TOPICS IN QUANTUM SHEAF COHOMOLOGY

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Dedicated to my parents

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ABSTRACT

TOPICS IN QUANTUM SHEAF COHOMOLOGY

Zhentao Lu

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Quantum sheaf cohomology generalizes the theory quantum cohomology, in the sense that it deals with a class of more general sheaves rather than the tangent bundle. In this thesis we study quantum sheaf cohomology of bundles on smooth projective toric varieties. The basic case is when the bundle is a deformation of the tangent bundle. We study the quantum correlators defined by the quantum sheaf cohomology. We give a mathematical proof of a formula that computes the quantum correlators in this case, confirming the conjecture in the physics literature. The next important case is when the bundle is of higher rank than the tangent bundle. We study bundles being deformations of $T \oplus \mathcal{O}$ where T is the tangent bundle and $\mathcal{O}$ is the trivial bundle. We give a rudimentary description of the classical and quantum sheaf cohomology ring in this case. We also discuss other interesting cases and the demanding from physics, as well as the connections between them and the previously mentioned ones.

Contents

Chapter 1

Introduction

The study of quantum sheaf cohomology (QSC) arises from the physics problem of understanding the Gauged Linear Sigma Model (GLSM), introduced by Witten [21]. There are two versions of GLSM, the (2,2) theory and the (0,2) theory, where (2,2) and (0,2) indicates the amount of supersymmetry of the theory. Both theories study maps from Riemann surfaces to compact Kähler manifolds. The former theory considers the manifold with its tangent bundle, while the latter considers more general bundles.

The study of the (2,2) theory is more mature and the associated quantum cohomology theory is studied by Batyrev [2] and Morrison-Plesser [16]. The main result there, the Toric Residue Mirror Conjecture (TRMC, see Equation (2.4.4)), is formulated in [1] based on [16], and is proved independently by Szenes-Vergne [20] and Borisov [4].

Quantum sheaf cohomology is associated to the study of the (0,2) theory, which is relatively new and many problems remain open. See [15] for a survey. The basic object studied here is a compact Kähler manifold V with an omalous[1] holomorphic vector bundle $\mathcal{E}$ on it. An important quantity in the (0,2) theory (as well as in the (2,2) theory) is the set of *correlators*. For cohomology elements in $H^1(V, \mathcal{E}^\vee)$, the classical correlator is a sheaf cohomology analog of the intersection number of divisors, while the quantum correlator is a weighted sum of classical correlators of the moduli spaces parametrized by effective curve classes of V.

Donagi, Guffin, Katz and Sharpe in [9] devoloped the mathematical theory of the quantum sheaf cohomology for any smooth projective toric variety X with a bundle $\mathcal{E}$ defined by the deformed toric Euler sequence (2.1.1). Bundles defined this way are naturally omalous, and they can be studied using Koszul complex. The quantum sheaf cohomology ring is defined by specifying the quantum Stanley-Reisner ideals. This enables the authors of [9] to define the quantum correlators with values in a one-dimensional complex vector space H^*.

From the physics side, McOrist and Melnikov formulated a conjectures about the quantum correlators in [17].

Conjecture 1.0.1. *For a toric variety V with a holomorphic vector bundle $\mathcal{E}$ defined by a deformed toric Euler sequence (See (2.1.1) below), the quantum correlator of σ_i's in $H^1(\mathcal{E}^\vee)$ can be computed by the following summation formula:*

[1] "Omalous" means "non-anomalous", i.e. the Chern classes satisfy $c_i(\mathcal{E}) = c_i(V), i = 1, 2$.

$$\langle \sigma_{i_1}, ..., \sigma_{i_s} \rangle^{quantum} = \sum_{\{u \in W^\vee | \tilde{v}_j(u) = q_j\}} \frac{\sigma_I}{\prod_{c \in [\Sigma(1)]} Q_c} \frac{\prod_{j=1}^r \tilde{v}_j}{\det_{j,k}(\tilde{v}_{j,k})}. \tag{1.0.1}$$

In the above formula, the quantities on the right hand side are constructed from the map in the deformed toric Euler sequence defining $\mathcal{E}$. We will give the precise definition of the notations later in this thesis.

The authors of [17] work in the physics theory of Coulomb branch and derive the formula (1.0.1) there. They then conjecture that the same formula holds for the geometric case as described in Conjecture 1.0.1.

Conjecture 1.0.1 has the feature that the quantum correlators take values in the complex numbers. Compared to the result in [9], it offers a specific identification of H^* to $\mathbb{C}$, and an effective way computing it.

In Chapter 2 we prove Conjecture 1.0.1 using results of [9]. We then explore the next important case, where the omalous bundle is a deformation of $T \oplus \mathcal{O}$, in Chapter 3. Lastly, in Chapter 4, we discuss the case when the variety is a complete intersection in a toric variety, as well as the QSC for Grassmannians. These two directions are motivated by physicists' interest: complete intersections contains the case of Calabi-Yau manifolds which are interested to string theorists; QSC on Grassmannians generalizes the current theory to the case with non-abelian gauge groups.

Chapter 2

QSC for Deformations of Tangent Bundles on Toric Varieties

In Section 2.1 we set up the basic notations, introduce the classical correlators, and review some results in the case where $\mathcal{E}$ is the tangent bundle of the toric variety.

In Section 2.2, we prove an integral formula that computes the classical correlators for $\mathcal{E}$ being a deformation of the tangent bundle T_V with small deformation parameters.

In Section 2.3, we define the quantum correlators following [9]. Then we use this to write the quantum correlator, which is the sum of all contributions of classical correlators from different moduli spaces parametrized by effective curve classes of the toric variety, into an integral form.

In Section 2.4, we show that Equation (1.0.1) can be written as an integral, which has the same integrand as the one in Section 2.3. However the two integrals are over different cycles. we will prove the equality of the two integrals, which proves Conjecture 1.0.1. The proof is inspired by the (2,2) case work [20] proving the TRMC. In the end we comment on the significance of this summation formula.

2.1 Preliminaries

2.1.1 The basic setting

Throughout this thesis, let V be an n-dimensional smooth projective toric variety with the fan Σ, and let $\Sigma(k)$ be the collection of all k-dimensional cones of the fan Σ. Each ray $i \in \Sigma(1)$ corresponds to a prime divisor D_i via the orbit-cone correspondence.

Let $\mathcal{E}$ be a holomorphic vector bundle of rank n on V, defined by the deformed toric Euler sequence (sometimes referred as *monadic deformation*, as $\mathcal{E}$ is the cokernel of two bundles that are direct sums of line bundles)

$$0 \to \mathcal{O} \otimes W^\vee \to \bigoplus_{i \in \Sigma(1)} \mathcal{O}(D_i) \to \mathcal{E} \to 0. \tag{2.1.1}$$

We will also make use of the dual sequence

$$0 \to \mathcal{E}^\vee \to \bigoplus_{i \in \Sigma(1)} \mathcal{O}(-D_i) \xrightarrow{E} \mathcal{O} \otimes W \to 0. \tag{2.1.2}$$

2.1.2 The definition of the polymology

Note that $W \cong H^1(\mathcal{E}^\vee)$ for all bundles $\mathcal{E}^\vee$ defined by different E maps. The space of cohomology $\bigoplus_{p,q} H^q(V, \wedge^q \mathcal{E}^\vee)$ together with cup product forms an associative algebra $H^*_{\mathcal{E}}(V)$ called the *polymology* of $\mathcal{E}$.

Recall that the cohomology of toric varieties can be described using Stanley-Reisner ideals:

$$H^*_{T_V}(V) \cong \mathrm{Sym}^* W/SR(V). \tag{2.1.3}$$

We can describe the polymology in a similar way.

Let E be the second map in (2.1.2) whose kernel defines $\mathcal{E}^\vee$. Then E is in $Hom(\oplus(-D_i), \mathcal{O} \otimes W) \cong H^0(\oplus \mathcal{O}(D_i)) \otimes W$. For each $i \in \Sigma(1)$ we have an expression in monomials

$$E_i = \sum_{m \in \Delta_i \cap M} a_{im} \chi^m, \tag{2.1.4}$$

where $\chi^m, m \in \Delta_i \cap M$ is a basis of $H^0(\mathcal{O}(D_i))$ and a_{im} takes values in W.

Since it is shown by [9] that the quantum sheaf cohomology does not depend on non-linear deformations, we can focus on the linear part of E_i defined by

$$E_i^{\mathrm{lin}} := \sum_{i' \in \{i' \in \Sigma(1) \ | \ D_{i'} \sim D_i\}} a_{ii'} x_{i'}, \tag{2.1.5}$$

where $D_{i'} \sim D_i$ means they are linearly equivalent divisors, and $x_{i'} \in H^0(\mathcal{O}(D_i))$ is the element in the homogeneous coordinate ring of the toric variety corresponding to a global section vanishing on $D_{i'}$.

For the divisor class c, we introduce the notation $Q_c = \det(a_{ij})$ with i, j running through all rays $k \in \Sigma(1)$ such that $[D_k] = c$.

An important notion in toric geometry is the primitive collection. A primitive collection $P \subset \Sigma(1)$ is a collection of rays such that no cone in Σ contains all the rays in P, but for any proper subset P' of P, there is a cone in Σ that contains all the rays in P'.

It can be shown that for each primitive collection P of rays in $\Sigma(1)$, if it contains one ray i, it has to contain all i' such that $D_{i'} \sim D_i$.

For any subset $S \subset \Sigma(1)$, let $[S] = \{[D_i] \mid i \in S\}$.

Define the deformed Stanley-Reisner ideal $SR(V, \mathcal{E}) \subset \mathrm{Sym}^* W$ to be the ideal generated by $\prod_{c \in [P]} Q_c$ with P running through all primitive collections of the fan, i.e.,

$$SR(V, \mathcal{E}) = \langle \prod_{c \in [P]} Q_c \mid P \text{ is a primitive collection} \rangle. \tag{2.1.6}$$

Then it is proved in [9] that the polymology of $\mathcal{E}$ satisfies:

$$H^*_{\mathcal{E}}(V) \cong \mathrm{Sym}^* W / SR(V, \mathcal{E}). \tag{2.1.7}$$

2.1.3 The classical correlators

For $i \in \Sigma(1)$, let $\alpha_i \in W$ be the first Chern class of the toric invariant divisor D_i (under the identification $W \cong H^1(\Omega)$) and denote $\mathfrak{U} = \{\alpha_i \mid i \in \Sigma(1)\}$. Let $\sigma_i \in W$ be a general element of W and $\sigma_I = \prod_{i \in I} \sigma_i \in \mathrm{Sym}^* W$.

