixed finite set of primes. We assume without loss of generality that p, q and all primes dividing m are in Z. This is harmless since, for $Z' \subset Z$, density of a subset of $\mathbb{P}^1(\mathbb{Q})$ in $\prod_{\ell \in Z} \mathbb{P}^1(\mathbb{Q}_\ell)$ implies density in $\prod_{\ell \in Z'} \mathbb{P}^1(\mathbb{Q}_\ell)$.

Call a subset $X \subset \mathbb{P}^1(\mathbb{Q})$ *close to zero* if for each $x \in X$, we can choose elements $\gamma \in \Delta_0 := \Delta_0(u^2, 2t)$ so as to get x arbitrarily close to 0 in $H := \prod_{\ell \in Z} \mathbb{P}^1(\mathbb{Q}_\ell)$. To get cusp density, we want to show that in fact the whole set $\mathbb{P}^1(\mathbb{Q})$ is close to zero, whence the lemma above implies the cusp set of Δ_0 is dense in H. First, note that

(a) If X is close to zero and $g \in \Delta_0$ then gX is close to zero.

(b) If $\{X_i\}$ is a collection of sets that are close to zero, then $\cup_i X_i$ is close to zero.

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We will use these without further mention.

First define $V_1 := t\mathbb{Z}$. One basis of neighborhoods of 0 in $\mathbb{P}^1(\mathbb{Q})$ in the topology from H is the collection of sets $n\mathbb{Z}$ for n a nonzero integer divisible by $q = t$. A given $x \in V_1$ can be mapped into any such $n\mathbb{Z}$ by some power of $T : z \mapsto z + t$. Therefore V_1 is close to zero.

Next, consider $S \in \Delta_0$, represented in $\mathrm{PGL}_2(\mathbb{Q})$ by $\begin{pmatrix} 0 & -m/p \\ 1 & 0 \end{pmatrix}$. This maps x to $-m/px$. Its effect on valuations is

$$
\begin{aligned}
v_p(x) = e &\quad\Rightarrow\quad v_p(Sx) = -1 - e \\
v_q(x) = e &\quad\Rightarrow\quad v_q(Sx) = v_q(m) - e \\
v_\ell(x) = e &\quad\Rightarrow\quad v_\ell(Sx) = v_\ell(m) - e, \qquad \ell \neq p, q.
\end{aligned}
$$

With this in mind set

$$
V_2 = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) < 0, \quad v_q(x) \leq 0, \quad v_\ell(x) \leq 0 \text{ for } \ell \in Z, \ell \neq p, q\}.
$$

Then

$$
SV_2 = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) \geq 0, \ v_q(x) \geq v_q(m), \ v_\ell(x) \geq v_\ell(m) \text{ for } \ell \in Z, \ell \neq p, q\}.
$$

Since q divides m, $v_q(m) \geq 1$, so $SV_2 \subset V_1$. Therefore V_2 is close to zero. Indeed, by the Chinese Remainder Theorem, we can do much better: let

$$
V_2' = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) < 0, \quad v_q(x) \leq 0\}.
$$

In other words, drop the restrictions in the definition of V_2 on primes in Z other than p and q.

Now suppose $x \in V_2'$. If $v_\ell(x) \leq 0$ for all ℓ in Z other than p and q, then $x \in V_2$. Otherwise, by the Chinese Remainder Theorem, we can apply some power of T to x to enforce this. Moreover, the inequalities $v_p(x) < 0$ and $v_q(x) \leq 0$ are invariant under T, so this implies $T^a x \in V_2$ for some $a \in \mathbb{Z}$. Therefore V_2' is close to zero. Note that V_2' is defined in terms of conditions on only two primes. We shall proceed similarly below, first finding a set that is close to zero and defined by a condition with respect to one prime, and then eliminating this condition to establish the final result.

Next we consider how $A \in \Delta_0$ treats values with specified p- and q-valuations. Since $u^2 = m/p$ and $t = q$, A is represented by the integral matrix $\begin{pmatrix} -m & -m \\ p(q-1) & m \end{pmatrix}$. Suppose x, written as a quotient r/s of coprime integers, has negative p-valuation, so $p \mid s$ and $p \nmid r$. Then

$$v_p(Ax) = v_p(m(r+s)) - v_p(p(q-1)r + ms) < 0$$

since $p \nmid m(r+s)$ and $p \mid s$. As A has order 2 in Δ_0, we infer that $v_p(x) < 0$ if and only if $v_p(Ax) < 0$.

To see how A interacts with v_q, compute

$$v_q(Ax) = v_q(m) + v_q(r+s) - v_q(p(q-1)r + ms).$$

If $v_q(x) \le 0$ then $q \nmid r$ so, since q divides m, we find that $v_q(p(q-1)r + ms) = 0$ and thus $v_q(Ax) \ge v_q(m)$. On the other hand, if $v_q(x) \ge v_q(m)$, then because $v_q(m)$ is positive, we see that q divides r but not s, whence

$$v_q(Ax) = v_q(m) - v_q(p(q-1)r + ms).$$

Now $v_q(r) \ge v_q(m)$, so $v_q(Ax) \le 0$. Therefore, since A is of order 2, A exchanges the sets of x with q-valuation, respectively, at most 0 and at least $v_q(m)$.

We use the behavior of A to eliminate the condition on v_p as follows. Through the computations with S above, we have that

$$SV_2' = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) \ge 0, \quad v_q(x) \ge v_q(m)\}$$

and so

$$ASV_2' = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_p(x) \ge 0, \quad v_q(x) \le 0\}.$$

Being an image of V_2' under $AS \in \Delta_0$, this set is close to zero, so its union with V_2', namely $V_3 := \{x \in \mathbb{P}^1(\mathbb{Q}) : v_q(x) \le 0\}$, is also close to zero.

Finally, note that

$$V_4 := V_3 \cup SV_3 = \{x \in \mathbb{P}^1(\mathbb{Q}) : v_q(x) \le 0 \quad \text{or} \quad v_q(x) \ge v_q(m)\}$$

is close to zero. If $x \in \mathbb{P}^1(\mathbb{Q})$ with $v_q(x) \le 0$, then $x \in V_4$. Otherwise, $v_q(x) \ge 1 = v_q(t)$ and so by applying some power T^a of T we can enforce the inequality

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$v_q(T^a x) \geq v_q(m)$. Therefore, every element of $\mathbb{P}^1(\mathbb{Q})$ can be pushed into V_4 by some $g \in \Delta_0$. We conclude that $\mathbb{P}^1(\mathbb{Q})$ is close to zero.

If $p = 1$, then the above proof holds with the following modifications: assume Z contains q and all primes dividing m; drop the conditions on v_p from the definitions of V_2 and V_2'; and, noting that V_2' then equals the set V_3 above, omit the analysis of the behavior of A.   ☑

By the definition of the product topology, the cusps of those $\Delta(u^2, 2t)$ satisfying the hypotheses of the preceding proposition are in fact dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$.

We remark that the process described by the above result somewhat parallels the use of a covering of $\mathbb{R}$ by killer intervals (§2.4) as done in [LR1]. Given a set of killer intervals corresponding to a pseudomodular group $\Delta(u^2, 2t)$, we can iteratively use elements of $\Delta(u^2, 2t)$, corresponding to the intervals, to send each rational to another rational of strictly lower denominator. For each rational number this process terminates since we ultimately reach the cusp ∞. In contrast, in Proposition 4.26 above, we are in effect producing a covering of $\mathbb{P}^1(\mathbb{Q})$ by sets open with respect to the product topology on a space $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$ and using this to show that each rational number can be moved arbitrarily close, in this product, to the cusp 0. As we said above and as we will soon show, the existence of one of our coverings does not imply pseudomodularity.

According to Proposition 3.6, the groups $\Delta(u^2, 2t)$ and $\Delta(-1 + t - u^2, 2t)$ are equivalent. In particular, the cusp set of one is dense in $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$ if and only if the cusp set of the other is as well. The above hypotheses on u^2 are that it has prime denominator and is congruent to 0 mod t. Then, through the noted equivalence, and using that t is assumed to be an integer, the proposition also applies to u^2 that are congruent to -1 mod t and have prime denominator.

To compare this to the results of Long and Reid in [LR1] that are reproduced in Tables A.1 and A.2, note that

(a) the set of cusps of each $\Delta(m/p, 4)$, with p an odd prime, is dense in the product of all $\mathbb{P}^1(\mathbb{Q}_\ell)$;

(b) the set of cusps of each $\Delta(m/p, 6)$, with p a prime distinct from 3 and with $m/p \not\equiv 1 \pmod 3$, is dense in the product of all $\mathbb{P}^1(\mathbb{Q}_\ell)$.

As promised, there are some groups, such as $\Delta(6/11, 6)$, that are neither pseudo-modular nor arithmetic—see Proposition 5.9 or Table A.2—but nevertheless have set of cusps dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$. This establishes the final result of this chapter.

