Special Value Formulae of

Rankin-Selberg L-Functions

A Dissertation

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2005

Supervisor of Dissertation

Graduate Group Chairperson

Acknowledgement

I would like to express my indebtness to Shou-Wu Zhang for his wonderful idea of using the trick of Eisenstein series as well as his fundamental papers [21] and [22] from which I learned his new language to compute Fourier coefficients. I also would like to thank my advisor Ted Chinburg for his constant support and encouragement. At last but not the least, I wish to express my heartiest thanks to Ye Tian who told me this interesting topic and constantly encourage me to work out the problem and shared his idea with me always. In fact this is the a part of a joint project with Ye Tian aiming to establishing a p-adic analogue of the special value formula of a Hida family of Hilbert newforms.

ABSTRACT

Haiping Yuan

Advisor: Ted Chinburg

In this paper, we prove a special value formula of level N of Rankin-Selberg L-function associated to a Hilbert modular form of higher weight and a ring class character of an totally imaginary quadratic extension of a totally real field. The formula relates the special value of the Rankin-Selberg L-functions at $s = \frac{1}{2}$ to the value of certain test form at some CM-point on a 0-dimensional Shimura Variety associated to a quaterion algebra. The formula generalizes the formula proved by Shou-Wu Zhang which is a vast generalization of classical Gross-Zagier formula. The proof is based on a formula (level ND) of Hui Xue combined with a technique of Eisenstein Series to compute the universal constants which arise in the comparison of both formulae of level N and ND.

Contents

1. **Introduction**

Let F be a totally real number field, and K/F a totally imaginary quadratic extension of relative discriminant d. Fix a Hilbert cusp newform ϕ of GL_2 over F of level N which generates a cuspidal automorphic representation $\pi(\phi)$ of $GL_2(\mathbb{A}_F)$, and a ring class character χ of K, i.e., a finite order Hecke character of K which is trivial on $\mathbb{A}_F^\times$ and $K^\times$. We denote by $c(\chi)$ the conductor of χ.

To ϕ and χ, one associates a Rankin-Selberg L-function $L(s, \phi, \chi)$ (following Jacquet), which has analytic continuation and satisfies the functional equation under $s \to 1 - s$. It is well known that the central critical value $L(1/2, \phi, \chi)$ is related to the "height" of certain CM divisor on some 0-dimensional Shimura Variety. In [8], Gross proves a formula expressing the special value in terms of the height of certain CM divisor on a 0-dimensional Shimura variety in the special case that $F = \mathbb{Q}$, and N and D are primes. A far reaching generalization is proposed by Gross ([7], [9]). There has been breakthrough recently on this conjecture. Shou-Wu Zhang([21] and [22]) proves a special (but still quite general) case of the conjecture. More precisely, let ϕ be a Hilbert cusp newform of weight $(2, \cdots, 2)$, of level N, and of the trivial character. Assuming that N, $c(\chi)$, and d are coprime to each other, he shows that the special value $L(1/2, \phi, \chi)$ can be expressed as the height of certain CM divisor of 0-dimensional Shimura variety, namely, an (torus) integral of certain form on the torus given by $K^\times$. His results generalize Gross' earlier results and give an refinement of Waldspurger's results [19] concerning the equivalence between the non-vanishing of $L(1/2, \phi, \chi)$ and the non-vanishing of the torus integral. We generalize Zhang's results to higher weight $(2k,\cdots,2k)$, based on Hui Xue's formula of level ND [20], thus partially proves Gross' conjecture.

Here are a little more details. Let ϕ be a Hilbert cusp newform over F of level N, of weight $(2k,\cdots,2k), k\geq 1$, and of trivial central character, see chapter 2 for the definition. The representation of $\mathrm{GL}_2(\mathbb{A}_F)$ spanned by ϕ is denoted by π. Fix a ring class character χ of $\mathbb{A}_K^\times/K^\times$ with conductor $c(\chi)$, i.e., $\chi:\mathbb{A}_K^\times/K^\times\mathbb{A}_F^\times\to\mathbb{C}^\times$ is a homomorphism. We denote by ω the quadratic character associated to K/F. The Rankin-Selberg L-function $L(s,\phi,\chi)$ has analytic continuation and satisfies the following functional equation

$$L(s,\phi,\chi)=(-1)^{\#\Sigma}\mathrm{N}_{F/\mathbb{Q}}(ND)^{1-2s}L(1-s,\phi,\chi),$$

where $D:=c(\chi)^2c(\omega)$ and Σ is the following set of places of F

$$\Sigma=\{\,v|v|\infty,\quad \text{or } v\nmid\infty\quad\text{and } \omega_v(N)=-1\,\}\,.$$

The point $s=\frac{1}{2}$ is the central critical point. We consider the special value of $L(s,\phi,\chi)$ at $s=\frac{1}{2}$ under the assumption that $\#\Sigma$ is even and that $N,c(\chi),c(\varepsilon)$ are coprime to each other. Then there exists a unique quaternion algebra B over F ramified exactly at the places in Σ. Let G be the algebraic group given by $B^\times/F^\times$. Associated to ϕ and χ, one can define two forms:

(1) the quasi-newform $\phi^\#$,

the unique form of level ND in the space π satisfying the relation

$$(\phi^\#,\phi_a)=\nu^*(a)(\phi^\#,\phi^\#),\quad (a|D)$$

where

$$\nu^*(a) = \begin{cases} \nu(a), & \text{if } a|c(\varepsilon); \\ 0, & \text{otherwise.} \end{cases}$$

and $\phi_a := \rho\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\phi$;

(2) a test form $\widetilde{\phi}$,

an automorphic form on M_U such that for each finite place v not dividing ND, $\widetilde{\phi}$ is the eigenform for Hecke operators T_v with the same eigenvalue as ϕ, where $M_U = G(F)\backslash G(\mathbb{A}_f)/U$ with U an order of B of reduced discriminant N.

Zhang [21] (and Hui Xue [20] for higher weight case) proves the following special value formula for $L(\frac{1}{2}, \phi, \chi)$ in terms of the quasi-newform $\phi^{\#}$:

$$\widehat{\phi^{\#}}(1)L(\frac{1}{2}, \phi, \chi) = \frac{C}{\sqrt{\mathrm{N_{F/\mathbb{Q}}}(c(\omega))}}||\phi^{\#}||^2_{U_0(ND)} \cdot |(\phi_\chi, \eta)|^2, \tag{1.1}$$

where $C = (\frac{2\cdot 4^{k-1}[(k-1)!]^2}{(2k-1)!})^g$, $\widehat{\phi^{\#}}(1)$ the first Fourier coefficient of the quasi-newform $\phi^{\#}$, ϕ_χ a toric newform on M_U suitably normalized, and $||\phi^{\#}||^2_{U_0(ND)}$ is computed as L^2-norm with respect to the Haar measure dg which is the product of the standard measure on $N(\mathbb{A}_F)A(\mathbb{A}_F)$, and the measure on the standard maximal compact group with

$$\mathrm{vol}(\mathrm{SO}(F_\infty)U_0(ND)) = 1.$$

In [8], Gross conjectures that the special value may be expressed in terms of the test form $\widetilde{\phi}$ in viewing of earlier work of Waldspurger [19] and taking advantage of $\widetilde{\phi}$ being of level N. Meanwhile various applications ([2], [18]) suggest that it would be more natural to have a formula expressing the special value in terms of ϕ and χ. Inspired by those motivations, Zhang proves the conjecture of Gross using the technique of continuous spectrum to deduce the formula of level N from his above formula in the case of $k = 1$. Our approach basically follows Zhang's. Notice that a test form $\widetilde{\phi}$ is a form in the space of $\pi' = \otimes_v \pi'_v$, the representation of $G(\mathbb{A})$ corresponding to π via Jacquet-Langlands correspondence. In the special case that $k = 1$, π'_v is trivial for $v|\infty$ so that one can ignore the archimedean part. This is treated by Zhang. In general, it is of finite dimension, since π'_v is irreducible representation of $G(F_v) \cong \mathrm{SO}_3(\mathbb{R})$, which is compact. In fact, the dimension is $2k-1$. The formula can be stated as follows.