Note that (2.1.2) implies that $\wedge^n \mathcal{E}^\vee \cong \mathcal{O}(-\sum_{i \in \Sigma(1)} D_i) \cong K$, the canonical bundle. Hence $H^n(V, \wedge^n \mathcal{E}^\vee)$ is one-dimensional. Identify $H^n(V, \wedge^n \mathcal{E}^\vee)$ with $\mathbb{C}$ by integrating over the fundamental class. For $\sigma_i, i \in I$, one can first take the image of σ_I in $H^*_{\mathcal{E}}(V)$, project to the degree (n,n) part $[\sigma_I]_n$, and define the *(classical) correlator* of $\sigma_i, i \in I$ to be the image of $[\sigma_I]_n$ in $\mathbb{C}$. Denote the correlator of $\sigma_i, i \in I$ by $\langle \sigma_I \rangle$.

2.1.4 An integral formula for the (2,2) classical correlators

Following the physicists' language, we call the case in which $\mathcal{E}$ is the tangent bundle T_V *the (2,2) case*. In this case the polymology of $\mathcal{E}$ is just the cohomology of holomorphic forms, and the ring structure can be computed by intersection theory (recall that V is a smooth projective toric variety).

In this section, we present an integral formula for the (2,2) classical correlators found by Szenes and Vergne in [20]. We will generalize this formula in Section 2.2.

First we need more notations.

Choose a maximal cone $\sigma \in \Sigma(n)$, and fix an order of $\alpha_{i_1}, ..., \alpha_{i_r}$ corresponding to rays that are not in σ. This fixes a translation invariant measure $d\mu$ on $W^\vee$, where $d\mu = d\alpha_{i_1} \wedge d\alpha_{i_2} \wedge ... \wedge d\alpha_{i_r}$.

For each prime toric divisor class $c \in [\Sigma(1)]$, let H_c be a hypersurface in $W^\vee$ defined by $H_c = \{u \in W^\vee \mid Q_c(u) = 0\}$. Let $U(\mathcal{E})$ be the complement of the union of $H_c, c \in [\Sigma(1)]$. In the current case where $\mathcal{E} = T_V$, $U(T_V)$ is the complement of the union of hyperplanes defined by $\alpha_i = 0, i \in \Sigma(1)$. Let $r = \dim W^\vee$.

We can then state the following theorem:

Theorem 2.1.1. *There is a homology class $h(T_V) \subset H_r(U(T_V), \mathbb{Z})$ such that the following integral computes the (2,2) classical correlators for any $\sigma_i \in W, i \in I$.*

$$\langle \sigma_I \rangle = \frac{1}{(2\pi i)^r} \int_{h(T_V)} \frac{\sigma_I}{\prod_{i \in \Sigma(1)} \alpha_i} d\mu. \tag{2.1.8}$$

Moreover, the homology class is represented by a disjoint union of tori with orientations, as described below.

Description of the homology class.

To describe $h(T_V)$, we first introduce the set $\mathcal{FL}(\mathfrak{U})$ of complete flags

$$F = \{F_0 = \{0\} \subset F_1 \subset F_2 \subset ... \subset F_{r-1} \subset F_r = W\}, \tag{2.1.9}$$

such that each F_j is generated by some $\alpha_i \in \mathfrak{U}$.

We say an ordered basis $\gamma^F = (\gamma_1^F, ..., \gamma_r^F)$ of W is *compatible* with F, if the following conditions hold:

(a) γ_j^F is rationally generated by $\alpha_i \in \mathfrak{U}$,

(b) $\{\gamma_m^F\}_{m=1}^j$ is a basis of F_j,

(c) $d\gamma_1^F \wedge ... \wedge d\gamma_r^F = d\mu$.

Fix a ξ in the Kähler cone $\mathfrak{c}$ of V. Let $\mathcal{FL}^+(\mathfrak{U}, \xi)$ be the set of those flags $F = \{F_j\} \in \mathcal{FL}(\mathfrak{U})$ such that ξ is in the interior of the cone spanned by κ_j, $j = 1, 2, ..., r$, where $\kappa_j = \sum_{\{\alpha_i \in \mathfrak{U} | \alpha_i \in F_j\}} \alpha_i$.

For each flag F, we always fix a compatible basis γ^F. Let $u_j = \gamma_j^F(u)$ for $u \in W^\vee$.

Consider the torus $T_F(\epsilon) = \{u \in W^\vee \mid |u_j| = \epsilon_j, j = 1, ..., r\}$. Let

$$Z(\epsilon) = \sum_{F \in \mathcal{FL}^+(\mathfrak{U}, \xi)} \nu(F) T_F(\epsilon), \tag{2.1.10}$$

where $\nu(F) = \pm 1$ depending on the orientation of κ_j. Szenes and Vergne prove the theorem by showing that $Z(\epsilon)$ represents $h(T_V)$ for ϵ in some specific neighbourhood of 0. We will specify the constraint for ϵ in Section 2.2.

2.2 An integral formula for (0,2) classical correlators

In this section we prove an integral formula which computes the classical correlators for $\mathcal{E}$ being a deformation of the tangent bundle T_V with small deformation parameters. The statement is Theorem 2.2.1 below.

2.2.1 The integral formula

We fix the same translation invariant measure $d\mu$ on $W^\vee$ as in Section 2.1.4. And recall that $U(\mathcal{E})$ is the complement of the union of all the hypersurfaces $H_c = \{u \in W^\vee \mid Q_c(u) = 0\}$ in $W^\vee$, for $c \in [\Sigma(1)]$. $r = \dim W^\vee$. Our first result generalizes the formula for (2,2) classical correlators:

Theorem 2.2.1. *For $\mathcal{E}$ being a small deformation of the tangent bundle T_V, there is a homology class $h(\mathcal{E}) \subset H_r(U(\mathcal{E}), \mathbb{Z})$ such that the following integral computes the (0,2) classical correlators for any $\sigma_i \in W, i \in I$:*

$$\langle \sigma_I \rangle = \frac{1}{(2\pi i)^r} \int_{h(\mathcal{E})} \frac{\sigma_I}{\prod_{c\in[\Sigma(1)]} Q_c} d\mu, \tag{2.2.1}$$

where Q_c's are factors of generators of $SR(V, \mathcal{E})$, as described in (2.1.6).

Moreover, $h(\mathcal{E})$ is represented by $Z(\epsilon)$ as described in (2.1.10).

Remark:

(a) There are constrains on ϵ_j.That is $N\epsilon_i < \epsilon_{i+1}, i = 1, 2, ..., r-1$, for a sufficiently large N, namely, N is larger than $N_0(F)$ which could be chosen as follows: For all $l \in \Sigma(1)$, write $\alpha_l = \sum a_{li}\gamma_i^F$. Define $N_0(F) = r \cdot (\max_l(\frac{1}{|a_{li}|}) \cdot (\max_i |a_{li}|))$.

(b) The integral vanishes on $SR(V, \mathcal{E})$. This consists the main part of the proof.

(c) It is shown in [9] that the quantum sheaf relations do not depend on the non-linear deformations. Hence the correlation functions, being linear functions vanishing on the ideal generated by the quantum sheaf relations, do not depend on the non-linear deformations.

2.2.2 Preparatory results

We state and prove some lemmas before we prove Theorem 2.2.1 in next section.

Lemma 2.2.2. *Fix a flag F and a collection $\{\alpha_l, l \in L\}$. If there exist a k, $1 \leq k \leq r$ such that F_k is not generated by elements of $\{\alpha_l, l \in L\}$ but every $F_i, i < k$ is, then $\prod_{l \in L} \alpha_L$ is non-zero on the region $\Omega = \{u; |u_k| \leq \epsilon_k, |u_i| = \epsilon_i, \text{for } i \neq k\}$.*

Proof:

For any $l \in L$, express α_l in terms of the basis γ_i^F, we have $\alpha_l = \sum a_{li} \gamma_i^F$. Order the sum in the descending order regarding i-index, and call the largest index s. Then $s \neq k$ by the definition of k.

Then, for $u \in \Omega, l \in L$,

$$\begin{aligned} |\alpha_l(u)| &= |\textstyle\sum_{i=1}^{s} a_{li} u_i| \\ &\geq |a_s| \cdot |u_s| - \textstyle\sum_{i=1}^{s-1} |a_i| \cdot |u_i| \\ &\geq |a_s| \epsilon_s - \textstyle\sum_{i=1}^{s-1} |a_i| \cdot \epsilon_i \\ &> \epsilon_{s-1}(N_0(F) \cdot |a_s| - r \cdot \max_i(|a_i|)) \\ &> 0, \end{aligned} \tag{2.2.2}$$

by the definition of $N_0(F)$. $\square$

Lemma 2.2.3. *Let $P \subset \Sigma(1)$ be a primitive collection.*

(a)

When $\{\alpha_i, i \in L = \Sigma(1) - P - J\}$ does not form a basis compatible with the flag F,

$$\int_{T_F(\epsilon)} \frac{\alpha_J}{\prod_{i \notin P} \alpha_i} d\mu = 0.$$

In particular, when $\{\alpha_i, i \in L\}$ does not generate W, we have

$$\int_{T_F(\epsilon)} \frac{f}{\prod_{i \in L} \alpha_i} d\mu = 0$$

for any $T_F(\epsilon)$.

(b) When $\alpha_i, i \in \Sigma(1) - P - J$ form a basis compatible with both the flag F and F', $\int_{T_{F'}(\epsilon)} \frac{\alpha_J}{\prod_{c \notin P} Q_c} = sgn(F', F) \cdot \int_{T_F(\epsilon)} \frac{\alpha_J}{\prod_{c \notin P} Q_c}$, where $sgn(F', F) = \pm 1$ depending

Proof:

(a) Let $L = \Sigma(1) - P - J$. $\{\alpha_i, i \in L\}$ satisfies the assumption of Lemma 2.2.2, hence $\prod_{l \in L} \alpha_L$, is non-zero on Ω.

Hence the integrand is defined on the region Ω. Note that $T_F(\epsilon) = \partial\Omega$. Hence $\int_{T_F(\epsilon)} \frac{f}{prod_{i \notin P} \alpha_i} d\mu = \int_\Omega d(\frac{f}{prod_{i \notin P} \alpha_i} d\mu) = 0$.

(b) In this case we will rename the indices of $\{\alpha_i, i \in \Sigma(1) - P - J\}$, so that F_j is generated by $\{\alpha_1, ..., \alpha_j\}$. If we write $\alpha_j(u) = \sum_{i=1}^{j} a_{ji} u_i$, then $|\alpha_j(u)/a_{jj}| = |u_j + \sum_{i=1}^{i=j-1} \frac{a_{ji}}{a_{jj}} u_i|$. This allows us to use a linear homotopy map to show that $|\alpha_j(u)/a_{jj}| = \epsilon_j$ is homotopic to $|u_j| = \epsilon_j$. Thus we conclude that $T_F(\epsilon) = \pm[T_\alpha]$, where $T_\alpha = \{u; |\alpha_j(u)/a_{jj}| = \epsilon_j\}$ and $[T_\alpha]$ is its homology class.