THEOREM 4.27 (*p*-adic Insufficiency). *Let $\Delta(u^2, 2t)$ be nonarithmetic. Then density of $\mathcal{C}(\Delta(u^2, 2t))$ in the product $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$, ranging over all primes ℓ, is not a sufficient condition for pseudomodularity.*

Chapter 5

Congruence Topologies

In the previous chapter, we examined the relation between having the cusp set $\mathcal{C}$ of $\Delta(u^2, 2t)$ equal $\mathbb{P}^1(\mathbb{Q})$ and having $\mathcal{C} \setminus \{\infty\}$, the set of finite cusps, be dense in two certain topologies on $\mathbb{Q}$. These topologies, which we now label $\mathcal{T}_{\text{adelic}}$ and $\mathcal{T}_{\text{p-adic}}$, are those induced by the inclusions of $\mathbb{Q}$ into $\mathbb{A}_{\mathbb{Q},f}$ and $\prod_{\text{all } \ell} \mathbb{Q}_\ell$. Theorem 4.2 says that, for nonarithmetic groups $\Delta(u^2, 2t)$, pseudomodularity is equivalent to density of the set of finite cusps in $\mathcal{T}_{\text{adelic}}$. The results of Section 4.2 indicate that infinitely many non-pseudomodular groups (spanning infinitely many commensurability classes) have sets of finite cusps not dense in $\mathcal{T}_{\text{p-adic}}$, but Theorem 4.27 implies cusp density in $\mathcal{T}_{\text{p-adic}}$ is not equivalent to pseudomodularity.

In this sense, the topology $\mathcal{T}_{\text{p-adic}}$ has too few open sets, although it gives several effective results obstructing pseudomodularity for certain families of Fricke groups. On the other hand, the adelic topology $\mathcal{T}_{\text{adelic}}$ has enough open sets. We might even say that it has too *many* open sets: every orbit of rationals is open in this topology and none of our effective results are direct consequences of Theorem 4.2. Thus, we are interested in finding a topology $\mathcal{T}$ on $\mathbb{Q}$ that is strictly between $\mathcal{T}_{\text{adelic}}$ and $\mathcal{T}_{\text{p-adic}}$, such that density of the set of finite cusps of a nonarithmetic $\Delta(u^2, 2t)$ in $\mathcal{T}$ is equivalent to pseudomodularity, and such that we can derive further effective results using $\mathcal{T}$.

Towards this end, we exhibit below a topology on $\mathbb{Q}$ that is finer than $\mathcal{T}_{\text{p-adic}}$ and yields strictly more obstructions to pseudomodularity than $\mathcal{T}_{\text{p-adic}}$.

5.1 Definition and properties

DEFINITION 5.1. Identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{P}^1(\mathbb{Z})$, considered as a subset of $\mathbb{Z}^2/\{\pm 1\}$. Let S be a set of primes. The diagonal embedding of $\mathbb{Z}^2/\{\pm 1\}$ in $(\prod_{\ell \in S} \mathbb{Z}_\ell^2)/\{\pm 1\}$, endowed with the product topology, induces a topology on $\mathbb{P}^1(\mathbb{Q})$ that we call the *S-congruence topology*. If S is the set of all primes, call the S-congruence topology the *full congruence topology*. If N is an integer, let the *N-congruence topology* be the S-congruence topology for which S is the set of primes dividing N.

Notice that for each prime ℓ that $A_\ell := \mathbb{Z}_\ell^2 \setminus \{(0,0)\}$ maps continuously into and surjectively onto $\mathbb{P}^1(\mathbb{Q}_\ell)$ via $(a,b) \mapsto a/b$. Using these we can build a continuous, surjective function

$$\left(\prod_{\text{all } \ell} A_\ell\right)/\{\pm 1\} \longrightarrow \prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell) : (a_\ell, b_\ell)_\ell \longmapsto (a_\ell/b_\ell)_\ell.$$

Since the map from $\mathbb{P}^1(\mathbb{Z})$ to the product in Definition 5.1 misses $(0,0)$ in all factors, it in fact maps into $(\prod_\ell A_\ell)/\{\pm 1\}$. It follows that the full congruence topology is finer than the topology on $\mathbb{P}^1(\mathbb{Q})$ induced by inclusion into the product $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$. Similarly, each S-congruence topology is finer than that induced by the product $\prod_{\ell \in S} \mathbb{P}^1(\mathbb{Q}_\ell)$.

As a consequence of this, in each result of Section 4.2 in which $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in a product of spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$ over a set S of primes ℓ, we have that $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q})$ given either the S-congruence or full congruence topology.

Now we show that all congruence topologies arise from *dense* inclusions of $\mathbb{P}^1(\mathbb{Q})$ into profinite topological spaces as with the case of products of spaces $\mathbb{P}^1(\mathbb{Q}_\ell)$. The elements used in our constructions here will be helpful in later proofs. (Note that the inclusions of $\mathbb{P}^1(\mathbb{Q})$ into $(\prod_\ell A_\ell)/\{\pm 1\}$ are not dense since we can choose nonempty open sets in A_ℓ containing no pairs in the image of $\mathbb{P}^1(\mathbb{Z})$.)

Suppose N is an integer at least 2 and K is a subgroup of $(\mathbb{Z}/N)^\times$. Let the space $P_K(N)$ be the set of pairs (a,b) in $(\mathbb{Z}/N)^2$ with $\gcd(a,b,N) = 1$, modulo scalar multiplication by elements of K, and endowed with the discrete topology. The class of (a,b) in $P_K(N)$ is denoted by $[a:b]_K$ or $[a:b]$ when the context is clear. In any case we will mention the modulus N as necessary. For the special

cases where $K = \langle -1 \rangle$, resp. $K = (\mathbb{Z}/N)^\times$, we adopt the notation $P_{-1}(N)$, resp. $P_*(N)$.

By definition, $P_*(N) = \mathbb{P}^1(\mathbb{Z}/N)$. Using the obvious reduction-mod-M maps $P_*(N) \to P_*(M)$ for M dividing N, we can form the topological inverse limit $\varprojlim_N P_*(N)$, which is homeomorphic to $\mathbb{P}^1(\hat{\mathbb{Z}}) \cong \prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$. Similarly, if for a set S of primes, $I(S)$ denotes the set of positive integers generated multiplicatively by primes in S, then $\varprojlim_{N \in I(S)} P_*(N)$ is identified with $\prod_{\ell \in S} \mathbb{P}^1(\mathbb{Q}_\ell)$.

We could attempt to perform the same construction using the spaces $P_1(N)$, but the desired maps $\mathbb{P}^1(\mathbb{Q}) \to P_1(N)$ sending r/s (r, s coprime integers) to $[r : s]_1$ are not well-defined since r and s are only unique modulo ± 1. Thus we elect to use the spaces $P_{-1}(N)$, into which analogously defined maps are well-defined and surjective. Accordingly, for a set of primes S, let $\mathbb{K}_S$ be the topological inverse limit $\varprojlim_{N \in I(S)} P_{-1}(N)$ and include $\mathbb{P}^1(\mathbb{Q})$ into it using the surjections $\mathbb{P}^1(\mathbb{Q}) \to P_{-1}(N)$. Using the definition of the congruence topologies and the inclusions $P_{-1}(N) \to (\mathbb{Z}/N)^2/\{\pm 1\}$, we see that $\mathbb{K}_S$ is a profinite space into which the inclusion of $\mathbb{P}^1(\mathbb{Q})$ induces the S-congruence topology. Paralleling Definition 5.1 above, we let $\mathbb{K}_N$ equal $\mathbb{K}_S$ for S is the set of primes dividing N and we let $\mathbb{K}$ denote $\mathbb{K}_S$ when S is the set of all primes.

We remark that we can identify $\mathbb{K}_S$ as a subspace of $(\prod_{\ell \in S} \mathbb{Z}_\ell^2)/\{\pm 1\}$:

$$\mathbb{K}_S = \Big\{ (\alpha, \beta) = \{(\alpha_\ell, \beta_\ell)\}_\ell \in \prod_{\ell \in S} \mathbb{Z}_\ell^2 : (\forall \ell \in S)(v_\ell(\alpha_\ell) = 0 \text{ or } v_\ell(\beta_\ell) = 0) \Big\}/\{\pm 1\}.$$

The valuation condition arises from the condition $\gcd(a, b, N) = 1$ in the definition of $P_{-1}(N)$. This product form also makes it clear that each projection $\mathbb{K} \to \mathbb{K}_S$, given by restricting the index set of the inverse limit in the definition of $\mathbb{K}$ to $I(S)$, is continuous and surjective. Consequently, wherever we find that a set of cusps is not dense in an S-congruence topology, so too is it not dense in the full congruence topology.

PROPOSITION 5.2. *All inclusions $\mathbb{P}^1(\mathbb{Q}) \to \mathbb{K}_S$ defined as above are dense.*

Proof. The topology on $\mathbb{K}_S = \varprojlim_{N \in I(S)} P_{-1}(N)$ is the weakest topology with respect to which the projections $\pi_N : \mathbb{K}_S \to P_{-1}(N)$ are continuous. Therefore the family of finite intersections of sets $\pi_N^{-1}(V_N)$, with V_N any subset of $P_{-1}(N)$, forms a basis of opens for $\mathbb{K}_S$.

Let W be one such intersection. The elements (α, β) of $\mathbb{K}_S$ in a set $\pi_N^{-1}(V_N)$ are exactly those that project to $[a : b]$ modulo N for some $[a : b] \in V_N$. Each $P_{-1}(N)$ is finite, so each V_N is finite. It follows that W consists of elements (α, β) of $\mathbb{K}_S$ satisfying a finite set of congruence conditions. In order that W be nonempty, the congruence conditions must be consistent with respect to divisibility of the moduli N.

Assuming W is nonempty, we use this consistency and the Chinese Remainder Theorem (the set of congruence conditions is finite) to find an integer $M \in I(S)$ and some pair $[a : b] \in P_{-1}(M)$ such that if $(\alpha, \beta) \in \mathbb{K}_S$ is congruent to $\pm(a, b)$ modulo M, then (α, β) is in W.