Theorem. *Assume that $\#\Sigma$ is even, then*

$$L(\frac{1}{2}, \phi, \chi) = \frac{C}{\sqrt{\mathrm{N}_{F/\mathbb{Q}}(D)}} ||\phi||^2_{U_0(N)} |(\widetilde{\phi}, P_\chi)|^2, \tag{1.2}$$

Here $P_\chi := \sum_{\sigma \in \mathrm{Gal}(H_c/K)} \chi^{-1}(\sigma)[P_c^\sigma]$ *with* P_c *a CM point in* $G(F)\backslash G(\mathbb{A}_f)/U$.

To explain the idea of the proof, one rewrites the formula (1.1) in the following way:

$$\frac{\widehat{\phi^\#}(1)}{\mathrm{N}_{F/\mathbb{Q}} c(\chi)} \frac{||\phi||^2_{U_0(N)}}{||\phi^\#||^2_{U_0(ND)}} \frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2} L(\frac{1}{2}, \phi, \chi) = \frac{C}{\sqrt{\mathrm{N}(D)}} ||\phi||^2_{U_0(N)} |(\widetilde{\phi}, P_\chi)|^2.$$

We define the archimedean components of the test form $\widetilde{\phi}$ to be same as those of toric newform ϕ_χ. Thus the norm of the ϕ_v at archimedean place is same as that of toric newform. Therefore to deduce the formula (1.2) from formula

(1.1), it suffices to prove that the product

$$\frac{\widehat{\phi^{\#}}(1)}{\mathrm{N}_{F/\mathbb{Q}}c(\chi)}\frac{||\phi||^2_{U_0(N)}}{||\phi^{\#}||^2_{U_0(ND)}}\frac{|(\widetilde{\phi},P_\chi)|^2}{|(\phi_\chi,\eta)|^2}=1. \tag{1.3}$$

Now the key idea is that instead of fixing the Hilbert cusp newform ϕ, one views ϕ as a form varying in the space $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F))$, a space which decomposes into a discrete part where the form ϕ lies in and a continuous part whose elements are continuous sums of Eisenstein series. A crucial step is that the left hand side of the equation (1.3) is actually a universal function of local parameters in the sense that it is independent of the form ϕ. More precisely, for a finite place $v|D$, there exists a rational function $Q_v(t)\in\mathbb{C}(t)$, which depends only on χ_v, $Q_v(0)=1$, and $Q_v(t)$ is regular for $|t|\le|\varpi_v|^{\frac{1}{2}}+|\varpi_v|^{-\frac{1}{2}}$ such that

$$\frac{\widehat{\phi^{\#}}(1)}{\mathrm{N}_{F/\mathbb{Q}}c(\chi)}\frac{||\phi||^2_{U_0(N)}}{||\phi^{\#}||^2_{U_0(ND)}}\frac{|(\widetilde{\phi},P_\chi)|^2}{|(\phi_\chi,\eta)|^2}=C(\chi)\prod_{v|D}Q_v(\lambda_v),$$

where $C(\chi)$ is a constant depending only on χ, and λ_v is the parameter in $L_v(s,\phi)=(1-\lambda_v|\varpi_v|^s+|\varpi_v|^{2s})^{-1}$. The hard part of the above is to show the ratio $\frac{|(\widetilde{\phi},P_\chi)|^2}{|(\phi_\chi,\eta)|^2}$ has the similar property which requires to carefully analyze both test form and toric newform. Thus replace the form ϕ by a form E in the continuous spectrum $L^2_{\mathrm{cont}}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F))$. The entire proof of the formula (1.1) can be carried over to $L(s,E,\chi)$ to obtain a similar special value formula for $L(s,E,\chi)$ through which the same constants occur. Now choosing the form E appropriately, one reduces it to computing the periods of Eisenstein series, a feasible computation. Finally it turns out that both $C(\chi)$ and $Q_v(t)$ are equal to 1 for $v|D$.

For the purpose of self-contained paper, in chapter 2, we collect the materials needed for our proof. We briefly describe the relationship between Hilbert modular forms and automorphic representations, adelic newform theory, Weil representation and Jacquet-Langlands correspondence, Rankin-Selberg L-functions (following Jacquet). In addition, we add a proof of a theorem concerning the isomorphism of the group $\mathrm{GL}_2 \times T/\Delta\mathbb{G}_m(F)$ with $GU(F)$, the group of F-rational points of a group of unitary similitude, which is a beginning part of programme proposed by Gross [9] refining Waldspurger's general results. We wish to come to this topic in future. Chapter 3 is a brief review of proof of Xue's formula of level ND and meanwhile we explain various term occurring in the formula in order to carry out the computation in chapter 4. We prove the final formula using the technique of Eisenstein series. In order not to interrupt the continuity of the proof, an appendix of the spectral decomposition of the L^2-space $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$ is added.

Notations.

We fix a totally real field F of degree g, and a totally imaginary quadratic field extension K of F. Let ϖ_v denote the uniformizer of $\mathcal{O}_v$, the integer ring of the completion of F at v, and $\mathbb{A} := \mathbb{A}_F$ be the ring of adeles of F.

Let ψ be a fixed additive character of $F\backslash\mathbb{A}$. Write $\psi = \otimes_v \psi_v$, and $\delta_v \in F_v^\times$ denotes the conductor of ψ_v, i.e., if v is finite, then $\delta_v^{-1}\mathcal{O}_v$ is the maximal fractional ideal of F_v such that $\psi_v|_{\delta_v^{-1}\mathcal{O}_v} \equiv 1$, if v is infinite, $\psi_v(x) = e^{2\pi\delta_v x}$. Set

$\delta = \prod \delta_v \in \mathbb{A}^\times$. Then one sees that

$$|\delta|^{-1} = d_F.$$

Unless specifically mentioned, we usually normalize a Haar measure on $\mathbb{A}$ such that $\mathrm{vol}(F\backslash\mathbb{A}) = 1$. The measure is decomposed as

$$dx = \otimes_v dx_v$$

such that dx_v is self-dual with respect to ψ_v, and the multiplicative measure $dx^\times = \otimes_v dx_v^\times$ has the property that

$$dx_v^\times = \frac{dx_v}{x_v} \quad \text{if} \quad v|\infty, \text{and}$$

$$\mathrm{vol}(\mathcal{O}_v^\times) = 1 \quad \text{if} \quad v \nmid \infty.$$

For the torus T given by $K^\times/F^\times$, a Haar measure is chosen so that

$$\mathrm{vol}(T(F_v)) = 1$$

if $v|\infty$. If B is a quaternion algebra over F which is ramified at all infinite places. We fix a Haar measure

$$dg = \otimes_v dg_v$$

such that $\mathrm{vol}(G(F_v)) = 1$ if $v|\infty$, and $\mathrm{vol}(U) = 1$ for some open compact subgroup $U \subset G(\mathbb{A}_f)$ depending on the subgroup U. For the group PGL_2, we choose the standard Haar measure $dg = \otimes_v dg_v$, i.e., $\mathrm{vol}(\mathrm{PGL}_2(\mathcal{O}_v)) = 1$ if $v \nmid \infty$ and $\mathrm{vol}(\mathrm{SO}_2(\mathbb{R})) = 1$ if $v|\infty$.

2. **Hilbert modular forms and automorphic representations**

In this section,we shall discuss the basic relationship between Hilbert modular forms and automorphic representations. We shall also collect some further results which are needed later.

2.1. **Hilbert modular forms.** We fix some notations and give the definition of Hilbert modular form. Then we discuss the representations generated by Hilbert modular forms.