Recall that Q_c is a deformation of $\prod_{j,[D_j]=c}\alpha_j$. We may group the α_j's for $j\in J$ in a similar fasion. Namely, we have

$$\alpha_J=\prod_c(\prod_{j\in J,[D_j]=c}\alpha_j). \tag{2.2.3}$$

Since $J\subset\Sigma(1)-P$, we have

$$\frac{\alpha_J}{\prod_{c\notin P}Q_c}=\frac{1}{\prod_{c\notin P}\frac{Q_c}{\prod_{j\in J,[D_j]=c}\alpha_j}}. \tag{2.2.4}$$

One observation is that r of the factors $Q_c/(\prod_{j\in J,[D_j]=c}\alpha_j)$ are of degree 1 as rational functions, and the others are of degree 0, since $\alpha_i, i\in\Sigma(1)-P-J$ form a basis of W. We denote the degree 1 factors as $\tilde{\alpha}_i$, $i=1,...,r$, and the degree 0 factors l_j. Rewrite $(\prod l_j)\tilde{\alpha}_1$ as $\tilde{\alpha}_1$ so that all l_j are absorbed. After re-indexing, we can assume that $\tilde{\alpha}_i$ is a small deformation of α_i. Then we claim that the integration result satisfies

$$|\int_{T_F(\epsilon)}\frac{\alpha_J}{\prod_{c\notin P}Q_c}d\mu|=|\int_{T_{\tilde{\alpha}}}\frac{1}{\prod_{i=1}^r\tilde{\alpha}_i}d\mu|, \tag{2.2.5}$$

where $T_{\tilde{\alpha}}=\{u;|\tilde{\alpha}_j(u)/a_{jj}|=\epsilon_j\}$. To see why this is true, we note that similarly to the above description of deforming T_F to T_α, we can deform T_α to $T_{\tilde{\alpha}}$, as long as $Q_c(\mathcal{E})$ is a sufficiently small deformation of $Q_c(T)=\prod_{i;[D_i]=c}\alpha_i$. Since $T_{\tilde{\alpha}}$ only differs by possibly an orientation from the permutation of α_i for different F's, we conclude that the result is independent of the flag F as long as $\{\alpha_i, i\in\Sigma(1)-P-J\}$ forms a basis compatible with F.

2.2.3 Proof of the Theorem

Since Szenes-Vergne [20] has proved the corresponding result in (2,2) case, the map

$$\sigma \mapsto \int_{T_F(\epsilon)} \frac{\sigma}{\prod_{c\in[\Sigma(1)]} Q_c} d\mu \tag{2.2.6}$$

is not identically zero for small deformations. So it suffices to prove that for $\sigma \in SR(V,\mathcal{E})$ with $\deg\sigma = n$,

$$\int_{h(\mathcal{E})} \frac{\sigma}{\prod_{c\in[\Sigma(1)]} Q_c} d\mu = \sum_{F\in\mathcal{FL}^+(\xi)} \nu(F) \int_{T_F(\epsilon)} \frac{\sigma}{\prod_{c\in[\Sigma(1)]} Q_c} d\mu = 0. \tag{2.2.7}$$

Since $SR(V,\mathcal{E})$ is generated by $\{\prod_{c\in P} Q_c | P \subset \Sigma(1)$ is a primitive collection$\}$, it suffices to prove that the above equality (2.2.7) is true for those σ of the form $\sigma = (\prod_{c\in P} Q_c)\cdot\sigma_J$, where P is a primitive collection, $J \subset \Sigma(1)$, and $|J| = n - |P|$. Now we compute this by deforming the corresponding (2,2) result.

Note that $\prod_{c\notin P} Q_c$ is a small deformation of $\prod_{i\notin P}\alpha_i$, we can write $\prod_{c\notin P} Q_c = \prod_{i\notin P}\alpha_i - \delta\tilde{\alpha}$, for some small $\delta\in\mathbb{C}$ and $\tilde{\alpha}\in \mathrm{Sym}^*\mathrm{W}$. So we have

$$\begin{aligned}
\int_{h(\mathcal{E})} \frac{\sigma}{\prod_{c\in[\Sigma(1)]} Q_c} d\mu &= \int_{h(\mathcal{E})} \frac{\alpha_J}{\prod_{c\notin P} Q_c} d\mu \\
&= \int_{h(\mathcal{E})} \frac{\alpha_J}{\prod_{i\notin P}(\alpha_i - \delta\tilde{\alpha})} d\mu \\
&= \int_{h(\mathcal{E})} \frac{\alpha_J}{\prod_{i\notin P}\alpha_i} (\sum_{n=0}^{\infty} (\frac{\delta\tilde{\alpha}}{\prod_{i\notin P}\alpha_i})^n) d\mu \\
&= \sum_{m=0}^{\infty} \int_{h(\mathcal{E})} \frac{(\delta\tilde{\alpha})^m \alpha_J}{(\prod_{i\notin P}\alpha_i)^{m+1}} d\mu.
\end{aligned} \tag{2.2.8}$$

Claim: for each monomial $\prod_{i\in K}\alpha_i$ such that $\frac{\prod_{i\in K}\alpha_i}{(\prod_{i\notin P}\alpha_i)^{m+1}}$ has degree $-r$,

$$\int_{h(\mathcal{E})}\frac{\prod_{i\in K}\alpha_i}{(\prod_{i\notin P}\alpha_i)^{m+1}}d\mu = 0. \tag{2.2.9}$$

Proof of the Claim: by Lemma 2.2.3, if the factors of the denominator do not generate W, the integration is 0. Hence it reduces to the case when the factors of the denominator generate W.

For $k \in K$, write $\alpha_k = \sum a_{k_l}\alpha_l$, where l runs through those indices appearing in the denominator. This reduces the integrand to

$$\frac{\prod_{i\in K}\alpha_i}{(\prod_{i\notin P}\alpha_i)^{m+1}}d\mu = \sum_l \frac{a_{kl}\cdot\prod_{i\in K, i\neq k}\alpha_i}{(\prod_{i\notin P}\alpha_i)^{m+1}/\alpha_l}d\mu. \tag{2.2.10}$$

Observe that as long as the remaining denominator of a summand generates W, we can repeat this process of expressing the numerator terms into linear combinations of the denominator terms and then canceling out a term. This process terminates after finite steps, and the final expression is a summation of terms of two types:

Type (i): terms with non-generating denominators.

Type (ii): terms with factors of the denominator generate W, while the numerator is a constant (degree 0).

Type (i) terms integrate to 0 by Lemma 2.2.3. So to prove the claim, it suffices to show that each Type (ii) term integrates to 0. Namely

$$\int_{h(\mathcal{E})} \frac{1}{\prod_{j=1}^{r} \alpha_{i_j}} d\mu = 0. \tag{2.2.11}$$

Type (ii) terms have denominators of degree r, since the cancellation process preserves the degree of the fraction. Note that for the factors of the degree r denominator to generate W which is r dimensional, these r factors have to be distinct. So, being factors of $(\prod_{i \notin P} \alpha_i)^{m+1}$, they are actually factors of $\prod_{i \notin P} \alpha_i$.

Hence we have

$$\begin{aligned} & \textstyle\int_{h(\mathcal{E})} \frac{1}{\prod_{j=1}^{r} \alpha_{i_j}} d\mu \\ = & \textstyle\int_{h(\mathcal{E})} \frac{\prod_{i \in L} \alpha_i}{\prod_{i \in \Sigma(1)-P} \alpha_i} \\ = & \textstyle\int_{h(\mathcal{E})} \frac{\prod_{i \in L \cup P} \alpha_i}{\prod_{i \in \Sigma(1)} \alpha_i} \\ = & \ 0 \end{aligned} \tag{2.2.12}$$

The last equality comes from Theorem 2.1.1 and the fact that $\prod_{i \in L \cup P} \alpha_i \in SR(V)$. This proves Equation (2.2.11). Hence the claim is proved.

The theorem then follows from the claim and Equation (2.2.8).

2.3 The first integral formula for quantum correlators

2.3.1 The quantum correlators

Let $d_c^{\beta_j}$ be the intersection number of β_j with any divisor in the divisor class c.

The *quantum correlator* of $\sigma_i \in W, i \in I$ is defined to be a summation over the GLSM moduli spaces $\mathcal{M}_\beta$ indexed by effective curves β:

$$\langle \sigma_I \rangle^{quantum} = \sum_\beta \langle \sigma_I F_\beta \rangle_\beta q^\beta, \tag{2.3.1}$$

where β runs over the lattice points in the Mori cone (generated by effective curve classes) of the toric variety V, and F_β is the four-Fermi term introduced in [9]:

$$F_\beta = \prod_{c \in [\Sigma(1)]} Q_c^{h^1(d_c^\beta)}. \tag{2.3.2}$$

Also, when β' dominates β, the correlators over different moduli spaces are related by the "exchange rate" $R_{\beta'\beta}$:

$$\begin{aligned} \langle \sigma_I F_\beta \rangle_\beta &= \langle \sigma_I F_\beta R_{\beta'\beta} \rangle_{\beta'}, \\ R_{\beta'\beta} &= \textstyle\prod_c Q_c^{h^0(d_c^{\beta'}) - h^0(d_c^\beta)}. \end{aligned} \tag{2.3.3}$$

Remark: This holds even when β is not effective. The re-definition of q^β: Instead of viewing q^β as a formal parameter, we now consider complex value of it. For each $z = \sum_{i=1}^{n+r} z_i\omega_i \in g$ (ω_i corresponds to the ray $i \in \Sigma(1)$), and each $\beta \in H_2(V, \mathbb{Z})$, define $q^\beta(z) = \prod_{i=1}^{n+r} z_i^{\langle\alpha_i,\beta\rangle}$, where $\alpha_i \in H^2(V, \mathbb{Z})$ corresponds to the ray $i \in \Sigma(1)$.

2.3.2 The Mori cone of smooth projective variety

The Mori cone of a variety is the closure of the cone effective curves. For projective toric variety V, the Mori cone $\overline{NE}(V)$ is a strongly convex rational polyhedral cone of full dimension in $N_1(V)$, the real vector space of proper 1-cycles modulo numerical equivalence. ([8], pp 292 - 295.) Theorem 6.4.11 of [8] gives a concrete description of the Mori cone for any projective simplicial toric variety, representing it by specifying a generator for each primitive collection of V.[1] As a corollary, we have:

Proposition 2.3.1. *The number of primitive collections of a simplicial projective toric variety V is no less than its Picard number.*

Proof: By the above mentioned theorem, the number of primitive collections is the same as the number of cone generators of the Mori cone. Since the Mori has dimension $r =$ the Picard number, the proposition is proved.

[1] The simplical condition is not necessary, as commented in [8], p 307.