Since $\gcd(a, b, M) = 1$, there is a matrix $\begin{pmatrix} \bar{a} & * \\ \bar{b} & * \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z}/M)$. The reduction-mod-$M$ map from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/M)$ surjects, so this matrix lifts to an integral unimodular matrix $\begin{pmatrix} r & * \\ s & * \end{pmatrix}$, whence $[r : s]$ is in W. (See [DS] Lemma 3.8.4 and its proof for these facts.) Since r and s are coprime, the inclusion of $\mathbb{P}^1(\mathbb{Q})$ into $\mathbb{K}_S$ hits W at r/s. This process is independent of the W chosen and so we have completed the proof. $\boxed{}$

Using the topology given by the projective limit $\mathbb{K}_S = \varprojlim_{N \in I(S)} P_{-1}(N)$ and the discrete topology on each set $P_{-1}(N)$, we see that the family of sets

$$U(r_0, s_0, N) := \left\{ \frac{r}{s} \in \mathbb{P}^1(\mathbb{Z}) : (r, s) \equiv \pm(r_0, s_0) \ (\mathrm{mod}\, N) \right\},$$

where N ranges over integers in $I(S)$ at least 2 and (r_0, s_0) ranges over pairs of residues mod N with $\gcd(r_0, s_0, N) = 1$, forms a basis for the S-congruence topology on $\mathbb{P}^1(\mathbb{Q})$. We will repeatedly use this notation below.

Of note is that, in contrast to the p-adic cases of Chapter 4, the Möbius action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ is not continuous with respect to all of the S-congruence topologies. This is essentially because, in congruence topologies, we keep track of congruence classes of coprime numerator and denominator of elements of $\mathbb{P}^1(\mathbb{Q})$, rather than just congruence classes of their quotients. This distinction is visible in the definition of the family of sets U above. We will give an explicit example of the lack of continuity, but first we show that continuity does hold if we use the *full* congruence topology, i.e., the congruence topology corresponding to the set of all primes.

PROPOSITION 5.3. *The fractional linear action of* $\mathrm{GL}_2(\mathbb{Q})$ *on* $\mathbb{P}^1(\mathbb{Q})$ *is continuous with respect to the full congruence topology.*

Proof. Checking continuity with respect to a set of generators of $\mathrm{GL}_2(\mathbb{Q})$ suffices. One generating set for $\mathrm{GL}_2(\mathbb{Q})$ is given by the matrices

$$\mu = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \left\{ m_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}_{p \text{ prime}}.$$

To show this, we first note that any rational matrix of positive determinant can be multiplied by some matrices $m_{p_i}^{n_i}$, $n_i \in \mathbb{Z}$, so as to produce a matrix in $\mathrm{SL}_2(\mathbb{Q})$. An elementary argument (see, e.g., §2 of [Men]) shows that $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$ is generated by τ, σ and

$$m_p \sigma m_p^{-1} \sigma^3 = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

Therefore $\mathrm{SL}_2(\mathbb{Q})$ is generated by τ, σ and the family of elements $m_p \sigma m_p^{-1} \sigma^3$ as p ranges over all primes and so, using μ to negate determinant if necessary, we see that $\mathrm{GL}_2(\mathbb{Q})$ indeed has the above generating set.

We next show that each generator (and its inverse) induces, through the group action, an open map on $\mathbb{P}^1(\mathbb{Q})$ with the full congruence topology.

A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Z})$ maps the basic open set $U(r_0, s_0, N)$ onto the open set $U(ar_0 + bs_0, cr_0 + ds_0, N)$, for if r/s is a fraction written lowest terms, so too is $(ar + bs)/(cr + ds)$. Therefore μ, τ, σ and their inverses act continuously on $\mathbb{P}^1(\mathbb{Q})$.

Now we prove that each map $x \mapsto m_p x$ from $\mathbb{P}^1(\mathbb{Q})$ to itself is open. Fix a basic open set $V = U(r_0, s_0, n)$. We show that for each such V and for each $x \in V$ there is an open set containing $m_p x$ and contained in $m_p V$. Let $x \doteq r/s$.

If p does not divide s then $m_p x \doteq pr/s$. Let W be the basic open $U(pr, s, pn)$, which contains $m_p x$. Suppose $a/b \in W$ is written in lowest terms. Then for some integers α and β,

$$(a, b) = (pr + \alpha pn, s + \beta pn).$$

It follows that

$$m_p^{-1} \cdot \frac{a}{b} = \frac{1}{p} \cdot \frac{pr + \alpha pn}{s + \beta pn} = \frac{r + \alpha n}{s + \beta pn},$$

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which is in lowest terms since a and b are coprime. Since this is in lowest terms, it lies in $U(r, s, n) = U(r_0, s_0, n) = V$, so that $a/b \in m_p V$. This holds regardless of choice of a/b in W, so $W \subset m_p V$.

On the other hand, if p divides s then $m_p x \doteq r/s'$ where $s' = s/p$. Here, $W := U(r, s', pn)$ contains $m_p x$. Again let $a/b \in W$ be in lowest terms, so that

$$(a, b) = (r + \alpha pn, s' + \beta pn)$$

for some integers α and β. Then

$$m_p^{-1} \cdot \frac{a}{b} = \frac{1}{p} \cdot \frac{r + \alpha pn}{s' + \beta pn} = \frac{r + \alpha pn}{s + \beta p^2 n}.$$

The only possible common factor of numerator and denominator here is p, but p does not divide r since p divides s, which is coprime to r. Therefore $m_p^{-1}(a/b)$ lies in $V = U(r, s, n)$. Again we find that $W \subset m_p V$.

Consequently, m_p is an open bijection, so m_p^{-1} is continuous. Transposing numerator and denominator in the preceding arguments and splitting the cases according to divisibility of r by p, we find that m_p is continuous. This completes the proof. ☒

Remark. This proposition implies that equivalence of groups $\Delta(u^2, 2t)$ respects the property of whether the set of cusps is dense in $\mathbb{K}$.

In the course of the above proof, we showed that for $V = U(r_0, s_0, n)$, both $m_p V$ and $m_p^{-1} V$ are unions of basic open sets of the form $U(\cdot, \cdot, pn)$. Therefore we have also proved the following result.

PROPOSITION 5.4. *Let R be a set of primes and let $\mathbb{Z}_R = \mathbb{Z}[p^{-1} : p \in R]$. Then the Möbius action of $\mathrm{GL}_2(\mathbb{Z}_R)$ on $\mathbb{P}^1(\mathbb{Q})$ is continuous with respect to the S-congruence topology for each set of primes S containing R.*

We now give an example of the general lack of continuity of the $\mathrm{GL}_2(\mathbb{Q})$ action for S-congruence topologies by demonstrating that $g := \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} : z \mapsto z + 1/2$ is not an open map in the 7-congruence topology. Select the open neighborhood

$$V := \left\{ \frac{r}{s} \in \mathbb{P}^1(\mathbb{Z}) : (r, s) \equiv \pm(1, 1) \pmod{7} \right\}$$

of 1 in $\mathbb{P}^1(\mathbb{Q})$. A system of basic opens around $g \cdot 1 = 3/2$ in $\mathbb{P}^1(\mathbb{Q})$ is given by the sets

$$U_j := \left\{ \frac{r}{s} \in \mathbb{P}^1(\mathbb{Z}) : (r, s) \equiv \pm(3, 2) \pmod{7^j} \right\}$$

as j ranges over the positive integers. For each $j \in \mathbb{N}$, $x_j := (7^j+3)/(7^j+2)$, which is already in lowest terms, is in U_j. In order that gV be open it must contain some U_j and thus some x_j. We show that no x_j is in gV. Indeed, if some x_j is in gV, then V contains

$$g^{-1}x_j = x_j - \frac{1}{2} = \frac{7^j+3}{7^j+2} - \frac{1}{2} = \frac{7^j+4}{2(7^j+2)}. \tag{5.1.1}$$

To show that this quotient is already written in lowest terms, assume p is a prime dividing both 7^j+4 and $2(7^j+2)$. The numerator is odd, so p cannot be 2, whence p divides 7^j+2. However, the difference between 7^j+2 and the numerator is itself 2, so p must be 2, forcing a contradiction. Hence the rightmost quotient in equation 5.1.1 is in lowest terms. This being the case, we can see immediately that $g^{-1}x_j$ is not in V: the numerator and denominator of $g^{-1}x_j$ as written above are each congruent to 4 modulo 7, and $(4,4) \not\equiv \pm(1,1)$ modulo 7. Therefore gV is indeed not an open set in the 7-congruence topology on $\mathbb{P}^1(\mathbb{Q})$, so the action of g^{-1} is not continuous. Notice that, had we computed the quotient x_j modulo 7, we would have found it congruent to $3/2$ mod 7 and therefore in some 7-adic neighborhood of $3/2 = g \cdot 1$.

We end this section by showing that the full congruence topology compares to other topologies of interest as claimed earlier.

PROPOSITION 5.5. *The full congruence topology, restricted to $\mathbb{Q}$, is strictly finer than $\mathcal{T}_{p\text{-adic}}$ and strictly coarser than $\mathcal{T}_{adelic}$.*

Proof. The full congruence topology is finer than $\mathcal{T}_{p\text{-adic}}$ by the remarks immediately following the definition of congruence topologies.