Let F be a totally real field of degree d, I the set of embedding $F \hookrightarrow \mathbb{R}$. The $\mathbb{A}$-points of GL_2 is denoted by $\mathrm{GL}_2(\mathbb{A})$. It's easily checked that one has

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(F_\infty),$$

where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^I$. Fix $k = (k_i) \in \mathbb{Z}^I$ such that each component $k_i \geq 2$ and such that all components have the same parity. Set $t = (1, \cdots, 1)$, and $z_0 = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathcal{H}^I$, where $\mathcal{H}$ stands for the Poincaré upper half plane. Let $m = k - 2t$ and we choose $v \in \mathbb{Z}^I$ such that $w_v \geq 0$, $w_v = 0$ for some v and $m + 2w = \mu t$ for some $\mu \in \mathbb{Z}^{\geq 0}$ For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\infty)$ and $z \in \mathcal{H}^I$, one defines

$$j_{k,w}(g, z) = (ad - bc)^{t-w-k}(cz + d)^k.$$

Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$. Following Hida [12] and Tayler [17], one gives

Definition 2.1.1 *A Hilbert modular form of weight* (k, w) *with respect to the group* U *is a function* $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$*, satisfying the following two conditions:*
(i) $(f|_k u)(x) := j_{k,w}(u_\infty, z_0)^{-1} f(\alpha x u) = f(x)$ *for* $\alpha \in \mathrm{GL}_2(F)$*, and* $u \in U \cdot (\mathbb{R}^\times \cdot \mathrm{SO}_2(\mathbb{R}))^I$*;*
(ii) For all $x \in \mathrm{GL}_2(\mathbb{A}_f)$*, the function* $f_x : \mathcal{H}^I \to \mathbb{C}$ *defined by*

$$u_\infty z_0 \mapsto f(x u_\infty) j_{k,w}(u_\infty, z_0)$$

for $u_\infty \in \mathrm{GL}_2(F_\infty)$ *is holomorphic. If* $F = \mathbb{Q}$*, one has to assume that the function* f_x *is holomorphic at cusps for each* $x \in \mathrm{GL}_2(\mathbb{A}_f)$*. In addition, if*

$$\int_{F\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da = 0$$

for all $g \in \mathrm{GL}_2(\mathbb{A})$*, then* f *is called a cuspidal Hilbert modular form.*

Remark 2.1.2: Using the formula in condition (i), one easily verifies that the function f_x in condition (ii) is well defined.

We denote by $\mathcal{M}_{k,w}(U)(\mathcal{S}_{k,w}(U))$ the space of (cuspidal) Hilbert modular forms of weight (k, w) with respect to group U.

In particular, the Hilbert modular form defined above is an automorphic form. Recall that an automorphic form is a function $\phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ with the following properties:

- $\phi(z\gamma g) = \omega(z)\phi(g)$ for $z \in Z(\mathbb{A}), \gamma \in \mathrm{GL}_2(F)$, where Z is the center of GL_2;
- ϕ is invariant under the right action of some open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$;

• For each $v|\infty$, ϕ is smooth in $g_v \in \mathrm{GL}_2(F_v)$ and $\mathrm{SO}_2(F_v)$-finite, i.e., $\phi(gr(\theta))$ form a finite dimensional vector space, where $r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$;

• For each $v|\infty$, ϕ is $\mathfrak{gl}_2(F_v)$-finite, i.e., for $X \in \mathfrak{gl}_2(F_v) \cong \mathcal{Z} :=$ center of the universal enveloping algebra of $\mathfrak{gl}_2(F_v)$, $X\phi$ form a finite dimensional vector space, where

$$X\phi(g) := \frac{d}{dt}\phi(g\exp(tX))\,|_{t=0}$$

• ϕ has moderate growth, i.e., for any compact subset Ω there exist positive numbers C, t, such that

$$\left|\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| < C(|a| + |a|^{-1})^t, \forall g \in \Omega.$$

Let $\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$ denote the space of automorphic forms with central character ω.

In addition, if

$$\int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) = 0, \forall g \in \mathrm{GL}_2(\mathbb{A}),$$

then one says that ϕ is *cuspidal*. And we denote by $\mathcal{A}_0(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$ the space of cuspidal automorphic forms.

The group $\mathrm{GL}_2(\mathbb{A}_f)$ acts on $\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$ via right translation, i.e., for $g \in \mathrm{GL}_2(\mathbb{A}_f)$, and $\phi \in \mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \omega)$,

$$\pi(g)\phi(x) := \phi(xg).$$

Unfortunately the group $\mathrm{GL}_2(F_\infty)$ doesn't act on the space $\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$, since $\pi(g)\phi$ may not be $\mathrm{SO}_2(F_v)$-finite, for $g\in\mathrm{GL}_2(F_\infty)$. But we do have actions of both $\mathrm{SO}_2(F_v)$ and $\mathfrak{gl}_2(F_v)$. For $X\in\mathfrak{gl}_2(F_v)$, define

$$X\phi(g):=\frac{d}{dt}\phi(g\exp(tX))\,|_{t=0}$$

One can show that if ϕ is $\mathrm{SO}_2(F_v)$-finite, then $X\phi$ is an automorphic form and is $\mathrm{SO}_2(F_v)$-finite. Moreover one requires that both actions are compatible in the following sense: for $g\in\mathrm{SO}_2(F_v), X\in\mathfrak{gl}_2(F_v)$ and $\phi\in\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$,

$$\pi(g)\pi(X)\pi(g^{-1})\phi=\pi(\mathrm{Ad}(g)X)\phi,$$

where $\mathrm{Ad}:\mathrm{GL}_2(F_v)\to\mathrm{Aut}(\mathfrak{gl}_2(F_v))$ is the adjoint representation of $\mathrm{GL}_2(F_v)$. A vector space V equipped with such representations of $\mathrm{SO}_2(F_v)$ and $\mathfrak{gl}_2(F_v)$ is called a $(\mathrm{SO}_2(F_v)$,$\mathfrak{gl}_2(F_v))$-*module.*

Definition 2.1.3 *A representation* π *of* $\mathrm{GL}_2(\mathbb{A})$ *or more precisely a representation of* $\mathrm{GL}_2(\mathbb{A}_f)$ *and a commuting* $(\mathrm{SO}_2(F_v)$,$\mathfrak{gl}_2(F_v))$*-module is called an automorphic representation if* π *is isomorphic to a subquotient of* $\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ (*a quotient of submodule of* $\mathcal{A}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$).

For the purpose of the paper, we are particularly interested in the following two open compact subgroups $U_0(N)$ and $U_1(N)$ of $\mathrm{GL}_2(\mathbb{A}_f)$. Let N be an ideal of F, define

$$U_0(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\widehat{\mathcal{O}}_F)\middle|c\equiv 0\pmod N\right\},$$

$$U_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \middle| d \equiv 1 \pmod N \right\},$$

For each $v|\infty$, let $r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. one easily checks that the Hilbert modular form f satisfies the following property:

$$f(gr(\theta)) = f(g)e^{k_v \theta i},$$

which inspires the following

Definition 2.1.4 *An automorphic form ϕ is said to have weight $k = (k_v)$, if for each $v|\infty$,*

$$\phi(gr(\theta)) = \phi(g)e^{k_v \theta i}.$$

Similar to classical Hilbert modular form, one says that ϕ is of level N, if

$$\phi(gu) = \phi(g), \forall u \in U_1(N).$$

As already mentioned in the Introduction, we are primarily interested in Hilbert modular forms with trivial central character of level N and weight k. Having defined Hilbert modular forms and automorphic representation, we now briefly review the classification of local admissible representations. See Gelbart [5] for more details.

2.2. **Classification of local admissible representations.** Let (π, V) be an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. It's well known that for each place v of F, there exists an irreducible admissible representation π_v of $\mathrm{GL}_2(F_v)$ such that π is isomorphic to the restricted tensor product of π_v:

$$\pi \cong \otimes_v \pi_v.$$

Thus it boils down to the local irreducible admissible representations of $\mathrm{GL}_2(F_v)$. One has complete classification of all irreducible admissible infinite-dimensional representations of $\mathrm{GL}_2(F_v)$. See [5] and [14] for details

I. If v is nonarchimedean place of F.

(1) *Principal series.* These are the representations induced from a quasi-character of the Borel subgroup determined by two quasi-characters of $F^\times$. Let $\mu_1, \mu_2 : F^\times \to \mathbb{C}^\times$, be two quasi-characters, Define the space of locally constant functions on $\mathrm{GL}_2(F_v)$

$$\begin{aligned}\mathcal{B}(\mu_1, \mu_2) &= \mathrm{Ind}_B^{\mathrm{GL}_2}(\mu_1\mu_2)\\ &= \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \middle| f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{d}{a}\right|^{\frac{1}{2}} f(g) \right\}\end{aligned}$$

The group $\mathrm{GL}_2(F_v)$ acts on $\mathcal{B}(\mu_1, \mu_2)$ via right translation and the resulting representation is called a principal series if it is irreducible and is denoted $\pi(\mu_1, \mu_2)$. Using the Iwasawa decomposition

$$\mathrm{GL}_2(F_v) = B(F_v)K_v,$$

where $B(F_v) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a, d \in F_v^\times, b \in F_v \right\}$ and $K_v = \mathrm{GL}_2(\mathcal{O}_v)$, one sees easily that such a representation is admissible.