2.3.3 The integral formula

Note that we can choose generators of the Mori cone. Since the dimension of the Mori cone equals the Picard number r, we can choose r generators $\beta_1, ..., \beta_r$ such that they generates all the lattice points in a possibly bigger cone containing the Mori cone. We also require that $\langle \beta_j, \sum_{i=1}^{n+r} D_i \rangle \geq 0$. Since outside the Mori cone the moduli space is simply empty and (2.3.3) still holds in this case, we can write the summation over the Mori cone as summation over this (possibly bigger) cone:

$$\begin{aligned} \langle \sigma_I \rangle^{quantum} &= \varinjlim_{B} \sum_{\beta \text{ dominated by } B} \langle \sigma_I \cdot F_\beta \cdot R_{B\beta} q^\beta \rangle_B \\ &= \varinjlim_{B} \langle \sigma_I \sum_\beta \prod_c Q_c^{h^0(d_c^B) - h^0(d_c^\beta) + h^1(d_c^\beta)} q^\beta \rangle_B \\ &= \varinjlim_{B} \langle \sigma_I \prod_c Q_c^{h^0(d_c^B)-1} \sum_\beta \prod_c Q_c^{-d_c^\beta} q^\beta \rangle_B \\ &= \lim_{N\to\infty} \langle \sigma_I \prod_c Q_c^{h^0(d_c^B)-1} \prod_{j=1}^r (\sum_{a_j=0}^N u_j^{a_j}) \rangle_B \\ &= \lim_{N\to\infty} \langle \sigma_I \prod_c Q_c^{h^0(d_c^B)-1} \prod_{j=1}^r \frac{1-u_j^{N+1}}{1-u_j} \rangle_B, \end{aligned} \tag{2.3.4}$$

where $u_j = \prod_c Q_c^{-d_c^{\beta_j}} q^{\beta_j}$, and we will later write q_j for q^{β_j}.

Now we have:

$$\begin{aligned} \langle \sigma_I \rangle^{quantum} &= \lim_{N\to\infty} \frac{1}{(2\pi i)^r} \int_{h(\mathcal{E})} \frac{1}{\prod_c Q_c^{h^0(d_c^B)}} \cdot \left(\sigma_I \prod_c Q_c^{h^0(d_c^B)-1} \prod_{j=1}^r \frac{1-u_j^{N+1}}{1-u_j} \right) d\mu. \\ &= \lim_{N\to\infty} \frac{1}{(2\pi i)^r} \int_{h(\mathcal{E})} \frac{1}{\prod_{c\in[\Sigma(1)]} Q_c} \cdot \left(\sigma_I \prod_{j=1}^r \frac{1-u_j^{N+1}}{1-u_j} \right) d\mu. \end{aligned} \tag{2.3.5}$$

Take a representative $Z(\epsilon)$ of $h(\mathcal{E})$ and take q_j sufficiently small, we will have $|u_j| < 1$ on $h(\mathcal{E})$. We also write $v_j = u_j^{-1} = \prod_c Q_c^{d_c^{\beta_j}} q_j^{-1}$, $\tilde{v}_j = \prod_c Q_c^{d_c^{\beta_j}}$. So we have $|v_j| > 1$ on $h(\mathcal{E})$. Hence

$$\begin{aligned}
\langle \sigma_I \rangle^{quantum} &= \lim_{N\to\infty} \frac{1}{(2\pi i)^r} \int_{Z(\epsilon)} \frac{\sigma_I}{\prod_c Q_c} \prod_{j=1}^{r} \frac{1-u_j^{N+1}}{1-u_j} d\mu \\
&= \lim_{N\to\infty} \frac{1}{(2\pi i)^r} \int_{Z(\epsilon)} \frac{\sigma_I}{\prod_{c\in[\Sigma(1)]} Q_c} \prod_{j=1}^{r} \frac{1-u_j^{N+1}}{1-u_j} d\mu \\
&= \frac{1}{(2\pi i)^r} \int_{Z(\epsilon)} \frac{\sigma_I}{\prod_c Q_c} \prod_{j=1}^{r} \frac{1}{1-u_j} d\mu \\
&= \frac{1}{(2\pi i)^r} \int_{Z(\epsilon)} \frac{\sigma_I \prod_j v_j}{\prod_{c\in[\Sigma(1)]} Q_c} \prod_{j=1}^{r} \frac{1}{v_j - 1} d\mu
\end{aligned} \tag{2.3.6}$$

Thus we have proved the following result:

Theorem 2.3.2. *Let $\mathcal{E}$ be a holomorphic vector bundle defined by the deformed toric Euler sequence (2.1.1) with small deformations. Let $Z(\epsilon)$ be a cycle representing $h(\mathcal{E})$ and $z \in (\mathbb{C}^*)^n$. Let $q_j = z^{\beta_j}$, $j = 1, ..., r$. For a fixed basis $\beta_1, ..., \beta_r$ and $z \in (\mathbb{C}^*)^n$ such that $|q_j| < \min_{u\in Z(\epsilon)} |\tilde{v}_j(u)|$ holds, we have*

$$\langle \sigma_I \rangle^{quantum} = \frac{1}{(2\pi i)^r} \int_{Z(\epsilon)} \frac{\sigma_I}{\prod_{c\in[\Sigma(1)]} Q_c} \frac{\prod_{j=1}^{r} \tilde{v}_j}{\prod_{j=1}^{r} (\tilde{v}_j - q_j)} d\mu. \tag{2.3.7}$$

2.4 Quantum correlator: summation formula

2.4.1 The Main Result

In McOrist-Melnikov [17], there is a summation formula for quantum correlators, for $\mathcal{E}$ defined by the deformed toric Euler sequence 2.1.1 with a linear deformation. The authors derive it from physics argument using Coulomb branch techniques. Using the quantum sheaf cohomology theory set up by [9], we state it as Conjecture 1.0.1 in the Introduction.

Denote the partial derivative of $\frac{\partial f(u)}{\partial u_k}$ by $f_{,k}$. Recall that $\tilde{v}_j = \prod_{c\in[\Sigma(1)]} Q_c^{d_c^{\beta_j}}$, $z \in (\mathbb{C}^*)^n$, $q_j = z^{\beta_j}$, $j = 1, ..., r$. $\beta_1, ..., \beta_r$ generates a cone containing the Mori cone. Let $z \in (\mathbb{C}^*)^n$ such that $|q_j| < \min_{u\in Z(\epsilon)} |\tilde{v}_j(u)|$ holds. The main result of this thesis is a mathematical proof of Conjecture 1.0.1, including possibly non-linear deformations:

Main Result. *Let $\mathcal{E}$ be a holomorphic vector bundle defined by the deformed toric Euler sequence (2.1.1) with small deformations such that Theorem 2.2.1 holds. Then*

$$\langle \sigma_{i_1}, ..., \sigma_{i_s} \rangle^{quantum} = \sum_{\{u\in W^\vee | \tilde{v}_j(u)=q_j\}} \frac{\sigma_I}{\prod_{c\in[\Sigma(1)]} Q_c} \frac{\prod_{j=1}^r \tilde{v}_j}{\det_{j,k}(\tilde{v}_{j,k})}. \tag{2.4.1}$$

holds for z in a complex open region contained in $(\mathbb{C}^*)^n$.

Remarks: (a) Let $h_q \in H_r(W^\vee - \{u \in W^\vee \mid \tilde{v}_j(u) = q_j\}, \mathbb{Z})$ be the homology of the real r-dimensional cycle defined by $\{u \in W^\vee \mid |\tilde{v}_j(u) - q_j| < \delta\}$, for a δ that is small enough. Then the above formula (2.4.1) can be written as

$$\langle \sigma_I \rangle^{quantum} = \frac{1}{(2\pi i)^r} \int_{h_q} \frac{\sigma_I}{\prod_{c\in[\Sigma(1)]} Q_c} \frac{\prod_{j=1}^r \tilde{v}_j}{\prod_{j=1}^r (\tilde{v}_j - q_j)} d\mu. \tag{2.4.2}$$

Equation (2.3.7) and (2.4.2) has exactly the same integrand.

(b) When $\mathcal{E}$ is the tangent bundle, (2.4.2) is reduced to

$$\langle \sigma_I \rangle^{quantum} = \frac{1}{(2\pi i)^r} \int_{h_q} \frac{\sigma_I}{\prod_{i\in\Sigma(1)} \alpha_i} \frac{\prod_{j=1}^r \tilde{v}_j}{\prod_{j=1}^r (\tilde{v}_j - q_j)} d\mu. \tag{2.4.3}$$

This resembles the conclusion of the hypersurface case of the "Toric Residue Mirror Conjecture" in (2,2) theory [1], which says[2] for anti-canonical hypersurface X (i.e. the fundamental class is dual to $\kappa = \sum_{i\in\Sigma(1)} \alpha_i$) in a Fano simplicial toric variety V of dimension n, the quantum correlator $\langle \sigma_{i_1} ... \sigma_{i_{n-1}} \rangle^{quantum}$ for $\sigma_i \in H^1(X, T_X^*)$ coming from the restriction of $H^1(V, T_V^*)$ is

$$\langle \sigma_{i_1} ... \sigma_{i_{n-1}} \rangle^{quantum} = \frac{1}{(2\pi i)^r} \int_{h_q} \frac{\sigma_I}{(1-\kappa)\prod_{i\in\Sigma(1)} \alpha_i} \frac{\prod_{j=1}^r \tilde{v}_j}{\prod_{j=1}^r (\tilde{v}_j - q_j)} d\mu. \tag{2.4.4}$$

The main goal of this section is to prove the Main Result by proving (2.4.2). Our proof is inspired by Szenes and Vergne's proof of (2.4.4).

We set up some notations in 2.4.2. Then we state the theorem and some lemmas. We prove the theorem in 2.4.3.

[2] We adopt the formulation of [20]. See Proposition 4.7 there.

2.4.2 Some preparations

We fix a bundle $\mathcal{E}_1$ such that Theorem 2.3.2 holds. Multiply the deformation parameters in the map defining $\mathcal{E}_1$ by t, we get a one parameter family $\mathcal{E}_t$. Then Theorem 2.3.2 holds for $|t| \leq 1$ and $\mathcal{E}_0 = T_V$.

Specifying the bundle dependence of the map $\tilde{v}_j = \prod_c Q_c^{d_c^{\beta_j}}$, we denote it by $\tilde{v}_j^{(t)}$.

We make some definitions generalizing those (2,2) case notations in [20] to the (0,2) case:

Define

$$\hat{Z}^{(t)}(\xi) = \{u \in U; |\tilde{v}_j^{(t)}| = e^{-\langle \xi, \beta_j \rangle}\}. \tag{2.4.5}$$

$\hat{Z}^{(t)}(\xi)$ can be viewed as the preimage of a torus $T(\xi) = \{y \in (\mathbb{C}^*)^r; |y_j| = e^{-\langle \xi, \beta_j \rangle}\}$, under the map $\tilde{v}^{(t)} = (\tilde{v}_1^{(t)}, ..., \tilde{v}_r^{(t)}) : U(\mathcal{E}) \to (\mathbb{C}^*)^r$.