To show strictness, let V be the set $U(0,1,5)$, which is open in the congruence topology and contains 0. One basis of open neighborhoods around 0 in $\mathcal{T}_{p\text{-adic}}$ is the family of sets

$$W_M := \mathbb{Q} \cap \left(\left(M \prod_{\ell | M} \mathbb{Z}_\ell \right) \times \prod_{\ell \nmid M} \mathbb{Q}_\ell \right) \subset \prod_{\text{all } \ell} \mathbb{Q}_\ell$$

as M ranges over positive integers. If 5 does not divide M, then $W_M \not\subset V$ since M itself is in W_M but not in V. If 5 does divide M then we choose, using Dirichlet's Theorem, a prime p not dividing M and congruent to 2 mod 5. Then M/p is in W_M but not in V, so again $W_M \not\subset V$. This establishes the first claim of strictness.

As to the second comparison, recall that the full congruence topology restricted to $\mathbb{Q}$ has as a basis of opens the sets

$$U(r_0, s_0, N) = \left\{ \frac{r}{s} \in \mathbb{Q} : r, s \in \mathbb{Z},\ \gcd(r, s) = 1,\ (r, s) \equiv \pm(r_0, s_0) \ (\mathrm{mod}\, N) \right\}$$

for r_0, s_0, N as above. If r/s is in $U(r_0, s_0, N)$ and in lowest terms, then so too is $r/s \pm N = (r \pm Ns)/s$. Thus $U(r_0, s_0, N)$, being invariant under translation by N, is open in the finite-adelic topology on $\mathbb{Q}$, by the remark following Lemma 4.1. As a consequence, the full congruence topology on $\mathbb{Q}$ is coarser than $\mathcal{T}_{\mathrm{adelic}}$.

We establish strictness by observing that $\mathbb{Z}$ is open in $\mathcal{T}_{\mathrm{adelic}}$ and contains no set $U(r_0, s_0, N)$. Indeed, if $U(r_0, s_0, N) \subset \mathbb{Z}$, then necessarily $s_0 = \pm 1$. However, $(r_0(N+1) - N)/(N+1)$, which is in lowest terms, is in $U(r_0, 1, N) = U(-r_0, -1, N)$ but not in $\mathbb{Z}$. $\boxed{}$

Similarly, for any set of primes S, the S-congruence topology on $\mathbb{Q}$ is strictly between the finite-adelic topology and the topology induced by diagonal inclusion of $\mathbb{Q}$ into $\prod_{\ell \in S} \mathbb{Q}_\ell$.

5.2 *S*-integral obstructions

In this section we will find a family of groups $\Delta(u^2, 2t)$ whose sets of cusps are not dense in the full congruence topology despite being dense in $\prod_{\mathrm{all}\ \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$.

We recall from Section 4.2.3 the groups $\Lambda(u^2, 2t)$, which are the kernels of the group homomorphisms $\Delta(u^2, 2t) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $g_1 \mapsto (1, 0)$, $g_2 \mapsto (0, 1)$. By a covering space argument (see Figure 5.1), each $\Lambda(u^2, 2t)$ is a 5-fold free product of $\mathbb{Z}$, with generators

$$g_1^2,\ g_2^2,\ g_2^{-1} g_1^2 g_2,\ g_2^{-1} g_1^{-1} g_2 g_1,\ g_1^{-1} g_2 g_1 g_2$$

that we hereafter label, respectively, $h_1, \ldots, h_5$. We collect some facts about the groups $\Lambda(u^2, 2t)$ for future reference.

LEMMA 5.6. *With* $\Lambda(u^2, 2t)$ *defined as above:*

(a) $\Lambda(u^2, 2t)$ *is contained in* $\mathrm{PSL}_2(\mathbb{Q})$;

(b) $\Lambda(u^2, 2t)$ *has four orbits of cusps, represented by* ∞, u^2, -1 *and 0; and*
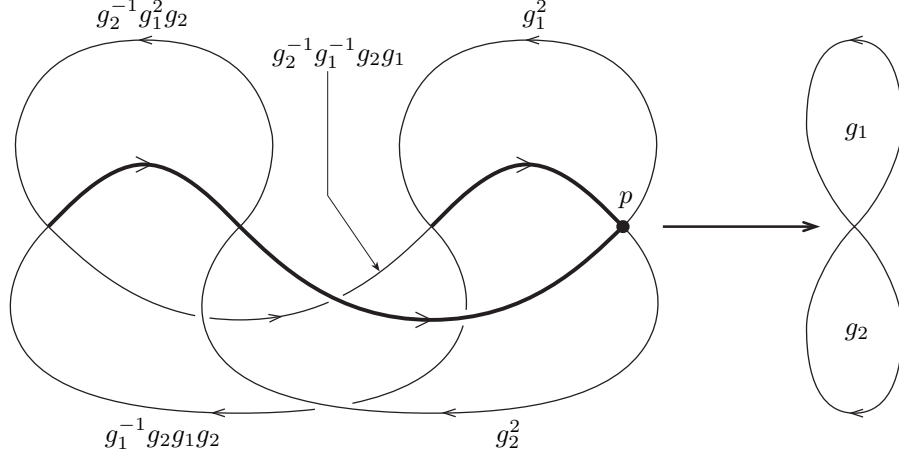
Figure 5.1: $(\mathbb{Z}/2)^2$-cover X of $S^1 \vee S^1$ with generators of $\pi_1(X, p) \cong \Lambda(u^2, 2t)$

(c) *if $\Lambda(u^2, 2t)$ has more than four orbits in its action on $\mathbb{P}^1(\mathbb{Q})$, then $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic.*

Proof. As to (a), g_1 and g_2 are represented by unimodular matrices of the form $\sqrt{r}M$ where r is rational and M has rational entries. Each h_i is therefore in $\mathrm{PSL}_2(\mathbb{Q})$.

As to (b), we have a 4-fold covering of compact Riemann surfaces $\Lambda(u^2, 2t)\backslash\mathcal{H}^* \to \Delta(u^2, 2t)\backslash\mathcal{H}^* =: X$. This covering is unramified away from the cusp of X because $\Delta(u^2, 2t)$ is free, and is unramified at the cusp of X because the stabilizer in $\Delta(u^2, 2t)$ of ∞, generated by $g_1 g_2^{-1} g_1^{-1} g_2$, equals the stabilizer in $\Lambda(u^2, 2t)$ of ∞. In particular, $\Lambda(u^2, 2t)$ always has four orbits of cusps. Taking the images under ∞ under each of the coset representatives 1, g_2, g_1^{-1}, $g_1^{-1}g_2$ of $\Lambda(u^2, 2t)$ in $\Delta(u^2, 2t)$, we find that the cusp orbits are represented by ∞, u^2, -1 and 0.

Finally, (c) follows from (b): if $\Lambda(u^2, 2t)$ has more than four orbits in its action on $\mathbb{P}^1(\mathbb{Q})$, then some rationals are not cusps, so by direct commensurability with $\Delta(u^2, 2t)$, $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic. ☐

We will show explicitly that the behavior described in part (c) occurs in the groups below.

To that end, let t be a prime integer and $u^2 = m/p$ with p a prime distinct from t and m an integer with $v_t(m) = 1$ and $v_p(m) = 0$. Set $\Lambda := \Lambda(u^2, 2t)$. We shall look at the image of Λ modulo pt, using that $\Lambda \subset \mathrm{PSL}_2(\mathbb{Q})$ and showing

along the way that the entries of the matrices h_i generating Λ are invertible modulo pt. For the following computations, we lift the h_i to matrices in $\mathrm{SL}_2(\mathbb{Q})$ using the representatives of the g_i given in $(*)$ of Section 2.3. First,

$$
\begin{aligned}
h_1 = g_1^2 &= \frac{1}{-1+t-u^2}\begin{pmatrix} (t-1)^2+u^2 & tu^2 \\ t & 1+u^2 \end{pmatrix} \\
&= \frac{1}{-p+pt-m}\begin{pmatrix} p(t-1)^2+m & tm \\ pt & p+m \end{pmatrix} \\
&\equiv -\frac{1}{p+m}\begin{pmatrix} p+m & tm \\ 0 & p+m \end{pmatrix} \\
&\equiv \begin{pmatrix} -1 & -tm/(p+m) \\ 0 & -1 \end{pmatrix}. \qquad\qquad (\mathrm{mod}\, pt)
\end{aligned}
$$

Here $p+m$ is invertible mod pt since $p \nmid m$ and $t \nmid p$. Also, $-tm/(p+m)$ is congruent to $0 \bmod t$ and $-t \bmod p$, whence

$$
h_1 \equiv \begin{pmatrix} -1 & -t \\ 0 & -1 \end{pmatrix} (\mathrm{mod}\, pt).
$$

Next,

$$
\begin{aligned}
h_2 = g_2^2 &= \frac{1}{-1+t-u^2}\begin{pmatrix} 1+u^2 & t \\ tu^{-2} & 1-2t+t^2u^{-2}+u^2 \end{pmatrix} \\
&= \frac{1}{-p+pt-m}\begin{pmatrix} p+m & pt \\ tp^2m^{-1} & p-2pt+t^2p^2m^{-1}+m \end{pmatrix} \\
&\equiv -\frac{1}{p+m}\begin{pmatrix} p+m & 0 \\ \frac{p^2}{m/t} & p+\frac{p^2t}{m/t}+m \end{pmatrix} \\
&\equiv \begin{pmatrix} -1 & 0 \\ -\frac{p^2}{(p+m)m/t} & -1 \end{pmatrix}. \qquad\qquad (\mathrm{mod}\, pt)
\end{aligned}
$$