(2) *Special representation.* If the above representation is not irreducible, then one must have $\mu(x) := \mu_1(x)\mu_2(x) = |x|^{\pm 1}$. If $\mu(x) = |x|^{-1}$, then $\pi(\mu_1, \mu_2)$ contains a one-dimensional invariant subspace and the representation induced on the quotient space is irreducible. If $\mu(x) = |x|$, then $\pi(\mu_1, \mu_2)$ contains an irreducibly invariant subspace of codimension one. In both cases, the irreducible subquotients of $\pi(\mu_1, \mu_2)$ are called special representation and denoted $\sigma(\mu_1, \mu_2)$.

(3) *Supercuspidal representation.* If an irreducible admissible representation is neither principal nor special, then it is called supercuspidal.

II. if v is archimedean.

(1) *principal series.* Similar to the non-archimedean case, one still defines the induced representation of a character of Borel subgroup to $\mathrm{GL}_2(F_v) \cong \mathrm{GL}_2(\mathbb{R})$. Let $\mu_1, \mu_2 \to \mathbb{C}^\times$, be two characters, $K_v = \mathrm{O}_2(F_v)$. Define the space of K_v-finite functions on $\mathrm{GL}_2(F_v)$:

$$\mathcal{B}(\mu_1, \mu_2) = \left\{ f : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C} \middle| f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \right) = \mu_1(a)\mu_2(d) \left| \frac{d}{a} \right|^{\frac{1}{2}} f(g) \right\}.$$

The Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ acts on $\mathcal{B}(\mu_1, \mu_2)$ by

$$X \cdot \phi(g) = \frac{d}{dt} \phi(g \exp(tx))|_{t=0},$$

where $X \in \mathfrak{gl}_2(\mathbb{R})$. The compact group K_v acts via the right translation, thus produces a $(\mathfrak{gl}_2(\mathbb{R}),\ K_v)$-module. If $\mu(x) := \mu_1\mu_2^{-1}(x) \neq \mathrm{sgn}(x)^{\varepsilon}|x|^{k-1}$, where $\varepsilon = 0$ or 1, and k is an integer of the same parity as ε, then the representation is irreducible and denoted by $\pi(\mu_1, \mu_2)$.

(2) *Discrete series.* If $\mu x = \mathrm{sgn}(x)^{\varepsilon}|x|^{k-1}$, then $\mathcal{B}(\mu_1, \mu_2)$ contains an unique nonzero subspace V_0 which is either finite dimensional or infinite dimensional depending that if $k > 1$ or $k < 1$. In this case, one denotes by $\sigma(\mu_1, \mu_2)$ and calls it discrete series.

The following strong multiplicity one theorem is extremely useful in applications.

Theorem 2.2.1 (strong multiplicity one) *If* $\pi = \otimes_v \pi_v$ *and* $\pi' = \otimes_v \pi'_v$ *are two cuspidal irreducible representations of* $\mathrm{GL}_2(\mathbb{A})$, *if* $\pi_v \cong \pi'_v$, *for almost all* v, *then* $\pi \cong \pi'$

2.3. **Newform theory.** The adelic analogue of classical Atkin-Lehner theory is recalled briefly in this section, meanwhile we shall discuss a modified notion of newform.

2.3.1. **Atkin-Lehner theory.** As in classical modular form case, if $N'|N$, one may embed the space of modular forms of level N' into the space of modular forms of level N. In automorphic forms, one has the similar results. Let

$\mathcal{A}_k(N, \omega)$ be the space of forms of weight k and level N and with central character ω, one defines the following two operators:

$$\phi \mapsto \pi \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \phi \quad (v \nmid \infty),$$

$$\phi \mapsto \pi \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \phi \quad (v|\infty).$$

The first one increases the level by order 1 at the place v, the second one increases weight by 2 at infinite place v. Thus one obtains an embedding:

$$\mathcal{A}_{k'}(N', \omega) \hookrightarrow \mathcal{A}_k(N, \omega),$$

if $N'|N$ and $k' \leq k$, i.e., $k_v - k'_v \geq 0, \forall v \in \infty$. Let $\mathcal{A}_k(N, \omega)$ be the subspace of those forms which come from lower level or lower weight, i.e., they are obtained by applying one of these two operators. To define newform. We need to define Hecke operators. For each finite place v, $v \nmid N$, One defines the Hecke operator T_v to be the characteristic function of the double coset $H_v := U_0(N) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_0(N)$, where ϖ_v is the idele whose v-th component is ϖ_v and 1 elsewhere. Recall that the Hecke operator acts on $\mathcal{A}_k(N, \omega)$ by

$$\pi \mapsto \mathrm{T}_v \cdot \phi(g) = \int_{H_v} \phi(gh) dh.$$

Similar to classical situation, for any ideal $\mathfrak{a}$, $(\mathfrak{a}, N) = 1$, the Hecke operator $\mathrm{T}_{\mathfrak{a}}$ on $\mathcal{A}_k(N, \omega)$ is the following

$$\mathrm{T}_{\mathfrak{a}}\phi(g) = \sum_{\substack{\alpha\beta=\mathfrak{a} \\ x \pmod{\mathfrak{a}}}} \phi\left(g\begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix}\right).$$

where α and β run through representatives of integral ideles modulo $\widehat{\mathcal{O}}_F^{\times}$ with trivial component at places dividing N such that $\alpha\beta$ generates $\mathfrak{a}$. A form $\phi \in \mathcal{A}_k(N, \omega)$ is called a newform if it is an eigenform under $\mathrm{T}_{\mathfrak{a}}$, for each ideal $\mathfrak{a}$ of F and there is no old form which has the same eigenvalues as ϕ.

In previous section, we already discussed the automorphic representation generated by an automorphic form. A natural question is that when the representation is irreducible. One has

Lemma 2.3.1.1 *Assume that ϕ is an eigenform for all Hecke operators $\mathrm{T}_v, v \nmid N$, then the representation π_ϕ generated by ϕ is irreducible.*

Proof. Let $\mathcal{H}_v$ be the Hecke algebra of $\mathrm{GL}_2(F_v)$. It's well known, see Bump [3] Proposition 4.6.5, that $\mathcal{H}_v$ is generated by $\mathrm{T}_v, \mathrm{R}_v$ and R_v^{-1}, where R_v is the characteristic function of the double coset

$$H'_v = U_0(N)\begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix}U_0(N).$$

Since $\mathrm{R}_v\phi(g) = \int_{H'_v} \phi(gh)dh = \omega(\varpi_v)\phi(g)$. Thus ϕ is also an eigenform under R_v with eigenvalue $\omega(\varpi_v)$. Hence ϕ is an eigenform of $\mathcal{H}_v$. Note that ϕ is determined by eigenvalues of R_v and T_v.

Now let V be an irreducible subrepresentation inside $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ such that the projection ϕ' of ϕ onto V is not zero. Since the projection is $\mathrm{GL}_2(\mathbb{A})$-equivariant, thus ϕ and ϕ' have the same eigenvalues under T_v and R_v at least at those places v such that both π_v and π'_v are spherical representations. Hence $\pi_v \cong \pi'_v$ by a well known fact in representation theory of p-adic groups, see Bump [3] theorem 4.6.3. Finally strong multiplicity one theorem 2.2.1 implies that $\pi \cong \pi'$, since π is spherical representation for almost all v. □

Remark 2.3.1.2 If f is a classical Hilbert eigenform of level N, weight k, one may easily show that the Hilbert modular form ϕ_f produced by f is an eigenform of $\mathcal{H}_v$ for all $v \nmid N$, and both have the same eigenvalues,thus corresponds to an irreducible representation of $\mathrm{GL}_2(\mathbb{A})$.