For $S \subset \{1, 2, ...n\}$ define

$$T_S(\xi, \eta) = \left\{ y \in (\mathbb{C}^*)^r; |y_j| = \begin{cases} \exp(-\langle \xi, \beta_j \rangle), \text{ if } j \in S, \\ \exp(-\langle \xi - \eta, \beta_j \rangle), \text{ if } j \notin S. \end{cases} \right\}$$

and $T_\delta(q) = \{y \in (\mathbb{C}^*)^r; |y_j - q_j| = \delta, j = 1, ..., r\}$. Let $Z_S^{(t)}(\xi, \eta)$ and $Z_\delta^{(t)}(q)$ be the pull-back of $T_S(\xi, \eta)$ and $T_\delta(q)$ respectively by $\tilde{v}^{(t)}$. Note that results about $\hat{Z}(\xi)$ apply to $\hat{Z}_S^{(t)}(\xi, \eta)$.

Remark about notations: In order to keep the notations clean, we omit the label (t) when $t = 1$, as well as (ξ, η), and simply write $Z_S, Z_\delta, \tilde{v}$.

Let $R(\xi, \eta)$ be the multi-dimensional annulus

$$R(\xi, \eta) = \{y = (y_1, ..., y_r) \in (\mathbb{C}^*)^r; \langle \xi - \eta, \beta_j \rangle < -\log |y_j| < \langle \xi, \beta_j \rangle, j = 1, ..., r\},$$

and let $W(\xi, \eta)$ be the pull-back of $R(\xi, \eta)$ by the map $q(z)$.

To simplify notation we will write Λ for $\dfrac{1}{(2\pi i)^r} \dfrac{\sigma_I}{\prod_{c \in [\Sigma(1)]} Q_c} \dfrac{\prod_{j=1}^r \tilde{v}_j}{\prod_{j=1}^r (\tilde{v}_j - q_j)} d\mu$.

We also recall the definition of τ-regularity from [20]:

Definition 2.4.1. ([20]) $\mathfrak{U} = \{\alpha_i \in W; i \in \Sigma(1)\}$. Define

$$\Sigma\mathfrak{U} = \left\{ \sum_{i \in \eta} \alpha_i; \eta \subset \Sigma(1)\} \right\},$$

which is the collection of partial sums of elements of $\mathfrak{U}$. For each subset $\rho \subset \Sigma\mathfrak{U}$ which generates W, we can write $\xi = \sum_{\gamma \in \rho} a_\gamma^\rho(\xi)\gamma$. Denote

$$\min(\Sigma\mathfrak{U}, \xi) = \min\{|a_\gamma^\rho(\xi)|; \rho \subset \Sigma\mathfrak{U}, \rho \text{ basis of } W, \gamma \in \rho\}.$$

We say $\xi \in W$ is τ-regular for $\tau > 0$ if $\min(\Sigma\mathfrak{U}, \xi) > \tau$.

The main theorem of this section is:

Theorem 2.4.1. *Let ξ be τ-regular for τ sufficiently large. For $z \in W(\xi, \eta)$, the* *following holds:*

$$\langle \sigma_I \rangle^{quantum} = \int_{h_q} \Lambda. \tag{2.4.6}$$

The requirement for the technical assumption for ξ being τ-regular for sufficiently large τ will be seen in Proposition 2.4.4.

We first collect some facts about the (2,2) case, and then state some lemmas before proving the theorem.

Proposition 2.4.2. *Let ξ be τ-regular for τ sufficiently large. In the (2,2) case, $\tilde{v}^{(0)}$ is regular over $(\tilde{v}^{(0)})^{-1}(R(\xi, \eta))$. Hence, $\tilde{v}^{(t)}$ is also regular over $(\tilde{v}^{(0)})^{-1}(R(\xi, \eta))$.*

Lemma 2.4.3. *Let ξ be τ-regular for τ sufficiently large. The (2,2) case map $\tilde{v}^{(0)}$ is proper from $(\tilde{v}^{(0)})^{-1}(R(\xi, \eta))$ to $R(\xi, \eta)$. Moreover, the (0,2) case map $\tilde{v}^{(t)}$ is proper on from a suitable region to $R(\xi, \eta)$ when $|t| \leq 1$.*

Proof: the (2,2) case is proved in [20][3]. The (0,2) case follows from deformation, as explained below:

Denote the compact set $(\tilde{v}^{(0)})^{-1}(\overline{R(\xi, \eta)})$ by K, and its boundary by ∂K.

Since $\mathcal{E}$ is a small deformation of T_V, we can pick (ξ', η') such that

(1) $R(\xi', \eta') \subsetneq R(\xi, \eta)$,

(2) $(\tilde{v}^{(t)})^{-1}(R(\xi', \eta')) \cap \partial K = \phi$, and

(3) $(\tilde{v}^{(t)})^{-1}(R(\xi', \eta')) \cap K \neq \phi$.

This then implies the properness of $\tilde{v}^{(t)}$ on $(\tilde{v}^{(t)})^{-1}(R(\xi', \eta')) \cap K$, which is the

[3]It is Proposition 5.15, the map $\tilde{v}^{(0)}$ is just p in [20].

‘suitable region’ in the lemma. $\square$

The technical assumption for ξ being τ-regular for sufficiently large τ is made to achieve the following result in (2,2) case:

Proposition 2.4.4. *(Theorem 6.2 of [20]) If τ is sufficiently large, then for any τ-regular $\xi \in \mathfrak{c}$, the set $\hat{Z}^{(0)}(\xi)$ is a smooth compact cycle whose homology class equals $h(\mathcal{E}) \in H_r(U(\mathcal{E}_0), \mathbb{Z}) = H_r(U(T_V), \mathbb{Z})$.*

Corollary 2.4.5. *The homology class of $\hat{Z}^{(t)}(\xi) \cap K$ is $h(\mathcal{E}_t) \in H_r(U(\mathcal{E}), \mathbb{Z})$.*

Proof: Lemma 2.4.3 shows the properness of $\tilde{v}^{(t)}$ on $(\tilde{v}^{(t)})^{-1}(R(\xi', \eta')) \cap K$. Since $\tilde{v}^{(t)}$ is regular on this region, the cycles $\hat{Z}^{(t)}(\xi) \cap K$ and $\hat{Z}^{(0)}(\xi)$ are homologous as preimages of $T(\xi) \subset R(\xi, \eta)$ under $\tilde{v}^t$. $\square$

Note: Since we always need to take the compact cycles and we can always do so by intersecting with K by Lemma 2.4.3, we will simply write Z for $Z \cap K$ in the rest of this thesis.

Lemma 2.4.6. *(a) $\int_{Z_\phi} \Lambda = \langle \sigma_I \rangle^{quantum}$.*

(b) $\int_{Z_S} \Lambda = 0$, when $S \neq \phi$.

Proof: By Corollary 2.4.5, Z_S represents $h(\mathcal{E}) \in H_r(U(\mathcal{E}), \mathbb{Z})$.

(a) For Z_ϕ, we have $|q_j| < |\tilde{v}_j(u)|$. By Theorem 2.3.2,

$$\langle \sigma_I \rangle^{quantum} = \int_{Z_\phi} \Lambda. \tag{2.4.7}$$

(b) For Z_S, $S \neq \phi$, without loss of generality assume the index $1 \in S$. Then for $u \in Z_S$ we have $|\tilde{v}_1(u)| = \exp(-\langle \xi, \beta_j \rangle) < |q_1|$. Hence Λ can be defined on $C = \{u \in U; |\tilde{v}_1(u)| = e^{-\langle \xi, \beta_1 \rangle}, |\tilde{v}_j(u)| = e^{-\langle \xi, \beta_j \rangle}, j \geq 2.\}$. Since $Z_S = \partial C$, this leads to

$$\int_{Z_S} \Lambda = \int_C d\Lambda = 0. \tag{2.4.8}$$

□

2.4.3 The proof of the Main Result

The proof makes use of Szenes-Vergne's proof for (2,2) case.

It is easy to show[4] that

$$\sum_{S \subset \{1,2,\ldots,n\}} (-1)^{|S|} T_S \text{ is homologous to } T_\delta(q)$$

in the open set $\{y \in (\mathbb{C}^*)^r; y_j \neq q_j \text{for } j = 1, \ldots, r\}$. The properness of $\tilde{v}$ by Lemma 2.4.3 then implies that $\sum (-1)^{|S|} Z_S$ is homologous to $Z_\delta(q)$ in $U(\mathcal{E}) \cap U(\beta, q)$. So

$$\int_{\sum(-1)^{|S|} Z_S} \Lambda = \int_{Z_\delta(q)} \Lambda. \tag{2.4.9}$$

Lemma 2.4.6 then implies that

$$\int_{Z_\phi} \Lambda = \int_{h_q} \Lambda. \tag{2.4.10}$$

[4] See Proposition 6.3 of [20].

This together with (2.4.7) finishes the proof.

Remark: It is worth pointing out that in the (2,2) case, the summation formula is further explained as a toric residue of the dual toric variety. This gives the (2,2) formula the meaning of mirror symmetry. In the (0,2) case, an explanation of the right hand side of this flavour is still lacking. In future work, we hope to describe a set of dual data and explain the right hand side as a "(0,2) toric residue" of the dual data, making the formula into a (0,2) mirror symmetry statement.

Chapter 3

QSC for Higher Rank Bundles on Toric Varieties

3.1 Introduction

In this chapter we show how to compute the quantum sheaf cohomology ring of bundles that are deformations of the direct sum of the tangent bundle and copies of the trivial bundles over a smooth complete toric variety.

We work over a smooth projective toric variety V. In [9], the authors show how to compute the quantum sheaf relations of deformations of tangent bundles over V. And in last chapter we computed the quantum correlators in that case. In this chapter, we set out to understand the case when $\mathcal{E}^\vee$ is a generic deformation of $T \oplus \mathcal{O}^m$.

3.2 T$\oplus\mathcal{O}$

3.2.1 Definition of $\mathcal{E}^\vee$

In this section we give the definition of the bundle $\mathcal{E}^\vee$ we are considering.