The value m/t is invertible modulo p since $p \nmid mt$. It is also invertible modulo t, for since t is a prime exactly dividing m, m/t and t are coprime integers. Therefore m/t is a unit modulo pt. The lower left entry of h_2 is congruent to $0 \bmod p$ and $-p/(m/t) \bmod t$, so

$$
h_2 \equiv \begin{pmatrix} -1 & 0 \\ \frac{-p}{m/t} & -1 \end{pmatrix} (\mathrm{mod}\, pt).
$$

Next,

$$
h_3 = g_2^{-1} g_1^2 g_2 = \frac{1}{-1+t-u^2} \begin{pmatrix} 1+t^2+u^2 & t(2t-u^2) \\ -t & 1-2t+u^2 \end{pmatrix}
$$

$$
= \frac{1}{-p+pt-m} \begin{pmatrix} p+pt^2+m & t(2pt-m) \\ -pt & p-2pt+m \end{pmatrix}
$$

$$
\equiv -\frac{1}{p+m} \begin{pmatrix} p+m & -tm \\ 0 & p+m \end{pmatrix}
$$

$$
\equiv \begin{pmatrix} -1 & tm/(p+m) \\ 0 & -1 \end{pmatrix} \equiv h_1^{-1}; \qquad (\mathrm{mod}\, pt)
$$

$$
h_4 = g_2^{-1} g_1^{-1} g_2 g_1 = \frac{1}{-1+t-u^2} \begin{pmatrix} 1-3t+u^2 & -2t \\ 2t & 1+t+u^2 \end{pmatrix}
$$

$$
= \frac{1}{-p+pt-m} \begin{pmatrix} p-3pt+m & -2pt \\ 2pt & p+pt+m \end{pmatrix}
$$

$$
\equiv -\frac{1}{p+m} \begin{pmatrix} p+m & 0 \\ 0 & p+m \end{pmatrix} \equiv -I; \qquad (\mathrm{mod}\, pt)
$$

$$
h_5 = g_1^{-1} g_2 g_1 g_2 = \frac{1}{-1+t-u^2} \begin{pmatrix} -1-u^2 & -t \\ 2t+tu^{-2} & -1+2t+t^2u^{-2}-u^2 \end{pmatrix}
$$

$$
= \frac{1}{-p+pt-m} \begin{pmatrix} -p-m & -pt \\ 2pt+tp^2m^{-1} & -p+2pt+t^2p^2m^{-1}-m \end{pmatrix}
$$

$$
\equiv -\frac{1}{p+m} \begin{pmatrix} -p-m & 0 \\ \frac{p^2}{m/t} & -p+\frac{p^2t}{m/t}-m \end{pmatrix}
$$

$$
\equiv \begin{pmatrix} 1 & 0 \\ -\frac{p^2}{(p+m)m/t} & 1 \end{pmatrix} \equiv -h_2^{-1}. \qquad (\mathrm{mod}\, pt)
$$

Using these data and the fact that the h_i generate Λ, we find that the image $\bar{\Lambda}$ of Λ in $\mathrm{SL}_2(\mathbb{Z}/pt)/\{\pm I\}$ is the subgroup generated by

$$
\alpha = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \frac{p}{m/t} & 1 \end{pmatrix}.
$$

Since m/t is a unit mod pt, the second of these generators has order t and so we can replace it with

$$
\beta = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.
$$

In $\mathrm{SL}_2(\mathbb{Z}/pt)/\{\pm I\}$, the elements α and β commute, so $\bar{\Lambda}$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/t$ through the homomorphism given by $\alpha \mapsto (1,0)$ and $\beta \mapsto (0,1)$.

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For any subgroup K of $(\mathbb{Z}/pt)^\times$ containing -1, the group $\mathrm{SL}_2(\mathbb{Z}/pt)/\{\pm I\}$, and hence its subgroup $\bar{\Lambda}$, acts on $P_K(pt)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [r:s]_K = [ar+bs:cr+ds]_K,$$

which is the usual action by fractional linear transformations modulo pt, considering the pairs in $P_K(pt)$ as "fractions" or column vectors. We shall see how Λ interacts with the pt-congruence topology by analyzing the action of $\bar{\Lambda}$ on the spaces $P_K(pt)$. First, we consider $P_{-1}(pt)$, as each $P_K(pt)$ is a quotient of this space.

PROPOSITION 5.7. *The group $\Lambda = \Lambda(u^2, 2t)$ has $2(p-1)(t-1) = 2\phi(pt)$ orbits in its action on $P_{-1}(pt)$. Specifically, there are:*

- *$(p-1)(t-1)/2$ orbits of size 1, consisting of $[r:s]$ with $t \mid r$, $p \mid s$;*

- *$(p-1)(t-1)/2$ orbits of size p, consisting of $[r:s]$ with $t \mid r$, $p \nmid s$;*

- *$(p-1)(t-1)/2$ orbits of size t, consisting of $[r:s]$ with $t \nmid r$, $p \mid s$;*

- *$(p-1)(t-1)/2$ orbits of size pt, consisting of $[r:s]$ with $t \nmid r$, $p \nmid s$.*

Proof. The divisibility properties on r and s given in the statement are fixed under multiplication by units in $\mathbb{Z}/pt$ and so give well-defined sets in $P_{-1}(pt)$. We consider the $\bar{\Lambda}$-orbits, which are of course identical to the Λ-orbits.

Since $|\bar{\Lambda}| = pt$ and p and t are distinct primes, the size of each $\bar{\Lambda}$-orbit in $P_{-1}(pt)$ is either 1, p, t, or pt; respectively, the stabilizer of an element of the $\bar{\Lambda}$-orbit is $\bar{\Lambda}$, $\langle \beta \rangle$, $\langle \alpha \rangle$, 1. We identify and count stabilizers and orbits as follows.

First, for $[r:s] \in P_{-1}(pt)$, α sends $[r:s]$ to $[r+ts:s]$, which is equal to $[r:s]$ if and only if $(r+ts,s) \equiv \pm(r,s) \pmod{pt}$. If $(r+ts,s) \equiv -(r,s)$ then, because $s \equiv -s$, one of these cases occurs:

$$(1)\ s \equiv 0, \quad (2)\ p = 2 \text{ and } s \equiv t; \quad (3)\ t = 2 \text{ and } s \equiv p.$$

If the first case holds, then $-r \equiv r + ts \equiv r$, so r must similarly be congruent to 0, p or t; regardless this contradicts that $\gcd(r,s,pt)$ must be 1. Should case (2) hold, then $r + t^2 \equiv -r \pmod{2t}$, which implies $t \mid 2r$. Since $t \neq 2$, this forces r and s to have a common divisor modulo pt, which cannot happen. Finally, if the third

case applies, then $r \equiv -r \pmod{2p}$ so p must divide both r and s, forcing another contradiction. Therefore $[r : s] = \alpha \cdot [r : s]$ if and only if $(r + ts, s) \equiv (r, s)$ modulo pt, which in turn is true exactly when p divides s.

Similarly, β stabilizes those $[r : s]$ with $t \mid r$. Using these facts, we now count the number of orbits by counting the number of elements stabilized by each subgroup of $\bar{\Lambda}$.

First, $\bar{\Lambda}$ fixes those $[r : s]$ with $t \mid r$ and $p \mid s$. There are p elements of $\mathbb{Z}/pt$ divisible by t. Of these, $p-1$ are not divisible by p and hence are valid r, since $p \mid s$ and we require that $\gcd(r, s, pt)$ is 1. Likewise there are $t-1$ choices for s. Therefore, taking identification under negation into account, there are $(p-1)(t-1)/2$ elements of $P_{-1}(pt)$ fixed by $\bar{\Lambda}$. Each of these forms a $\bar{\Lambda}$-orbit of size 1.

Next, $\langle\alpha\rangle$ is the stabilizer of those $[r : s]$ with $p \mid s$ and $t \nmid r$. To count, first write $[r : s] = [r : ps_0]$. There are t possible choices for s_0. Since p divides s, it cannot also divide r, so r must be a unit mod pt. This enforces the condition $\gcd(r, s, pt) = 1$, so the total number of elements whose stabilizer is $\langle\alpha\rangle$ is $t(p-1)(t-1)/2$. Each of these lies in a $\bar{\Lambda}$-orbit of size t, since α has order p.

This argument is symmetric: the subgroup $\langle\beta\rangle$ is the stabilizer of the $[r : s]$ with $p \nmid s$ and $t \mid r$. There are $p(p-1)(t-1)/2$ such elements $[r : s]$, and each one lies in an orbit of size p.

Finally, those elements $[r : s]$ with trivial stabilizer have $p \nmid s$ and $t \nmid r$. Regardless of the choices of r and s, the GCD of r, s and pt is 1 since all nontrivial common divisors are eliminated by the divisibility conditions given. Thus we can choose r and s independently. There are $p(t-1)$ choices for r and $t(p-1)$ choices for s, so there are $pt(t-1)(p-1)/2$ elements of $P_{-1}(pt)$ with trivial stabilizer, each in a $\bar{\Lambda}$-orbit of size pt. $\boxtimes$

Recall that whenever $P_K(N)$ is defined and K contains -1, we have a well-defined projection $\mathbb{P}^1(\mathbb{Q}) \to P_K(N)$ given by $r/s \mapsto [r : s]_K \mod N$ where r and s are coprime integers. Also recall that if $\Lambda = \Lambda(u^2, 2t)$ has more than four orbits on $\mathbb{P}^1(\mathbb{Q})$, then $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic. With these both in mind, we may hope that the result of the proposition above gives a lower bound for the number of Λ-orbits on $\mathbb{P}^1(\mathbb{Q})$.