The converse of the above lemma also holds. It is the adelic analogue of classical Atkin-Lehner theory proved by Casselman [4]. To describe that, we need to introduce a few notions.

I. Let F be a nonarchimedean local field with uniformizer ϖ, (π, V) be a admissible irreducible representation of $\mathrm{GL}_2(F)$ with central character ω. For any $c \geq 0$, one defines

$$U_0(\varpi^c) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \middle| c \equiv 0 \pmod{\varpi} \right\},$$

$$U_1(\varpi^c) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi^c} \right\}.$$

A vector v of V is said to have *level* ϖ^c if v is invariant under $U_1(\varpi^c)$.

Definition 2.3.1.3 *The order $o(\pi)$ of π is the minimal nonnegative integer c such that V has nonzero vector of level ϖ^c.*

Theorem 2.3.1.4 [Casselman] *(1) Let*

$$V((\varpi^c)) = \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \middle| f\left(g\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \omega(d)f(g), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\varpi^c) \right\},$$

then $V((\varpi^c))$ is one dimensional. Let v_π be a basis.
(2) If $c \geq o(\pi)$, then the space of vectors of level ϖ^c is of dimension $c - o(\pi) + 1$, and is generated by

$$v_i := \pi\begin{pmatrix} \varpi^{-i} & 0 \\ 0 & 1 \end{pmatrix} v_\pi, \quad i = 0, \cdots, c - o(\pi).$$

Proof. We only give the proof of the case that the representation is principal series. See Casselman [4] for other type of representations. So assume that the representation (π, V) is a principal series $\pi(\mu_1, \mu_2)$, where μ_i is a quasi-character of $F^\times$, $i = 1, 2$. Recall that it is the space of locally constant functions of $\mathrm{GL}_2(F)$ and

$$V = \left\{ f : \mathrm{GL}_2(F) \to \mathbb{C} \middle| f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d) \left|\frac{a}{d}\right|^{\frac{1}{2}} f(g) \right\}$$

Let n be the order of the representation π, using the Iwasawa decomposition of $\mathrm{GL}_2(F) = B(F) \cdot \mathrm{GL}_2(\mathcal{O}_F)$, one can write the space $V((\varpi^n))$

$$= \left\{ f : \mathrm{GL}_2(\mathcal{O}_F) \to \mathbb{C} \middle| f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = \mu_1(a)\mu_2(d)f(g)\mu_1\mu_2(d') \right\},$$

where $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U_0((\varpi^n))$.

We claim that $n \geq n_1 + n_2$, where n_i is the order of μ_i, i.e., n_i is the minimal nonnegative integer such that $\mu_i|_{1+(\varpi^{n_i})} \equiv 1$, $i = 1, 2$. The Bruhat decomposition of $\mathrm{GL}_2(F) = B(F) \coprod B(F) \cdot w \cdot B(F)$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, implies that a function f of $V((\varpi^n))$ is determined by $f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$. So take $0 \neq f \in V((\varpi^n))$, let $\Phi(x) = f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$. We look at the action of following elements: $\begin{pmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ (\varpi^n) & 1 \end{pmatrix}$, on the function $\Phi(x)$. Three conditions are obtained:

(1) $\Phi(x) = \Phi(x + b), \quad \forall b \in (\varpi^n)$;

(2) $\Phi(ax) = \mu_2(a)\Phi(x), \quad \forall a \in \mathcal{O}_F^\times$;

(3) $\Phi(x) = \mu(cx + 1)^{-1}|cx + 1|^{-1}\Phi(\frac{x}{cx+1}), \forall c \in (\varpi^n)$, where $\mu := \mu_1 \cdot \mu_2^{-1}$.

We may assume that either n_1 or $n_2 \geq 1$, since otherwise the claim is automatically true.

Case 1: $n_1, n_2 \geq 1$.

(i) Take $a \in 1 + \varpi^{n_2-1}$, if $x(a-1) \in \mathcal{O}_F$, then $\Phi(x) = 0$, i.e., $\Phi(x) = 0$ if $x \in \varpi^{-n_2+1}$. So we get an upper bound for ord(x), $x \in Supp(\Phi)$.

(ii) $\Phi(\lambda x) = \mu(\lambda)^{-1}|\lambda|^{-1}\Phi(x), \quad \forall \lambda \in F^\times$, s.t., ord$(\lambda^{-1} - 1) \geq n+$ ord(x). In particular, if $n+$ ord$(x) \geq 0$, then $\Phi(\lambda x) = \mu(\lambda)^{-1}|\lambda|^{-1}\Phi(x), \quad \forall \lambda \in \mathcal{O}^\times_{n+ord(x)}$. Assume first that $n+$ ord$(x) \geq 0, \quad x \in Supp(\Phi)$, then $\mu(\lambda)^{-1}|\lambda|^{-1}\Phi(x) = \mu_2(\lambda)\Phi(x)$, i.e., $n+$ord$(x) \geq n_1$, since c$(\mu_1) = n_1$. which implies that $n \geq n_1 + n_2$. Secondly, suppose that ord$(x) < -n$. Recall

$$\Phi(\lambda x) = \mu(\lambda)^{-1}|\lambda|^{-1}\Phi(x), \quad \forall \lambda \in F^\times, s.t., ord(\lambda^{-1} - 1) \geq n + ord(x).$$

In particular, the above formula holds for any $\lambda \in \mathcal{O}_F^\times$. Hence one has

$$\Phi(ax) = \mu_2(a)\Phi(x) = \mu_1^{-1}(x)\mu_2(x)\Phi(x) \Longrightarrow \mu_1(x) = 1, \quad \forall x \in \mathcal{O}_F^\times \Longrightarrow c(\mu_1) = 0.$$

A contradiction!

Case 2: $n_1 \geq 1, \quad n_2 = 0$.

Claim: $n+$ ord$(x) \geq 0, \quad \forall x \in Supp(\Phi)$, then $\Phi(\lambda x) = \mu^{-1}(\lambda)|\lambda|^{-1}\Phi(x), \quad \lambda \in \mathcal{O}_F^\times \Longrightarrow \mu_1(\lambda) = 1$, hence $n_1 = 0$. A contradiction.

The other parts are similar. $\Phi(\lambda x) = \mu(\lambda)^{-1}|\lambda|^{-1}\Phi(x) = \mu_2(x)\Phi(x), \quad \forall \lambda \in \mathcal{O}^\times_{ord(x)+n}$. Hence one gets $ord(x) + n \geq n_1$, i.e., $Supp(\Phi) = \varpi^{n_1-n}\mathcal{O}_F$. We want: $n \geq n_1$. If $n < n_1$, then $Supp(\Phi) \subseteq \varpi$. A contradiction, since $\Phi(x+b) = \Phi(x), \quad \forall b \in \mathcal{O}_F$.