Let V be a smooth, complete, n dimensional toric variety. Recall the Toric Euler sequence for the cotangent bundle of V is:

$$0 \to \Omega \to \oplus_{i=1}^{n+r}\mathcal{O}(-D_i) \xrightarrow{e} \mathrm{Pic(V)} \otimes \mathcal{O} \to 0. \text{ [1]} \tag{3.2.1}$$

We set $Z = \oplus_{i=1}^{n+r}\mathcal{O}(-D_i)$ and $W = \mathrm{Pic(V)}\otimes\mathbb{C}$. To define $\mathcal{E}^\vee$, we add a trivial bundle $\mathcal{O}$ to Z, and deform the map $(e, 0) : Z\oplus\mathcal{O} \to W\otimes\mathcal{O}$ to $\varepsilon = (\varepsilon', \varepsilon_0) : Z\oplus\mathcal{O} \to W\otimes\mathcal{O}$. Now we define $\mathcal{E}^*$ to be the kernel of ε if it is a vector bundle, i.e. $\mathcal{E}^*$ fits in the following short exact sequence:

$$0 \to \mathcal{E}^\vee \to Z \oplus \mathcal{O} \to W \otimes \mathcal{O} \to 0. \tag{3.2.2}$$

Let $i : Z \to Z \oplus \mathcal{O}$ be the inclusion $z \mapsto (z, 0)$, then the following diagram of

[1] [8]p363 vp387

exact sequences commutes:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Z \arrow[r, "\varepsilon'"] \arrow[d, "i"'] & W \otimes \mathcal{O} \arrow[d, equal] & \\
0 \arrow[r] & \mathcal{E}^* \arrow[r] & Z \oplus \mathcal{O} \arrow[r, "\varepsilon"] \arrow[d] & W \otimes \mathcal{O} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.2.3}$$

Thus snake lemma implies $0 \to Ker\varepsilon' \to \mathcal{E}^* \to \mathcal{O} \to Coker\varepsilon' \to 0$. We further restrict ourselves to the case when $Coker\varepsilon' = 0$. Thus we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}_0^\vee \arrow[r] \arrow[d] & Z \arrow[r, "\varepsilon'"] \arrow[d, "i"'] & W \otimes \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}^* \arrow[r] \arrow[d] & Z \oplus \mathcal{O} \arrow[r, "\varepsilon"] \arrow[d] & W \otimes \mathcal{O} \arrow[r] & 0 \\
 & \mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.2.4}$$

3.2.2 $h^{p,p}$

We first quote a theorem from [9] regarding the vanishing of cohomology of a particular type of line bundles.

Theorem 3.2.1. *Let D_i, $i = 1, ..., k$ be toric invariant divisors of V.*

(i) If $\cap_{i=1}^k D_i$ is nonempty, then $H^j(\mathcal{O}(-\sum_{i=1}^k D_i)) = 0$ for all j.

(ii) If K is a primitive collection, then

$h^{k-1}(\mathcal{O}(-\sum_{i=1}^k D_i)) = 1$ *and* $H^j(\mathcal{O}(-\sum_{i=1}^k D_i)) = 0$ *for* $j \neq k-1$.

Theorem 3.2.2. *For a generic bundle $\mathcal{E}^\vee$ fitting in the above diagram of short exact sequences, we have*

(i)For $p \leq \frac{n+1}{2}$, $h^p(V, \wedge^p \mathcal{E}^\vee) \cong h^{p,p}_{prim}$, where $h^{p,p}_{prim}$ is the dimension of the primitive cohomology of V;

(ii) For $p \geq \frac{n+1}{2}$, $H^{p-1}(V, \wedge^p \mathcal{E}^\vee) \cong H^{n+1-p}(V, \wedge^{n+1-p} \mathcal{E}^\vee)^*$;

(iii) $H^q(V, \wedge^p \mathcal{E}^\vee)$ vanishes for all other (p, q).

Note that when n is odd and $p = \frac{n+1}{2}$, all $H^q(\wedge^p \mathcal{E}^\vee)$ vanishes since there is no primitive cohomology of that dimension. Proof of the Theorem: (i) For any positive integer p, the Koszul resolution of

$$0 \to \mathcal{E}_0^\vee \to \mathcal{E}^\vee \to \mathcal{O} \to 0 \tag{3.2.5}$$

is

$$0 \longrightarrow \wedge^p \mathcal{E}_0^\vee \longrightarrow \wedge^p \mathcal{E}^\vee \xrightarrow{d_p} \wedge^{p-1} \mathcal{E}^\vee \longrightarrow \ldots \longrightarrow \mathcal{O} \longrightarrow 0, \tag{3.2.6}$$

which is a long exact sequence. In particular, this shows that $Ker\ d_p = \wedge^p \mathcal{E}_0^\vee$, $\forall p$.

Hence for each p, we have a short exact sequence

$$0 \to \wedge^p \mathcal{E}_0^\vee \to \wedge^p \mathcal{E}^\vee \to \wedge^{p-1} \mathcal{E}_0^\vee \to 0. \tag{3.2.7}$$

This gives rise to a long exact sequence of cohomology:

$$\begin{array}{l} \ldots \longrightarrow H^i(\wedge^p \mathcal{E}_0^\vee) \longrightarrow H^i(\wedge^p \mathcal{E}^\vee) \longrightarrow H^i(\wedge^{p-1} \mathcal{E}_0^\vee) \\ \longrightarrow H^{i+1}(\wedge^p \mathcal{E}_0^\vee) \longrightarrow H^{i+1}(\wedge^p \mathcal{E}^\vee) \longrightarrow H^{i+1}(\wedge^{p-1} \mathcal{E}_0^\vee) \\ \longrightarrow H^{i+2}(\wedge^p \mathcal{E}_0^\vee) \longrightarrow H^{i+2}(\wedge^p \mathcal{E}^\vee) \longrightarrow H^{i+2}(\wedge^{p-1} \mathcal{E}_0^\vee) \end{array} \tag{3.2.8}$$

Recall that $\mathcal{E}_0^\vee$ is a deformation of cotangent bundle. By semicontinuity, for a generic $\mathcal{E}_0^\vee$, $h^q(\wedge^p \mathcal{E}_0^\vee) \leq h^q(\wedge^p \Omega)$.

Since our variety V is toric, $h^q(\wedge^p \Omega)$ vanishes when $p \neq q$.[2] Hence $H^q(\wedge^p \mathcal{E}_0^\vee) = 0$ when $p \neq q$. This further implies that $H^p(\wedge^p \mathcal{E}_0^\vee) \cong H^p(\wedge^p \Omega)$. So we have a long exact sequence:

$$0 \to H^{p-1}(\wedge^p \mathcal{E}^\vee) \to H^{p-1}(\wedge^{p-1} \mathcal{E}_0^\vee) \xrightarrow{\delta} H^p(\wedge^p \mathcal{E}_0^\vee) \to H^p(\wedge^p \mathcal{E}^\vee) \to 0. \tag{3.2.9}$$

When $[\delta] \in Ext^1(\mathcal{O}, \Omega)$ is a Kaehler class, the map δ is conducted by a Lefschetz operator, so it is injective when $p \leq \frac{n+1}{2}$ and surjective when $p \geq \frac{n+1}{2}$, by the Hard Lefschetz Theorem. Since injectivity and surjectivity are open conditions and

[2]See [8] Theorem 9.3.2 in the section *Vanishing Theorems II.*

they are true for any Kaehler class, it is true generically. (To be precise: It is only obviously true for $\Omega \to \mathcal{E}^\vee \to \mathcal{O}$, but then it is true by semicontinuity that for a generic extension $\mathcal{E}_0^\vee \to \mathcal{E}^\vee \to \mathcal{O}$ and $p \leq \frac{n+1}{2}$, $H^{p-1}(\wedge^p \mathcal{E}^\vee) = 0$, and similarly for $p \geq \frac{n+1}{2}$.)

Part (ii) is just Serre duality. □

3.2.3 Realistic check: Euler characteristic

Let's check about the holomorphic Euler characteristic of $\wedge^p \mathcal{E}^\vee$. To be precise, we will check that the holomorphic Euler characteristic of $\wedge^p \mathcal{E}^\vee$ agrees with that of $\wedge^p(\mathcal{E}_0^\vee \oplus \mathcal{O})$.

By (3.2.7), we know the holomorphic Euler characteristic $\chi(\wedge^p \mathcal{E}^\vee) = \chi(\wedge^p \mathcal{E}_0^\vee) + \chi(\wedge^{p-1} \mathcal{E}_0^\vee)$. As for $\chi(\wedge^p(\mathcal{E}_0^\vee \oplus \mathcal{O}))$, since $\wedge^p(\mathcal{E}_0^\vee \oplus \mathcal{O}) \cong \wedge^p \mathcal{E}_0^\vee \oplus \wedge^{p-1} \mathcal{E}_0^\vee$, we have $\chi(\wedge^p(\mathcal{E}_0^\vee \oplus \mathcal{O})) = \chi(\wedge^p \mathcal{E}_0^\vee) + \chi(\wedge^{p-1} \mathcal{E}_0^\vee)$. So indeed they are the same.

3.3 Description of $H^p(V, \wedge^p \mathcal{E}^\vee)$

We want to describe $H^p(V, \wedge^p \mathcal{E}^\vee)$ for $p < \frac{n+1}{2}$ in term of a quotient of $S^p W$.

We first look at the case $p = 2$.

3.3.1 An example: $p = 2$

From the short exact sequence(3.2.2), we have the Koszul resolution:

$$0 \longrightarrow \wedge^2 \mathcal{E}^\vee \longrightarrow \wedge^2(Z \oplus \mathcal{O}) \longrightarrow (Z \oplus \mathcal{O}) \otimes W \longrightarrow S^2 W \otimes \mathcal{O} \longrightarrow 0 \quad (3.3.1)$$

This can be broken into two short exact sequences

$$0 \longrightarrow \wedge^2 \mathcal{E}^\vee \longrightarrow \wedge^2(Z \oplus \mathcal{O}) \longrightarrow K_1 \longrightarrow 0 \quad (3.3.2)$$

and

$$0 \longrightarrow K_1 \longrightarrow (Z \oplus \mathcal{O}) \otimes W \longrightarrow S^2 W \otimes \mathcal{O} \longrightarrow 0, \quad (3.3.3)$$

from which we can write down the long exact sequences of cohomology:

$$\begin{aligned} 0 &\longrightarrow H^0(\wedge^2 \mathcal{E}^\vee) \longrightarrow H^0(\wedge^2(Z \oplus \mathcal{O})) \longrightarrow H^0(K_1) \\ &\longrightarrow H^1(\wedge^2 \mathcal{E}^\vee) \longrightarrow H^1(\wedge^2(Z \oplus \mathcal{O})) \longrightarrow H^1(K_1) \\ &\longrightarrow H^2(\wedge^2 \mathcal{E}^\vee) \longrightarrow H^2(\wedge^2(Z \oplus \mathcal{O})) \longrightarrow H^2(K_1) \\ &\longrightarrow \dots \end{aligned} \quad (3.3.4)$$

and

$$
\begin{tikzcd}
0 \arrow[r] & H^0(K_1) \arrow[r] & H^0((Z \oplus \mathcal{O}) \otimes W)) \arrow[r] & H^0(S^2W \otimes \mathcal{O}) \\
\arrow[r] & H^1(K_1) \arrow[r] & H^1((Z \oplus \mathcal{O}) \otimes W)) \arrow[r] & H^1(S^2W \otimes \mathcal{O}) \\
\arrow[r] & H^2(K_1) \arrow[r] & H^2((Z \oplus \mathcal{O}) \otimes W)) \arrow[r] & H^2(S^2W \otimes \mathcal{O}) \\
\arrow[r] & \dots
\end{tikzcd}
\tag{3.3.5}
$$

Now we can do some direct computations:

By Theorem 3.2.2, $H^i(\wedge^2\mathcal{E}^\vee) = 0$ for any $i \neq 2$.