Unfortunately, there is a complication, which we illustrate by example. Let

$A = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ for some prime p. Choose a positive integer N coprime to p and have A act on $\mathbb{P}^1(\mathbb{Q})$ and on $P_{-1}(N)$ by fractional linear transformation. Suppose $x = r/s$ in $\mathbb{P}^1(\mathbb{Q})$ is in lowest terms with $p \nmid s$. Then $Ax \doteq (p^2 r)/s$. In $P_{-1}(N)$, $A \cdot [r : s] = [pr : qs]$ where q is the inverse of $p \bmod N$. Now Ax does not map to this tuple under the projection $\mathbb{P}^1(\mathbb{Q}) \to P_{-1}(N)$, as long as $p \not\equiv \pm 1 \pmod{N}$; i.e., the actions do not together respect reduction mod N. The problem here is that the reduction modulo N is applied to numerator and denominator of the rational number written in lowest terms. When the fractional linear action is interpreted as matrix multiplication applied to column vectors, we interpret Ax as being represented by $(pr, p^{-1}s)$, which does project to $[pr : qs]$ modulo N. The lowest terms representation $(p^2 r, s)$, however, will not map to $[pr : qs]$ in $P_{-1}(N)$ since the only scalars giving the equivalence in $P_{-1}(N)$ are ± 1, neither of which is congruent to p mod N. A remedy to this problem (for the matrix A) is to enlarge the set of permitted scalars by choosing a subgroup K of $(\mathbb{Z}/N)^\times$ containing $\bar{p}$ and using $P_K(N)$ in place of $P_{-1}(N)$.

So armed, we let $\Lambda = \Lambda(m/p, 2t)$ be as above and let $H = \mathrm{PSL}_2(\mathbb{Z}[q^{-1} : q \nmid pt])$ with q ranging over primes. This H contains Λ. Since we can reduce elements of H modulo pt, it acts on $\mathbb{P}^1(\mathbb{Q})$ and on $P_K(pt)$ by fractional linear transformations as above. In order that the diagram

$$
\begin{array}{ccc}
\mathbb{P}^1(\mathbb{Q}) & \xrightarrow{\ H-\mathrm{action}\ } & \mathbb{P}^1(\mathbb{Q}) \\
\downarrow & & \downarrow \\
P_K(pt) & \xrightarrow{\ H-\mathrm{action}\ } & P_K(pt)
\end{array}
$$

commutes when the H-action is restricted to Λ, it suffices to have $K \subset (\mathbb{Z}/pt)^\times$ contain -1 and all primes that occur in denominators of entries of matrices in Λ. Once we have that the Λ-actions on $\mathbb{P}^1(\mathbb{Q})$ and $P_K(pt)$ are equivariant with respect to the projection $\mathbb{P}^1(\mathbb{Q}) \to P_K(pt)$, we know that the number of Λ-orbits on $\mathbb{P}^1(\mathbb{Q})$ is at least the number of Λ-orbits on $P_K(pt)$.

So we can make use of this fact, we next compute the number of Λ-orbits on $P_K(pt)$ for an arbitrary K containing -1.

PROPOSITION 5.8. *Let $K \subset (\mathbb{Z}/pt)^\times$ be a subgroup containing -1. Then the number of Λ-orbits on $P_K(pt)$ is $4(p-1)(t-1)/|K|$, with $(p-1)(t-1)/|K|$ each*

of sizes 1, p, t and pt.

Proof. The set $P_K(pt)$ is the quotient of $P_{-1}(pt)$ by scalar multiplication by elements of $K/\langle -1 \rangle$. Since scalar multiplication by units in $\mathbb{Z}/pt$ preserves the divisibility properties with respect to p and t in the statement of Proposition 5.7, and because $K/\langle -1 \rangle$ acts freely on $P_{-1}(pt)$, the quotient induces an n-to-1 map on Λ-orbits, where $n = |K|/2$. ☒

So as to maximize the potential number of Λ-orbits on $\mathbb{P}^1(\mathbb{Q})$, we choose the minimal K such that we still get the Λ-equivariance described above. This K is the subgroup of $(\mathbb{Z}/pt)^\times$ generated by -1 and all primes that occur in denominators of entries of matrices in Λ. We so obtain an obstruction to pseudomodularity. We call it an *S-integral obstruction* since, in our computations above, we used the inclusion of Λ into some group $\mathrm{PSL}_2(\mathbb{Z}_S)$ where $\mathbb{Z}_S$ is the ring of S-integers, i.e., those rational numbers that are integral at all primes outside (a finite set) S.

PROPOSITION 5.9 (*S*-integral Obstruction I). *Suppose t and p are distinct primes and $u^2 = m/p$, where m is coprime to p and $v_t(m) = 1$. Let K be the subgroup of $(\mathbb{Z}/pt)^\times$ generated by -1 and all primes dividing denominators of entries of matrices in $\Lambda := \Lambda(m/p, 2t) \subset \mathrm{PSL}_2(\mathbb{Q})$. If $K \neq (\mathbb{Z}/pt)^\times$, then $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic, and its set of cusps is not dense in $\mathbb{K}_{pt}$.*

Proof. If K is a proper subgroup of $(\mathbb{Z}/pt)^\times$, then $|K| < (p-1)(t-1) = \phi(pt)$. Thus, by the preceding proposition, the number of Λ-orbits on $P_K(pt)$ exceeds 4. By hypothesis on K, Λ respects the projection $\pi : \mathbb{P}^1(\mathbb{Q}) \to P_K(pt)$, so Λ also has more than 4 orbits in its action on $\mathbb{P}^1(\mathbb{Q})$. By Lemma 5.6 (c), $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic.

As to the set of cusps not being dense in $\mathbb{K}_{pt}$, let $\mathcal{N}$ be the set of elements of $P_K(pt)$ that are not in the Λ-orbits of $\pi(\infty)$, $\pi(u^2)$, $\pi(0)$ and $\pi(-1)$. By Lemma 5.6 (b), $\pi^{-1}(\mathcal{N})$ is nonempty and contains no cusps. If we compose the continuous projection $\mathbb{K}_{pt} \to P_{-1}(pt)$ given by the definition of $\mathbb{K}_{pt}$ and the quotient map $P_{-1}(pt) \to P_K(pt)$ to form the continuous mapping $\theta : \mathbb{K}_{pt} \to P_K(pt)$, we have that $\theta^{-1}(\mathcal{N}) \cap \mathbb{P}^1(\mathbb{Q}) = \pi^{-1}(\mathcal{N})$. Thus $\pi^{-1}(\mathcal{N})$ is open in $\mathbb{P}^1(\mathbb{Q})$ in the *pt*-congruence topology, and this completes the proof. ☒

Remark. The set $\pi^{-1}(\pi(\infty))$ may contain points that are not in the orbit of ∞ under Λ and are therefore not cusps. Thus we have *not* proved that the set of cusps is closed in the *pt*-congruence topology.

Example. Set $(u^2, 2t) = (6/11, 6)$. The group $\Lambda(6/11, 6)$ lies in $\mathrm{PSL}_2(\mathbb{Z}[1/2])$. The subgroup K of $(\mathbb{Z}/33)^\times$ generated by -1 and 2 has order 10, less than $\phi(33) = 20$, since $2^5 \equiv -1 \pmod{33}$. Thus $\mathcal{C}(\Delta(6/11, 6))$ is not dense in $\mathbb{P}^1(\mathbb{Q})$ given the 33-congruence topology.

Now we suppose $v_t(m) > 1$ and otherwise leave unchanged the hypotheses on u^2 and t above. We can still produce obstructions in a fashion analogous to that above, but since m/t is no longer a t-adic unit, our reductions modulo pt of the generators h_i of $\Lambda = \Lambda(m/p, 2t)$ fail. However, we can still reduce mod p and obtain an action of Λ on $P_{-1}(p)$. By the computational work above, we find that the image $\bar{\Lambda}$ of Λ in $\mathrm{SL}_2(\mathbb{Z}/p)/\{\pm I\}$ is the cyclic order-p subgroup generated by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and hence generated by $\delta := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, since p and t are distinct primes.

PROPOSITION 5.10. *Let u^2, p, t be as above with $p \neq 2$. Then the group $\Lambda = \Lambda(u^2, 2t)$ has $p - 1$ orbits in its action on $P_{-1}(p)$. Specifically, there are:*

- *$(p-1)/2$ orbits of size 1, consisting of $[r : s]$ with $s \equiv 0 \pmod{p}$;*

- *$(p-1)/2$ orbits of size p, consisting of $[r : s]$ with $s \not\equiv 0 \pmod{p}$.*

If $K \subset (\mathbb{Z}/p)^\times$ is a subgroup containing -1, then the number of Λ-orbits on $P_K(p)$ is $2(p-1)/|K|$, with $(p-1)/|K|$ each of sizes 1 and p.

Proof. The group $\bar{\Lambda}$ is simple, so stabilizers for the $\bar{\Lambda}$-action are either trivial or $\bar{\Lambda}$ itself. We have that $[r : s] = \delta \cdot [r : s] = [r + s : s]$ in $P_{-1}(p)$ if and only if s is zero in $\mathbb{Z}/p$. There are $(p-1)/2$ elements $[r : s] \in P_{-1}(p)$ with $s \equiv 0$; these constitute the singleton orbits. The cardinality of $P_{-1}(p)$ is $(p^2 - 1)/2$, so there are $p(p-1)/2$ elements $[r : s]$ of $P_{-1}(t)$ with $s \not\equiv 0$. This completes the claimed identification of orbits.