Thus one can view μ_1, μ_2 as characters of $(\mathcal{O}_F/(\varpi^n))^\times$, which implies that the space $V((\varpi^n))$ is isomorphic to the space of functions ψ on $\mathrm{GL}_2(\mathcal{O}_F/(\varpi^n))$

satisfying the same condition as those in $V((\varpi^n))$, since

$$U(\varpi^n) = \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_F \middle| \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varpi^n} \right\}$$

is normal in both $\mathrm{GL}_2(\mathcal{O}_F)$ and $U_0(\varpi^n)$. We denote by $\overline{B}$ the image in $\mathrm{GL}_2(\mathcal{O}_F/(\varpi^n))$ of the Borel subgroup $B(\mathcal{O}_F)$ and can easily show that

$$\mathrm{GL}_2(\mathcal{O}_F/(\varpi^n) = \coprod_{i=0}^{n} \overline{B} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \overline{B}.$$

Therefore, a function ψ is determined by the value at $\overline{B} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \overline{B}$. To end the proof, we just need to know what function ψ on some $\overline{B} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \overline{B}$ satisfies the above condition. the condition can be translated into the following one: if

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^i & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \pmod{\varpi^n},$$

then $\mu_1(a)\mu_2(d) = \mu_1(d')\mu_2(d')$. For given a, d, a', d' there exist x, x' for which the equation holds if and only if the following equations have solution: $d \equiv d' \pmod{\varpi^i}$, $a \equiv a' \pmod{\varpi^i}$, $a' \equiv d \pmod{\varpi^{n-i}}$, $d - d' = a' - a$, which is equivalent to (1) ϖ^i lies in the conductor μ_1 and (2) ϖ^{n-i} is contained in the conductor of μ_2. Therefore, one sees that the minimal such n is exactly $n_1 + n_2$. For a given $c \geq n$, there are exactly $c - n + 1$ such distinct functions satisfying conditions, which form a basis of $V((\varpi))$. $\square$

II. If F is an archimedean local field, we already defined the notion of weight, which is the analogue of order for archimedean place. The weight of a representation π is the smallest nonnegative integer such that π has a nonzero vector of weight k. In fact, from the classification of $(\mathfrak{gl}_2(\mathbb{R}), K_v)$-modules, one knows that for any integer, the space of vectors of weight n is one dimensional of $|n| > k, n \equiv k \pmod 2$, zero otherwise.

Back to the number field case. Thus if $\pi = \otimes_v \pi_v$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{A})$, since π_v is irreducible, applying the above theorem, one obtains a unique line of newforms for each place v, $v \nmid \infty$. Globally, there exists a unique newvector up to a scaler, which generates the representation π. Therefore there exists one-one correspondence between newforms of level N and irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$.

2.3.2. ***Gross-Prasad theory and toric newform.*** We need a modified notion of newform as well as test form theory of Gross and Prasad, which occur in the formula. As a motivation, we first describe the theory of Waldspurger.

2.3.2.1 *Theory of Waldspurger.* Let F be a nonarchimedean local field, K be a quadratic extension of F (including the split case $K = F \oplus F$). Let T denote the torus of $K^\times$ embedded in $\mathrm{GL}_2(F)$. We denote $G = B^\times / F^\times$, where B is the quaternion algebra over F into which K is embedded. Let (π, V) be an irreducible admissible, infinite dimensional representation of $\mathrm{GL}_2(F)$, and χ be a quasi-character of $K^\times$. We assume that the central character ω of π is equal to $\chi^{-1}|_{F^\times}$, i.e., the subgroup $\Delta F^\times$ embedded diagonally in $\mathrm{GL}_2(F) \times T$

acts trivially on $V \otimes \mathbb{C}$. One considers the space of $\Delta F^\times$-invariant linear form $\ell : V \otimes \mathbb{C} \to \mathbb{C}$. Using Gelfand pairings, one can show that such a space is at most one dimensional if it exists. Waldspurger and Tunnel gave a criterion for a nonzero $\Delta F^\times$-invariant linear form to exist. To state Waldspurger and Tunnel's criterion, let σ_1 be the 2-dimensional representation of Deligne-Weil group of F associated to π by local Langlands correspondence, and σ_2 be the two-dimensional representation of Weil group of F which is induced from the quasi-character $\chi; K^\times \to \mathbb{C}^\times$. Then $\det\sigma_1 = \omega$, and $\det\sigma_2 = \alpha_{K/F} \cdot \chi|_{F^\times}$, where $\alpha_{K/F}$ is the quadratic character associated to K/F. The four-dimensional representation of the Deligne-Weil group has local root number $\epsilon(\sigma_1 \otimes \sigma_2) = \pm 1$.

The condition that $\epsilon(\sigma_1 \otimes \sigma_2) \neq \alpha_{K/F} \cdot \omega(-1)$ implies that the representation (π, V) is square-integrable, and K is a field, thus by local Jacquet-Langlands correspondence, (π, V) corresponds to an irreducible infinite dimensional representation (π', V') of $G(F)$. Similarly one considers the representation $V' \otimes \mathbb{C}$ of the group $G(F) \times \mathrm{GL}_1(K)$, and can show that the space of $\Delta K^\times$-invariant linear form $\ell' : V' \otimes \mathbb{C} \to \mathbb{C}$ is at most one. Waldspurger and Tunnel's criterion for both cases is

Theorem 2.3.2.1(Waldspurger, Tunnel) *There is a nonzero $\Delta K^\times$- invariant linear form $\ell : V \otimes \mathbb{C} \to \mathbb{C}$, if and only if*

$$\epsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1).$$

There is a non-zero $\Delta K^\times$-invariant linear form $\ell' : V' \times \mathbb{C} \to \mathbb{C}$ if and only if

$$\epsilon(\sigma_1 \otimes \sigma_2) = -\alpha_{K/F} \cdot \omega(-1).$$

Globally, one defines the global nonzero linear form if locally it exists for each place v. The significance of the existence of such a nonzero linear form is the following theorem due to Waldspurger.

Theorem 2.3.2.2 *There is a global nonzero linear form if and only if $L(\frac{1}{2}, \pi, \chi) \neq 0$.*

2.3.2.2 *Theory of Gross-Prasad.* If a local nonzero linear form ℓ exists, the vector $v \in V \otimes \mathbb{C}$ such that $\ell(v) \neq 0$ is called a *test form.* Gross and Prasad gave a concrete realization of such test vector under the assumption:

Either π is a unramified principal series of $\mathrm{GL}_2(F)$ or χ is an unramified quasi-character of $K^\times$.

I. If π is an unramified principal series, then $B \cong \mathrm{M}_2(F)$, let R be a maximal order in $\mathrm{M}_2(F)$ optimally containing the order $\mathcal{O}_{c(\chi)}$ of K. In this case, their result reads

Proposition 2.3.2.3 *If (π, V) is an umramified principal series, then there is a unique line L fixed by $R^\times \times \mathcal{O}_{c(\chi)}^\times$. If ℓ is any nonzero $\Delta K^\times$-invariant linear* *form, then* $\ell(v) \neq 0, \forall v \in L$.

II. If χ is unramified. When $\epsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1)$, let R_n be an order of reduced discriminant $(\varpi)^n$ in $\mathrm{M}_2(F)$ containing $\mathcal{O}_K$, where n is the conductor of π. When $\epsilon(\sigma_1 \otimes \sigma_2) = -\alpha \cdot \omega(-1)$, the condition forces $n \geq 1$. Let R'_n be an order of reduced discriminant $(\varpi)^n$ in B containing $\mathcal{O}_K$.

Proposition 2.3.2.4 *Assume that* χ *is an unramified quasi-character of* $K^\times$*, when* $n(\pi) \geq 2$*, assume further that the extension* K/F *is unramified.*
If $\epsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1)$*, the open compact subgroup* $R_n^\times \times \mathcal{O}_K^\times$ *fixes a unique line* L*, if* ℓ *ia a nonzero* $\Delta K^\times$*-invariant linear form, then* $\ell(v) \neq 0, \forall v \in L$.
If $\epsilon(\sigma_1 \otimes \sigma_2) = -\alpha_{K/F}\omega(-1)$*, the group* ${R'_n}^\times \times \mathcal{O}_K^\times$ *fixes a unique line* L'*. If* ℓ' *is a nonzero* $\Delta K^\times$*-invariant linear form, then* $\ell'(v) \neq 0, \forall v \in L'$.

If v is archimedean, then the representation π' is of finite dimensional and the torus $T(F)$ is compact. One shows that the fixed subspace of $T(F)$ in π' is of one dimension. And one defines a test vector to be any fixed vector up to a scalar.