For $\wedge^2(Z \oplus \mathcal{O}) \cong \wedge^2 Z \oplus Z$: Recall $Z = \oplus_i \mathcal{O}(-D_i)$, D_i's are all the toric invariant divisors. Hence by Theorem 3.2.1, for $i \neq 1$, $H^i(\wedge^2(Z \oplus \mathcal{O})) \cong H^i(Z) \oplus H^i(\wedge^2 Z) = 0$, and $H^1(\wedge^2(Z \oplus \mathcal{O})) \cong H^1(\wedge^2 Z)$. We define $P^1 = H^1(\wedge^2 Z)$.

Now the long exact sequence (3.3.4) becomes

$$0 \to H^0(K_1) \to 0 \to P^1 \to H^1(K_1) \ \to H^2(\wedge^2\mathcal{E}^\vee) \to 0 \tag{3.3.6}$$

So we have $H^0(K_1) = 0$ and

$$H^2(\wedge^2\mathcal{E}^\vee) \cong H^1(K_1)/P^1. \tag{3.3.7}$$

For $(Z \oplus \mathcal{O}) \otimes W$, $H^0((Z \oplus \mathcal{O}) \otimes W) = W$ and $H^i((Z \oplus \mathcal{O}) \otimes W) = 0$ for $i > 0$.

For $S^2W \otimes \mathcal{O}$, $H^0(S^2W \otimes \mathcal{O}) = S^2W$, and $H^i(S^2W \otimes \mathcal{O}) = 0$ for $i > 0$.

Now the long exact sequence (3.3.5) becomes

$$0 \to W \to S^2W \to H^1(K_1) \to 0 \tag{3.3.8}$$

Hence

$$H^1(K_1) \cong S^2W/W := S^2\overline{W}. \tag{3.3.9}$$

Combine (3.3.7) and (3.3.9) we have

$$H^2(\wedge^2\mathcal{E}^\vee) \cong S^2\overline{W}/P^1. \tag{3.3.10}$$

3.3.2 A second example: $p = 3$

For $p = 3 < \frac{n+1}{2}$, we will brief repeat the calculations for $p = 2$ in order to fill in the missing feature of the general $H^p(\wedge^p\mathcal{E}^\vee)$.

We begin with the Koszul resolution:

$$0 \to \wedge^3\mathcal{E}^\vee \to \wedge^3(Z \oplus \mathcal{O}) \to \wedge^2(Z \oplus \mathcal{O}) \otimes W \to (Z \oplus \mathcal{O}) \otimes S^2W \to S^3W \otimes \mathcal{O} \to 0, \tag{3.3.11}$$

break it into short exact sequences

$$0 \longrightarrow \wedge^3\mathcal{E}^\vee \longrightarrow \wedge^3(Z \oplus \mathcal{O}) \longrightarrow K_2 \longrightarrow 0, \tag{3.3.12}$$

$$0 \longrightarrow K_2 \longrightarrow \wedge^2(Z \oplus \mathcal{O}) \otimes W \longrightarrow K_1 \longrightarrow 0, \tag{3.3.13}$$

and

$$0 \longrightarrow K_1 \longrightarrow (Z \oplus \mathcal{O}) \otimes S^2W \longrightarrow S^3W \otimes \mathcal{O} \longrightarrow 0. \tag{3.3.14}$$

Similarly to the $p = 2$ case, we have the induced long exact sequences of cohomology, and we can do direct computations to single out non-vanishing pieces of $H^i(\wedge^3\mathcal{E}^\vee)$, $H^i(\wedge^3(Z \oplus \mathcal{O}))$, $H^i(\wedge^2(Z \oplus \mathcal{O}) \otimes W)$, $H^i((Z \oplus \mathcal{O}) \otimes S^2W)$, and $H^i(S^3W \otimes \mathcal{O})$. Since no new feature appears to this stage and everything is parallel to $p = 2$ case via Theorem 3.2.2 and Theorem 3.2.1, we will simply summarize the possibly non-vanishing homology in the following diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^3\mathcal{E}^\vee & \longrightarrow & \wedge^3(Z \oplus \mathcal{O}) & \longrightarrow & K_2 & \longrightarrow & 0 \\
H^1: & & 0 & & P^1 & & H^1(K_2) & & \\
H^2: & & 0 & & P^2 & & H^2(K_2) & & \\
H^3: & & H^3(\wedge^3\mathcal{E}^\vee) & & 0 & & 0 & &
\end{array} \tag{3.3.15}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2 & \longrightarrow & \wedge^2(Z \oplus \mathcal{O}) \otimes W & \longrightarrow & K_1 & \longrightarrow & 0 \\
H^0: & & 0 & & 0 & & H^0(K_1) & & \\
H^1: & & H^1(K_2) & & P^1 \otimes W & & H^1(K_1) & & \\
H^2: & & H^2(K_2) & & 0 & & 0 & &
\end{array} \tag{3.3.16}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & (Z\oplus\mathcal{O})\otimes S^2W & \longrightarrow & S^3W\otimes\mathcal{O} & \longrightarrow & 0 \\ H^0: & & H^0(K_1) & & S^2W & & S^3W & & \\ H^1: & & H^1(K_1) & & 0 & & 0 & & \\ H^2: & & 0 & & 0 & & 0 & & \end{array} \tag{3.3.17}$$

Note that $P^2 := H^2(\wedge^3 Z)$.

Here comes the new feature: The map $H^1(K_2) \to H^1(\wedge^2(Z\oplus\mathcal{O})\otimes W)$ is injective. We will prove this later as a lemma. With this understood, we conclude that

$$Ker(H^1(K_1)\to H^2(K_2)) \cong Coker(H^1(K_2)\to P^1\otimes W) \cong \frac{P^1\otimes W}{P^1\otimes\mathbb{C}} \cong P^1\otimes\overline{W}. \tag{3.3.18}$$

Hence we have a description of $H^3(\wedge^3\mathcal{E}^\vee)$ now:

$$H^3(\wedge^3\mathcal{E}^\vee) \cong H^2(K_2)/P^2 \cong \frac{H^1(K_1)}{P^2\oplus P^1\otimes\overline{W}} \cong \frac{S^3\overline{W}}{P^2\oplus P^1\otimes\overline{W}}. \tag{3.3.19}$$

Note that $\overline{W} = Ker(\mathbb{C}\to W)$.

Lemma 3.3.1. *The map $H^1(K_2)\to H^1(\wedge^2(Z\oplus\mathcal{O})\otimes W)$ is injective.*

This will recur in the general p case.

Proof: The definition of K_2 fit it into the following commutative diagram:

$$\begin{tikzcd} \wedge^3(Z\oplus\mathcal{O}) \arrow[rr, "\varepsilon"] \arrow[dr] & & \wedge^2(Z\oplus\mathcal{O})\otimes W \\ & K_2 \arrow[ur, "i_2"'] & \end{tikzcd} \tag{3.3.20}$$

where $\varepsilon : \wedge^3(Z\oplus\mathcal{O}) \to \wedge^2(Z\oplus\mathcal{O})\otimes W$, as a map in the Koszul resolution, is induced by the map $\varepsilon : Z\oplus\mathcal{O} \to W\otimes\mathcal{O}$.

From (3.3.15) we see that $H^1(K_2) \cong H^1(\wedge^3(Z\oplus\mathcal{O})) \cong P^1$. So to proof the injectivity of $i_{2*} : H^1(K_2) \to H^1((\wedge^2(Z\oplus\mathcal{O})\otimes W)$ is equivalent to proof the injectivity of $\varepsilon_* : H^1(\wedge^3(Z\oplus\mathcal{O})) \to H^1(\wedge^2(Z\oplus\mathcal{O})\otimes W)$. By Theorem 3.2.1, $H^1(\wedge^3(Z\oplus\mathcal{O})) \cong H^1(\wedge^2 Z)$ is generated by $\{s_I | I \in \mathcal{P}_2\}$, where $\mathcal{P}_2$ is the set of all length-2 primitive collections of toric invariant divisors. $\varepsilon_*(s_I) = s_I \otimes w_0$, where $w_0 = \varepsilon(0,1) = \varepsilon_0(1) \in W$.

Say $\varepsilon_*(\sum_I c_I s_I) = 0$, where $c_I \in \mathbb{C}$. Then $\sum_I c_I s_I \otimes w_0 = (\sum_I c_I s_I)\otimes w_0 = 0$. Thus $\sum_I c_I s_I = 0$[3], which means ε_* is injective. $\square$

3.3.3 General Description of $H^p(V, \wedge^p\mathcal{E}^\vee)$

Theorem 3.3.2. *Let $\mathcal{E}^\vee$ be a bundle as defined above, i.e. fitting into Diagram 3.2.4 of short exact sequences, and be generic in the sense that Theorem 3.2.2's*

[3] We are making use our assumption that $\mathcal{E}^\vee$ does not split, which means $w_0 \neq 0$.

conclusions hold. Then for $p < \frac{n+1}{2}$,

$$H^p(V, \wedge^p \mathcal{E}^\vee) \cong \frac{S^p\overline{W}}{\oplus_{j=2}^{p} P^{j-1} \otimes S^{p-j}\overline{W}}, \tag{3.3.21}$$

where $S^p\overline{W} = Coker(S^{p-1}W \to S^pW)$*, and the map* $S^{p-1}W \to S^pW$ *is induced by* $\varepsilon_0 = \varepsilon|_{\mathcal{O}} : \mathcal{O} \to W \otimes \mathcal{O}$*, and* $P^{j-1} := H^{j-1}(\wedge^j Z)$.