The final claim follows immediately by the same arguments given in the proof of Proposition 5.8. $\boxtimes$

Remark. If $p = 2$, then there are two Λ-orbits in $P_{-1}(p)$, namely $\{[0:1]\}$ and its complement $\{[1:0], [1:1]\}$. The obstruction we produce below fails for this case as there are too few orbits.

As above we fix K to be the subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ generated by -1 and all primes in denominators of entries of matrices in Λ. This ensures that the projection $\mathbb{P}^1(\mathbb{Q}) \to P_K(p)$ is Λ-equivariant. Consequently, if the number of Λ-orbits on $P_K(p)$ exceeds 4, i.e., if $|K| < (p-1)/2$ then $\Delta(m/p, 2t)$ is neither pseudomodular nor arithmetic. However, we can tighten this bound by carefully tracking cusps, as in the proof of the following result.

PROPOSITION 5.11 (*S*-integral Obstruction II). *Let t and p be distinct primes with p odd and $u^2 = m/p$, where m is coprime to p and $v_t(m) > 1$. Let K be the subgroup of $(\mathbb{Z}/p)^\times$ generated by -1 and all primes dividing denominators of entries of matrices in $\Lambda := \Lambda(m/p, 2t) \subset \mathrm{PSL}_2(\mathbb{Q})$. If $K \neq (\mathbb{Z}/p)^\times$, then $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic, and its set of cusps is not dense in $\mathbb{K}_p$.*

Proof. From Lemma 5.6, the four cusp orbits of Λ are represented by -1, 0, m/p and ∞. First, -1 and 0 map to $[-1:1]$ and $[0:1]$ in $P_{-1}(p)$. Since the generator δ of $\bar{\Lambda}$ sends $[-1:1]$ to $[0:1]$, the images of -1 and 0 are in the same Λ-orbit in $P_K(p)$. Next, notice that the images $[1:0]$ of ∞ and $[m:0]$ of $m/p = u^2$ are fixed by Λ as elements of $P_{-1}(p)$. Thus the images of ∞ and u^2 are in the same Λ-orbit in $P_K(p)$ if and only if $\overline{m} \in K$ as a residue mod p.

If $\overline{m} \in K$, then the images of Λ-orbits of cusps in $\mathbb{P}^1(\mathbb{Q})$ project to two Λ-orbits in $P_K(p)$. Hence the inequality $2(p-1)/|K| > 2$, or $|K| < p - 1$, is sufficient to obstruct pseudomodularity.

When $\overline{m}$ is not in K, the Λ-orbits of cusps project to three Λ-orbits in $P_K(p)$, in which case the inequality $|K| < 2(p-1)/3$ suffices to obstruct pseudomodularity.

Since $|K|$ divides $p - 1$, either of these inequalities is equivalent to having K be a proper subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$. The density statement follows by an argument analogous to that in Proposition 5.9. $\boxtimes$

Example. Set $(u^2, 2t) = (4/17, 4)$. The group $\Lambda(4/17, 4)$ lies in $\mathrm{PSL}_2(\mathbb{Z}[1/26])$. The subgroup K of $(\mathbb{Z}/17)^\times$ generated by -1, 2 and 13 has order 8, less than $\phi(17) = 16$, since $2^4 \equiv 13^2 \equiv -1 \pmod{17}$. Thus $\mathcal{C}(\Delta(4/17, 4))$ is not dense in the

17-congruence topology on $\mathbb{P}^1(\mathbb{Q})$. Note also that this shows that not all groups of the form $\Delta(u^2, 4)$ where u^2 has prime denominator have cusp set $\mathbb{P}^1(\mathbb{Q})$. Thus the pattern suggested by the entries of Table A.1 where u^2 has prime denominator cannot hold in general.

Note that the groups to which Propositions 5.9 and 5.11 apply have cusp sets dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$ by Proposition 4.26. Therefore we have achieved a nontrivial extension of the obstructions of Section 4.2.

THEOREM 5.12. *There exist Fricke groups with rational cusps whose sets of cusps are not dense in $\mathbb{K}$ but are nevertheless dense in $\prod_{all \ \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$.*

5.3   Other points of view

In this section we briefly describe the congruence topologies from some alternative points of view that may allow for more general definitions and investigations.

We can use congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ to describe open sets in the congruence topology. Let $\Gamma(N)$ denote the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N.

LEMMA 5.13. *Suppose r/s is in $\mathbb{P}^1(\mathbb{Z})$ and let N be a positive integer such that $\gcd(r, s, N) = 1$. Then $U(r, s, N)$ is equal to the orbit $\Gamma(N) \cdot (r/s)$.*

Proof. Basic computations imply that if r/s and r'/s' are in $\mathbb{P}^1(\mathbb{Z})$, then $\Gamma(N) \cdot (r/s) = \Gamma(N) \cdot (r'/s')$ if and only if $(r, s) \equiv \pm(r', s')$ modulo N; see, e.g., Proposition 3.8.3 in [DS]. This completes the proof.   ☑

Since $U(r, s, N)$ is the inverse image under the projection $\mathbb{P}^1(\mathbb{Q}) \to P_{-1}(N)$ of $[r : s]$, the above lemma identifies $P_{-1}(N)$ with the $\Gamma(N)$-orbits on $\mathbb{P}^1(\mathbb{Q})$. Moreover, we find that for a set S of primes, the collection

$$\{\Gamma(N)x : x \in \mathbb{P}^1(\mathbb{Q}), \ N \in I(S)\}$$

is a basis of open sets in the S-congruence topology on $\mathbb{P}^1(\mathbb{Q})$.

Having translated the definition of the congruence topologies into a description in terms of subgroups of the modular group, we can easily generalize it by choosing

different subgroups. Indeed, if we let $\mathcal{F}$ be a family of finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, then we can define a topology associated to $\mathcal{F}$ by declaring that each set in the family

$$\{\Gamma x : x \in \mathbb{P}^1(\mathbb{Q}), \ \Gamma \in \mathcal{F}\}$$

is open. Equivalently, we can map $\mathbb{P}^1(\mathbb{Q})$ to the projective limit $\varprojlim_{\Gamma \in \mathcal{F}} \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ in which each $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$ is given the discrete topology and endow $\mathbb{P}^1(\mathbb{Q})$ with the weakest topology such that said map is continuous. If $\mathcal{F}$ is the family of principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, then we recover the full congruence topology. Otherwise, we find other topologies on $\mathbb{P}^1(\mathbb{Q})$ in which we can naturally ask if sets of cusps are dense.

Of particular note is that above, in our analysis of the action of Λ with respect to the pt-congruence topology, we are unable to work directly with the sets $P_{-1}(pt)$, since the projection $\mathbb{P}^1(\mathbb{Q}) \to P_{-1}(pt)$ is not necessarily Λ-equivariant. In other words, though we have a homomorphism ρ from Λ to $\mathrm{SL}_2(\mathbb{Z}/pt)/\{\pm I\}$, the diagram

$$
\begin{array}{ccc}
\mathbb{P}^1(\mathbb{Q}) & \xrightarrow{\ g \cdot - \ } & \mathbb{P}^1(\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Gamma(pt)\backslash\mathbb{P}^1(\mathbb{Q}) & \xrightarrow{\ \rho(g) \cdot - \ } & \Gamma(pt)\backslash\mathbb{P}^1(\mathbb{Q})
\end{array}
$$

does not commute for all $g \in \Lambda$. This diagram motivates one of our open questions below.

Our solution to this commutativity problem above is to use in place of $P_{-1}(pt) = \Gamma(pt)\backslash\mathbb{P}^1(\mathbb{Q})$ the set $P_K(pt)$ where K is the subgroup of $(\mathbb{Z}/pt)^\times$ generated by -1 and the set R of primes appearing in the denominators of entries of matrices in Λ. Denoting by $\mathbb{Z}_R$ the ring $\mathbb{Z}[\ell^{-1} : \ell \in R]$, we claim that $P_K(pt)$ is the set of $\Gamma_R(pt)$-orbits on $\mathbb{P}^1(\mathbb{Q})$, where $\Gamma_R(pt)$ is the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_R)$ of level pt. Thus, our computations regarding Λ in Section 5.2 above are less directly related to orbits of $\Gamma(pt)$ than they are to $\Gamma_R(pt)$, which is notably *not* a Fuchsian group.

Using the inclusion $\Lambda \hookrightarrow \mathrm{PSL}_2(\mathbb{Z}_R)$, we may be tempted to use other finite-index, normal subgroups of $\mathrm{SL}_2(\mathbb{Z}_R)$ to produce density obstructions in a manner analogous to that above. However, by a result of Serre [Ser2], every finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z}_R)$ is a congruence subgroup, assuming that R is nonempty. Therefore, using

finite-index subgroups of $\mathrm{SL}_2(\mathbb{Z}_R)$ in the above contexts is no more general than using congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

5.4 Open questions

Finally, we collect some open questions motivated by the previous section and our earlier work. Any group $\Delta(u^2, 2t)$ mentioned here is assumed to be nonarithmetic.

(1) Is the density of the $\mathcal{C}(\Delta(u^2, 2t))$ in $\mathbb{K}$ a sufficient condition for pseudomodularity? Equivalent to this density condition are the statements:

   (a) For all integers N at least 2 and all $(a, b) \in (\mathbb{Z}/N)^2$ with $\gcd(a, b, N) = 1$, there is an $r/s \in \mathcal{C}(\Delta(u^2, 2t))$, written in lowest terms, such that (r, s) is congruent to (a, b) modulo N.