2.3.2.3 *Toric newform.* In the formula of level ND, there is a modified notion of test form called toric newform, a form having character χ under the action of $T(F)$. The existence and uniqueness of such a form is guaranteed by the following

Lemma 2.3.2.5 [21]

(1) *If v is non-archimedean place of F, the χ-isotypic component $\pi'_{v,\chi}$ of π'_v under the action of Δ_v (see* [22] *for definition) is one-dimensional.*
(2) *If v is archimedean place of F, then the subspace of forms fixed by $T(F_v)$ in π'_v is of one dimension.*

2.4. **Weil representation and Jacquet-Langlands correspondence.** We review the constructions of theta series associated to a character χ of K and Jacquet-Langlands correspondence. For the purposes of the paper, it is sufficient to use Weil representation following Shimizu [15] to give construction directly, even though both constructions are special cases of much more general theory of theta lifting. So we first describe Weil representation, then give explicit constructions of theta series and Jacquet-Langlands correspondence. For details, see [5], [15].

2.4.1. *Weil representation.* In this subsection, we let F denote a non-archimedean local field. For our purposes, let V denote

(1) either a separable quadratic extension L of F equipped with a norm map q or
(2) the unique quaternion division algebra B over F with q the reduced norm.

In either case, let $x \to x^\sigma$ denote the canonical involution of V. Then

$$q(x) = x \cdot x^\sigma, \quad \forall x \in V$$

and

$$\mathrm{tr}(x) = x + x^\sigma, \quad \forall x \in V.$$

Let's fix a non-trivial additive character τ of F. V can be identified with its dual by the pairing

$$< x, y >= \tau(\mathrm{tr}(xy)),$$

since $(x, y) \to \mathrm{tr}(xy)$ is a non-degenerate bilinear form on V.

Let $S(V)$ denote the space of Schwartz-Bruhat functions on V. Recall that for each $\Phi \in S(V)$, the Fourier transform $\hat{\Phi}$ of Φ is defined to be

$$\hat{\Phi}(x) = \int_V \Phi(y) < x, y > dy$$

where Haar measure dy is chosen so that

$$(\hat{\Phi})^\wedge(x) = \Phi(-x).$$

The Weil representation is associated to the pairing (q, V). To describe it, we first construct a representation $r(s)$ of $\mathrm{SL}_2(F)$ in $S(V)$. Since elements of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generate $\mathrm{SL}_2(F)$. It suffices to describe the actions of

$$r\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right), r\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right), \text{and} \quad r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

on $S(V)$. One has

$$r\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \tau(uq(x))\Phi(x),$$

$$r\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)\Phi(x) = \omega(\alpha)|\alpha|^{\frac{1}{2}}\Phi(\alpha x),$$

$$r\left(\begin{pmatrix} 0 & 1 \\ 01 & 0 \end{pmatrix}\right)\Phi(x) = \gamma \cdot \hat{\Phi}(x^{\sigma}).$$

Here ω is the non-trivial character of $F^{\times}/q(V^{\times})$ if $V = L$ and the trivial character of $F^{\times}$ if $V = B$, and $\gamma = -1$ if $V = B$ and $|\gamma| = 1$ if $V = L$.

The existence of such a representation is proved by Weil, Shalika, and Tanaka. The representation may depend on the character τ. One may extend the representation to a representation of the group G_+ consisting of elements in $\mathrm{GL}_2(F)$ with determinant in $q(V^{\times})$. This group is of index 2 or 1 in $\mathrm{GL}_2(F)$ depending on whether $V = L$ or $V = B$. For $a = q(h) \in q(V^{\times})$, set

$$r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \Phi(xh).$$

One can show that this gives rise to a representation denoted by $r(n)$ of G_+.

Finally the induced representation

$$r(g) = \mathrm{Ind}_{G_+}^{\mathrm{GL}_2(F)}(r(n))$$

is called the *Weil representation* of $\mathrm{GL}_2(F)$ associated to (q, V). A remarkable thing is that it is independent of τ. This Weil representation is the bulk of the constructions we are working on which associates to each finite dimensional irreducible representation π' of $V^\times$ an irreducible representation π of $\mathrm{GL}_2(F)$.

So suppose that (π', H) is a finite-dimensional irreducible representation of $V^\times$. Consider the space

$$S(V) \otimes H$$

on which $\mathrm{SL}_2(F)$ acts. Here $\mathrm{SL}_2(F)$ acts on H trivially. One may view elements of $S(V) \otimes H$ as functions on V valued on H. We are interested in the subspace

$$\{\Phi \in S(V) \otimes H | \Phi(xh) = \pi'(h^{-1})\Phi(x), \forall h \in V^\times, q(h) = 1\}.$$

One can show the subspace is invariant under $\mathrm{SL}_2(F)$. The resulting representation is denoted by $r_{\pi'}$. Now following the same procedure as we did to Weil representation, i.e., we extend the representation $r_{\pi'}$ to a representation of G_+ by requiring

$$r_{\pi'}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = |h|^{\frac{1}{2}}\pi'(x)\Phi(xh)$$

if $a = q(h)$ for some $h \in V^\times$. And Moreover

$$r\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\chi_{\pi'}(a)I$$

for $a \in F^\times$ and $\chi_{\pi'}$ the central character of π'. The induced representation, still denoted by $r_{\pi'}$, $\mathrm{Ind}_{G_+}^{\mathrm{GL}_2(F)}(r_{\pi'})$ has the remarkable property.

Theorem 2.4.1. *Assume that $V = L$.*

(1) *If there is no character* λ *of* $F^\times$ *such that* $\chi = \lambda \cdot q$ *then* $r_{\pi'}(g)$ *is a supercuspidal representation of* $\mathrm{GL}_2(F)$*.*

(2) *If* $\chi = \lambda \cdot q$ *for some character* λ *of* $F^\times$ *then* $r_{\pi'}(g)$ *is equivalent to the principal series* $\pi(\lambda, \lambda\omega)$*.*

Theorem 2.4.2. *Assume that* $V = B$*.*

(1) *The representation* $r_{\pi'}(g)$ *decomposes as the direct sum of* $d = \dim(\pi')$ *mutually equivalent irreducible representations* $\pi(\pi')$ *of* $\mathrm{GL}_2(F)$*;*

(2) *Each* $\pi(\pi')$ *is supercuspidal if* $d > 1$ *and special if* $d = 1$*;*

(3) *All supercuspidal and special representations of* $\mathrm{GL}_2(F)$ *are obtained in this way. More precisely, the map*

$$\pi' \to \pi(\pi')$$

gives a one-to-one correspondence between the equivalence classes of finite-dimensional irreducible representations of $V^\times$ *and the equivalence classes of special and supercuspidal representations of* $\mathrm{GL}_2(F)$*.*

Remark 2.4.3. In the above, we assume that F is non-archimedean. Now assume that F is archimedean local field $\mathbb{R}$.

Case 1. $V = L = \mathbb{C}$.

If χ is not of the form $\lambda \cdot q$ with λ a character of $\mathbb{R}$, then

$$\chi(z) = (z\bar{z})^r z^m \bar{z}^n$$

with $r \in \mathbb{C}$, m and n two integers, one zero and other positive. In this case, $r_\chi = \sigma(\mu_1, \mu_2)$. Here

$$\mu_1(t) = |t|^{2r} t^{m+n} \mathrm{sgn}(t)$$

and

$$\mu_1 \mu_2^{-1}(t) = t^{m+n} \mathrm{sgn}(t).$$

If $\chi = \lambda \cdot q$ with $m + n = 0$, then

$$r_{\pi'} = \pi(\mu_1, \mu_1 \cdot \mathrm{sgn}).$$

Case 2. $V = B = \mathbb{H}$.

Identifying $\mathbb{H}$ with matrices $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a, b \in \mathbb{C}$ then $q(h) = \det(h)$. Every irreducible finite-dimensional representation π' of $\mathbb{H}^\times$ has the form

$$\pi'(h) = q(h)^r \rho_n(h)$$

Where $r \in \mathbb{C}$, and ρ_n is the n-th symmetric tensor product of the standard representation of $\mathrm{GL}_2(\mathbb{C})$. Let μ_1, μ_2 be characters of $\mathbb{R}^\times$ defined by

$$\mu_1(\alpha) = |\alpha|^{r+n+\frac{1}{2}}$$

$$\mu_2(\alpha) = |\alpha|^{r-\frac{1}{2}} \mathrm{sgn}(\alpha)^n.$$

define

$$\pi = \sigma(\mu_1, \mu_2).$$

In particular, in our special case that the automorphic representation $\pi = \otimes_v \pi_v$ is generated by Hilbert cusp newform ϕ of weight $(2k, \cdots, 2k)$ with trivial central character. The local representation π_v for $v|\infty$ is a discrete series $\sigma(p, t)$, where $p = 2k-1, t = 0$, see [5]. So one can determine n. It's easy to see that $n = 2k-2$. Hence π'_v for $v|\infty$ is a finite dimensional representation of $\mathrm{SO}_3(\mathbb{R})$. Its dimension is $2k-1$.