Proof:

From the short exact sequence(3.2.2) we build the Koszul resolution:

$$0 \to \wedge^p \mathcal{E}^\vee \to \wedge^p(Z \oplus \mathcal{O}) \to \ldots \to \wedge^j(Z \oplus \mathcal{O}) \otimes S^{p-j}W \to \mathcal{O} \times S^pW \to 0 \tag{3.3.22}$$

and further break it into short exact sequences

$$0 \to K_j \to (\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j}W \to K_{j-1} \to 0, \tag{3.3.23}$$

for $j = 0, \ldots, p$. Note that $K_p = \wedge^p \mathcal{E}^\vee$, $K_0 = S^pW \otimes \mathcal{O}$. Now we study the induced long exact sequences in cohomology:

$$\begin{array}{l}\dots \longrightarrow H^{j-3}(K_j) \longrightarrow H^{j-3}((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W) \longrightarrow H^{j-3}(K_{j-1}) \\ \longrightarrow H^{j-2}(K_j) \longrightarrow H^{j-2}((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W) \longrightarrow H^{j-2}(K_{j-1}) \\ \longrightarrow H^{j-1}(K_j) \longrightarrow H^{j-1}((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W) \longrightarrow H^{j-1}(K_{j-1}) \\ \longrightarrow H^{j}(K_j) \longrightarrow \dots \end{array} \tag{3.3.24}$$

First, by Theorem 3.2.1, $H^i((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W) \cong H^i(\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W$ vanishes for all i except $i = j-2$ and $i = j-1$.

Then we see $H^{j-3}(K_{j-1}) \hookrightarrow H^{j-2}(K_j) \hookrightarrow \dots \hookrightarrow H^{p-2}(K_p) = 0$, Hence $H^{j-3}(K_{j-1}) = 0$.

Now we can list all possible non-vanishing terms

$$\begin{array}{ccccc} 0 \to & K_j \to & (\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j} W \to & K_{j-1} \to & 0. \\ H^{j-3}: & 0 & 0 & 0 & \\ H^{j-2}: & 0 & P^{j-2} \otimes S^{p-j} W & H^{j-2}(K_{j-1}) & \\ H^{j-1}: & H^{j-1}(K_j) & P^{j-1} \otimes S^{p-j} W & H^{j-1}(K_{j-1}) & \\ H^{j}: & H^{j}(K_j) & 0 & 0 & \end{array} \tag{3.3.25}$$

and rewrite the long exact sequence 3.3.24 as

$$
0 \longrightarrow P^{j-2} \otimes S^{p-j} \xrightarrow{\cong} H^{j-2}(K_{j-1}) \xrightarrow{0} H^{j-1}(K_j) \tag{3.3.26}
$$

$$
\xrightarrow{\varepsilon_*} P^{j-1} \otimes S^{p-j}W \longrightarrow H^{j-1}(K_{j-1}) \xrightarrow{\delta} H^j(K_j) \longrightarrow 0,
$$

where, as before, $P^j := H^j(\wedge^{j+1}(Z))$.

Next, we claim the map $H^{j-2}(K_{j-1}) \to H^{j-1}(K_j)$ is always the zero map. (This will be proved as Lemma 3.3.3 below.) Hence $H^{j-2}(K_{j-1}) \cong P^{j-2} \otimes S^{p-j}W$, and $H^{j-1}(K_j) \hookrightarrow P^{j-1} \otimes S^{p-j}W$. Hence

$$
Coker\ \varepsilon_* \cong \frac{P^{j-1} \otimes S^{p-j}W}{H^{j-1}(K_j)} \cong \frac{P^{j-1} \otimes S^{p-j}W}{P^{j-1} \otimes S^{p-j-1}W} \cong P^{j-1} \otimes S^{p-j}\overline{W}. \tag{3.3.27}
$$

This will give us the desired description:

$$
H^j(K_j) \cong \frac{H^{j-1}(K_{j-1})}{Ker\ \delta} \cong \frac{H^{j-1}(K_{j-1})}{Coker\ \varepsilon_*} \cong \frac{H^{j-1}(K_{j-1})}{P^{j-1} \otimes S^{p-j}\overline{W}}. \tag{3.3.28}
$$

Repeat this for all j, we get

$$
H^p(V, \wedge^p \mathcal{E}^\vee) \cong \frac{S^p\overline{W}}{\oplus_{j=2}^p P^{j-1} \otimes S^{p-j}\overline{W}},
$$

which is exactly Equation (3.3.21). □

Lemma 3.3.3. *The map $H^{j-2}(K_{j-1}) \to H^{j-1}(K_j)$ is always the zero map. Hence*

$H^{j-2}(K_{j-1}) \cong P^{j-2} \otimes S^{p-j}W$, *and* $H^{j-1}(K_j) \hookrightarrow P^{j-1} \otimes S^{p-j}W$.

Proof: By induction. When $j = p$, we know that $H^{p-1}(K_p) = 0$. So the lemma is true.

Assume the lemma is true for $j+1$, then

$$H^{j-1}(K_j) \cong H^{j-1}((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j}W) \cong P^{j-1} \otimes S^{p-j-1}W. \tag{3.3.29}$$

To show it is true for j, it suffices to show that the map $i_2 : H^{j-1}(K_j) \to P^{j-1} \otimes S^{p-j}W$ is injective. Note that we have the commutative diagram

$$\begin{tikzcd} (\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j}W \arrow[rr, "\varepsilon"] \arrow[dr] & & (\wedge^{j-1} Z \oplus \wedge^{j-2} Z) \otimes S^{p-j+1}W \\ & K_{j-1} \arrow[ur, "i_2"] & \end{tikzcd} \tag{3.3.30}$$

from the definition of K_{j-1}. By (3.3.29), it suffices to show that $\varepsilon_* : H^{j-1}((\wedge^j Z \oplus \wedge^{j-1} Z) \otimes S^{p-j}W) \to H^{j-1}((\wedge^{j-1} Z \oplus \wedge^{j-2} Z) \otimes S^{p-j+1}W)$ is injective.

We take a close look at the map ε_*: by Theorem 3.2.1, the domain, $P^{j-1} \otimes S^{p-j-1}W$ is generated by $\{s_I \otimes w^J | I \in \mathcal{P}_j, w^J \in S^{p-j-1}W$, where $\mathcal{P}_j$ is the set of all length-2 primitive collections of toric invariant divisors. $\varepsilon_*(s_I \otimes w^J) = s_I \otimes w_0 \otimes^s w^J$, where the '$s$' in $\otimes^s$ indicates a symmetric tensor product.

Now we can check the injectivity easily: say $\varepsilon_*(\sum_I c_I s_I \otimes w^I) = 0$ (note that we have summed up the J indices for fix I), where $c_I \in \mathbb{C}$. Then $\sum_I c_I s_I \otimes w^I \otimes^s w_0 = (\sum_I c_I s_I \otimes w^I) \otimes^s w_0 = 0$. Thus $\sum_I c_I s_I \otimes w^I = 0$[4], which means ε_* is injective. $\square$

[4]We are making use our assumption that $\mathcal{E}^\vee$ does not split, which means $w_0 \neq 0$.

3.4 The Ring Structure

By Theorem 3.2.2, the dimension of the possible non-vanishing ones of $H^q(\wedge^p\mathcal{E}^\vee)$ are: (Let $h^{p,q} = \dim H^q(\wedge^p\mathcal{E}^\vee)$)

$$\begin{array}{|ll|ll|}
\hline
\multicolumn{2}{|c|}{n=2m} & \multicolumn{2}{c|}{n=2m+1} \\
\hline
h^{0,0} & & h^{0,0} & \\
h^{1,1} & & h^{1,1} & \\
\dots & & \dots & \\
h^{m,m} & h^{m+1,m} & h^{m,m} & \\
 & h^{m+2,m+1} & & h^{m+2,m+1} \\
 & \dots & & h^{m+3,m+2} \\
 & h^{n+1,n} & & \dots \\
 & & & h^{n+1,n} \\
\hline
\end{array} \tag{3.4.1}$$

The two diagrams are centrosymmetric by the duality part of Theorem 3.2.2.

So the cup-length is at most $\frac{n+2}{2}$.

We want to understand the multiplicative structure of $H^*(\wedge^*\mathcal{E}^\vee)$.

First we introduce the following notion: For a ring R (commutative with 1), let the *double* of R be a ring

$$R^{(2)} = R \oplus R^*, \tag{3.4.2}$$

u v = u v or v(u) or 0 We then define $R_0 = \bigoplus_{p=0}^{\lfloor \frac{n+1}{2} \rfloor} H^p(\wedge^p\mathcal{E}^\vee)$, and $R = R_0^{(2)}$.

It is clear that the cohomology ring is R and it only remains to describe R_0, which is done by the following theorem:

Theorem 3.4.1. *Under the isomorphism in the description (3.3.21) of* $H^p(\wedge^p\mathcal{E}^\vee)$*,* *the cup product* $H^p(\wedge^p \mathcal{E}^\vee) \times H^q(\wedge^q \mathcal{E}^\vee) \to H^{p+q}(\wedge^{p+q} \mathcal{E}^\vee)$ *is just the multiplication of the symmetric algebra.*

Proof: Similar to the T case. $\square$

3.5 Further questions

There are a few questions remaining: First, we only get results for generic deformations. We can ask further whether this is true for all deformations. Next, we need to do the quantum part. Also, we believe similar results on generic deformations of $T \oplus \mathcal{O}^s$, for $s \leq r = rank(Pic(V))$. (For larger s, $\mathcal{E}^\vee$ splits.)

Chapter 4

Further Discussions

4.1 QSC for complete intersections in toric varieties

We want to extend the theory of quantum sheaf cohomology to omalous bundles over complete intersections in toric varieties. These include many Calabi-Yaus that are of interest for string compactifications.

The first case is when $X \subset V$ is a hypersurface. The tangent bundle of X is the cohomology of the following *monad sequence*, i.e. a (non-exact) complex of vector bundles, each a direct sum of line bundles:

$$0 \to \mathcal{O}_X \otimes W^\vee \to \oplus \mathcal{O}_X(D_i) \to \mathcal{N}_{X/V} \to 0, \tag{4.1.1}$$

where $\mathcal{N}_{X/V}$ is the normal bundle.

The cohomology of a small deformation of this sequence will be an omalous bundle $\mathcal{E}$ on X. The authors of [9] conjecture that the "toric part" of the quantum sheaf cohomology ring structure of $\mathcal{E}$ could be described similarly to the toric variety case.

We are currently working on this conjecture. This should enable us to compute quantum correlators and to prove the (0,2) quantum restriction conjecture in [18]. The argument will presumably generalize those given in [4][12][20] for the (2,2) case when $\mathcal{E}$ is the tangent bundle. Another consistency check arises from the observation that certain toric varieties can be realized as complete intersections in others; in such cases, the quantum cohomology computed here must reproduce the answer obtained in Chapter 2.

4.2 QSC for Grassmannians

The theory of quantum cohomology on Grassmannians has been studied by many authors ([3, 19]). Accordingly, the quantum sheaf cohomology analogy will be interesting to both mathematicians and physicists. An ongoing project [10] studies this problem, aiming to generalize the quantum cohomology ring

$$QH^*X = \mathbb{Z}[c_1, ..., c_l, q]/(\sigma_{k+1}, ..., \sigma_{n-1}, \sigma_n + (-1)^l q) \tag{4.2.1}$$

of $X = Gr(k, n)$ in [3] to omalous bundles.

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