   (b) Each set $\Gamma(N)x$, for N a positive integer and $x \in \mathbb{P}^1(\mathbb{Q})$, contains a cusp of $\Delta(u^2, 2t)$.

(2) Suppose that for each (optionally normal) subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite-index and for each $x \in \mathbb{P}^1(\mathbb{Q})$, the set Γx contains a cusp of $\Delta(u^2, 2t)$. Is $\Delta(u^2, 2t)$ pseudomodular?

   (a) If not, what if we allow Γ to range over all arithmetic Fuchsian groups that have cusps?

   (b) More generally, fix a non-pseudomodular $\Delta := \Delta(u^2, 2t)$ and let $\mathcal{A}$ be the commensurability class of an arbitrary Fricke group with rational cusps, chosen so that $\mathcal{A}$ does not contain Δ. Is there an $x \in \mathbb{P}^1(\mathbb{Q})$ and a Γ in $\mathcal{A}$ such that Γx contains no cusps of Δ?

(3) Suppose $\Delta(u^2, 2t)$ is not pseudomodular. Is there a finite-index, normal subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ and a homomorphism ρ from $\Lambda(u^2, 2t)$ to $\mathrm{PSL}_2(\mathbb{Z})/\Gamma$ such that $|\Gamma\backslash\mathbb{P}^1(\mathbb{Q})| > 4$ and the diagram

$$
\begin{array}{ccc}
\mathbb{P}^1(\mathbb{Q}) & \xrightarrow{\;g\cdot-\;} & \mathbb{P}^1(\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) & \xrightarrow{\rho(g)\cdot-} & \Gamma\backslash\mathbb{P}^1(\mathbb{Q})
\end{array}
$$

commutes for all $g \in \Lambda$?

(4) Can we effectively identify and enumerate *all* cases where the cusp set of $\Delta(u^2, 2t)$ is not dense in $\prod_{\text{all } \ell} \mathbb{P}^1(\mathbb{Q}_\ell)$? Are there groups $\Delta(u^2, 2t)$ whose cusp sets are dense in every product $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$ but not dense in some product $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q) \times \mathbb{P}^1(\mathbb{Q}_r)$?

Appendix A

Tabular Data

We reproduce the data of the Tables 5.1 and 5.2 of [LR1] that list results about the cusps of $\Delta(u^2, 2t)$ for $t = 2, 3$ and u^2 having small denominator. We also indicate the groups to which our results apply.

The "structure" column duplicates the data from Long and Reid's tables. The "obstruction" column gives the following data.

- Where a space X is given, the set of cusps of the group in question is not dense in X.

  - If the space is decorated with an asterisk, then the special (i.e., rational) fixed point given in the structure column is not a member of any of the invariant sets from our applicable results.

  - If a space $\mathbb{K}_N$ is listed then additionally the set of cusps is dense in all products $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$; i.e., none of our weaker obstructions apply.

- Where a hyphen is given, the set of cusps is known to be a proper subset of $\mathbb{P}^1(\mathbb{Q})$ but is nevertheless dense in all products $\prod_\ell \mathbb{P}^1(\mathbb{Q}_\ell)$. In these cases, the behavior of the cusp set with respect to the congruence topologies is yet unclear.

- Where the word "none" is given, the set of cusps is known to equal $\mathbb{P}^1(\mathbb{Q})$.

Since knowing the status of the cusp set of $\Delta(u^2, 2t)$ with respect to the p-adic topologies is equivalent—by Proposition 3.6—to knowing that of $\Delta(-1+t-u^2, 2t)$, we omit the entries of Long and Reid's Table 5.1 having $u^2 > 1/2$.

The third table below compactly summarizes results from Sections 4.2 and 4.3. There, the "obstruction" column serves the same purpose as that of the first two tables.

Table A.1: Cusp set data for $\Delta(u^2, 2t)$, $t = 2$

$0 < u^2 \leq 1/2$	structure [LR1]	obstruction
1/2	arithmetic	none
1/3	arithmetic	none
1/4	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2)$
1/5	arithmetic	none
2/5	pseudomodular	none
1/6	special point 3/2	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_3)$
1/7	conjecturally pseudomodular	-
2/7	conjecturally pseudomodular	-
3/7	pseudomodular	none
1/8	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2)$
3/8	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2)$
1/9	special point 1/3	$\mathbb{P}^1(\mathbb{Q}_3)$
2/9	special point 1/3	$\mathbb{P}^1(\mathbb{Q}_3)$
4/9	special point 2/3	$\mathbb{P}^1(\mathbb{Q}_3)$
1/10	special point 7/2	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
3/10	special point 1/5	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
1/11	conjecturally pseudomodular	-
2/11	conjecturally pseudomodular	-
3/11	pseudomodular	none
4/11	conjecturally pseudomodular	-
5/11	conjecturally pseudomodular	-
...	...	...
4/17	-	$\mathbb{K}_{17}$

Table A.2: Cusp set data for $\Delta(u^2, 2t)$, $t = 3$

$0 < u^2 \leq 1$	structure [LR1]	obstruction
1	arithmetic	none
1/2	arithmetic	none
1/3	special point 1	$\mathbb{P}^1(\mathbb{Q}_3)$
2/3	special point 1/3	$\mathbb{P}^1(\mathbb{Q}_3)$
1/4	special point $-5/8$	$\mathbb{P}^1(\mathbb{Q}_2)$
3/4	special point 3/2	$\mathbb{P}^1(\mathbb{Q}_2)$
1/5	arithmetic	none
2/5	special point 1/7	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_5)$
3/5	conjecturally pseudomodular	-
4/5	conjecturally pseudomodular	-
1/6	special point $-1/35$	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_3)$
5/6	special point $-17/24$	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_3)$
1/7	special point $-37/14$	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_7)^*$
2/7	conjecturally pseudomodular	-
3/7	special point 3/4	-
4/7	special point 2/7	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_7)$
5/7	pseudomodular	none
6/7	special point 5/3	-
1/8	special point 1/14	$\mathbb{P}^1(\mathbb{Q}_2)$
3/8	special point $-15/2$	$\mathbb{P}^1(\mathbb{Q}_2)$
5/8	special point 7/4	$\mathbb{P}^1(\mathbb{Q}_2)$
7/8	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2)$
1/9	special point $-100/117$	$\mathbb{P}^1(\mathbb{Q}_3)^*$
2/9	special point 545/1521	$\mathbb{P}^1(\mathbb{Q}_3)^*$
4/9	special point $-52/9$	$\mathbb{P}^1(\mathbb{Q}_3)^*$
5/9	special point $-5/16$	$\mathbb{P}^1(\mathbb{Q}_3)^*$
7/9	special point 29/9	$\mathbb{P}^1(\mathbb{Q}_3)^*$
8/9	special point $-205/9$	$\mathbb{P}^1(\mathbb{Q}_3)^*$
continued on next page...		

	...continued from previous page	
$0 < u^2 \leq 1$	structure [LR1]	obstruction
1/10	special point 5/52	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
3/10	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
7/10	special point 1/2	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
9/10	special point 6/5	$\mathbb{P}^1(\mathbb{Q}_2) \times \mathbb{P}^1(\mathbb{Q}_5)$
1/11	conjecturally pseudomodular	-
2/11	special point $-266/4717$	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_{11})$
3/11	undecided	-
4/11	special point 1/5	-
5/11	special point $-1778/741$	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_{11})$
6/11	special point 69/11	$\mathbb{K}_{33}$
7/11	special point 149/136	-
8/11	special point $-79/93$	$\mathbb{P}^1(\mathbb{Q}_3) \times \mathbb{P}^1(\mathbb{Q}_{11})$
9/11	conjecturally pseudomodular	-
10/11	special point 1/3	-

Table A.3: Summary of p-adic density obstructions of §4.2 and §4.3

hypotheses	p-adic obstruction	result
$v_p(t) \geq 2$		
• $v_p(u^2) = 2v_p(t) - 1$	p-flip-borderline	Prop. 4.23
• $2 \leq v_p(u^2) \leq 2v_p(t) - 2$	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.20
• $v_p(u^2) = 1$		
○ p odd	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.21
$v_p(t) \geq 1$		
• $v_p(u^2) \geq 1$		
○ $t = p$, u^2 with prime denom.	none	Prop. 4.26
• $v_p(u^2) = 1$	p-flip-borderline	Prop. 4.23
• $v_p(u^2) = 0$		
○ $p > 3$, $u^2 \not\equiv -1$ (p)	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.8
○ $p = 3$, $u^2 \equiv 1$ (3)	3-borderline	
	and 3-flip-borderline	Prop. 4.24
○ $t = p$, u^2 with prime denom., $u^2 \equiv -1$ (p)	none	Prop. 4.26
• $v_p(u^2) = -1$		
○ p odd	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.6
$v_p(t) \geq 0$		
• $v_p(u^2) = -1$	p-borderline	
• $v_p(u^2) \leq -2$	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.4
$v_p(t) < 0$		
• $v_p(u^2) = 2v_p(t) - 1$	p-borderline	
• $v_p(u^2) \leq 2v_p(t) - 2$	$\mathbb{P}^1(\mathbb{Q}_p)$	Prop. 4.4
Two-prime obstructions		
• p-borderline, q-borderline	$\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$	Prop. 4.10
• p-flip-borderline, q-flip-borderline	$\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$	Prop. 4.10

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