2.4.2. *Theta series and Jacquet-Langlands correspondence.* Now we are able to construct automorphic representation associated to a character of χ of K and Jacquet-Langlands correspondence. So we assume that F is a number field.

Case 1. $(V = L)$

Let χ be a character of $\mathbb{A}_L^\times / L^\times$. Write

$$\chi = \otimes_w \chi_w.$$

We shall attach to χ an automorphic representation

$$\pi(\chi) = \otimes_v \pi_v.$$

of $\mathrm{GL}_2(\mathbb{A}_F)$. The local representation π_v is constructed as follows.

(1) If v splits in L, write $v = w_1 w_2$ in $\mathcal{O}_L$. Thus we may view both characters χ_{w_1} and χ_{w_2} as characters of F_v. Define

$$\pi_v = \pi(\chi_{w_1}, \chi_{w_2}).$$

(2) If there is only one prime w lying above v. Then L_w is a genuine quadratic extension of F_v. Now to χ_w, apply Weil representation. We define

$$\pi_v = \pi(\chi_w).$$

One can show that $\otimes\pi_v$ defines an irreducible unitary representation of $\mathrm{GL}_2(\mathbb{A}_F)$. The newform θ_χ of this representation is called the *theta series* associated to χ. It is easy to see that

$$L(s,\pi(\chi)) = L(s,\chi).$$

Here the right hand of the equation is the Hecke L-series associated to χ.

Case 2. (V=B)

Let G be the algebraic group $B^\times$ over F. Assume that π' is an irreducible representation of $G(\mathbb{A}_F)$. Write

$$\pi' = \otimes_v \pi'_v.$$

To π', we may associate an irreducible representation π of $\mathrm{GL}_2(\mathbb{A}_F)$.

(1) If B in unramified at v. Then $B_v \cong \mathrm{M}_2(F_v)$. Define

$$\pi_v = \pi'_v.$$

(2) If B is ramified at v. Then π'_v is finite-dimensional, since π'_v is irreducible and $G(F_v)$ is compact modulo its center. Now apply Weil representation, and define

$$\pi_v = \pi(\pi'_v).$$

To sum up,

Theorem 2.4.4.(Jacquet-Langlands correspondence) *To each irreducible unitary representation* $\pi' = \otimes_v \pi'_v$ *of* $G(\mathbb{A}_F)$*, one associates an irreducible unitary* *representation* $\pi = \otimes_v \pi_v$ *of* $\mathrm{GL}_2(\mathbb{A}_F)$, *where* $\pi_v = \pi'_v$ *if* v *is not ramified, and* $\pi_v = \pi(\pi'_v)$ *if* v *is ramified. Moreover*

(1) π *is cuspidal for* $\mathrm{GL}_2(\mathbb{A}_F)$ *if* π' *is (greater than one dimensional) cuspidal for* $G(\mathbb{A}_F)$.

(2) *The mapping*

$$\pi' \to \pi,$$

restricted to the collection of (greater than one dimensional) cuspidal representations on $G(\mathbb{A}_F)$ *is one-to-one correspondence onto the collection of all equivalence classes of cuspidal representations* $\otimes_v \pi_v$ *on* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that* π_v *is square-integrable for those* v *at which* B *is ramified.*

2.5. **Rankin-Selberg L-functions.** The aim of this subsection is to review theory of Rankin-Selberg L-function associated to an automorphis representation and a character χ of K. Before that, we first go over the L-function associated to a single automorphic representation. Our references are [**?**], [14].

2.5.1. *L-function associated to an cuspidal automorphic representation.* Let $\pi = \otimes \pi_v$ be a cuspidal irreducible representation of $\mathrm{GL}_2(\mathbb{A})$. Let's fix an additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$. The L-series $L(s,\pi)$ of π is defined as $L(s,\pi) = \prod_v L_v(s,\pi_v)$, which satisfies functional equation with $\epsilon(s,\pi) = \prod_v \epsilon_v(s,\pi_v,\psi_v)$. So we start with local version. One defines local L-factor $L_v(s,\pi_v)$ and $\epsilon_v(s,\phi,\psi_v)$ for each type of local irreducible representation of $\mathrm{GL}_2(F_v)$ as follows:

I. v is nonarchimedean place.

(1) if π_v is a principal series, $\pi_v = \pi(\mu_1, \mu_2)$, with $\mu_i : F^\times \to \mathbb{C}^\times$, quasi-characters, then

$$L_v(s, \pi_v) = (1 - \mu_1(\varpi_v)|\varpi_v|^s)^{-1}(1 - \mu_2(\varpi_v)|\varpi_v|^s)^{-1},$$

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \mu_1, \psi_v)\epsilon(s, \pi_v, \psi_v).$$

(2)if π_v is a special representation, $\pi_v = \sigma(\mu)$, define

$$L(s, \pi_v) = (1 - \mu(\varpi_v)|\varpi_v|^s)^{-1}$$

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \mu_1, \psi_v)\epsilon(s, \mu_2, \psi_v)\frac{L(1-s, \mu_1^{-1})}{L(s, \mu_2)},$$

if one writes $\mu_1 = \mu \cdot |\cdot|^{\frac{1}{2}}$, and $\mu_2 = \mu \cdot |\cdot|^{-\frac{1}{2}}$.

(3) if π_v is supercuspidal, one defines

$$L(s, \pi_v) = 1.$$

II. v is archimedean place, we assume that v is real.

(1) if π_v is principal series, $\pi_v = \pi(\mu_1, \mu_2)$, then

$$L(s, \pi_v) = L(s, \mu_1)L(s, \mu_2),$$

$$\epsilon(s, \pi_v) = \epsilon(s, \mu_1, \psi_v)\epsilon(s, \mu_2, \psi_v).$$

(2) if π_v is discrete series, $\pi_v = \sigma(p, t)$, one defines

$$L(s, \pi_v) = (2\pi)^{-s-\frac{t+p}{2}}\Gamma(s + \frac{t+p}{2}),$$

$$\epsilon(s, \pi_v) = i^{p+1-n_1-n_2}i^{s_1+s_2},$$

if $\mu_j(x) = |x|^{s_j}\mathrm{sgn}(x)^{n_j}$.

The L-function $L(s,\pi)$ is only defined for $\mathrm{Re}(s) \gg 0$, so one has to have analytic continuation to the whole complex plane in order to have applications.

Theorem 2.6.1.1(Jacquet-Langlands) $L(s,\pi)$ *can be continued to a holomorphic function on the entire complex plane and satisfies the functional equation*

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\pi).$$

Let f be a classical Hilbert newform of weight $(k,\cdots,k)$, level N and of character ψ, $k \geq 2$. Using the type of local irreducible representation of π_f described before, one sees easily that

$$L(s,\pi) = \prod_v L(s,\pi_v),$$

where

$$L_v(s,\pi_f) = (1-\lambda_v|\varpi_v|^s + \psi(\varpi_v)|\varpi_v|^{2s})^{-1}, \quad \text{if } v \nmid \infty,$$

$$L_v(s,\pi_f) = (2\pi)^{-s-\frac{k-1}{2}}\Gamma(s+\frac{k-1}{2}), \quad \text{if } v|\infty,$$

where $\lambda_v|\varpi_v|^{-\frac{k-1}{2}}$ is the eigenvalue of T_v acting on f, $\forall v \nmid \infty$.

Classically, see Shimura [15], one defines the L-series of f as

$$\begin{aligned} L(s,f) &= \prod_v (1-\lambda_v|\varpi_v|^{-\frac{k-1}{2}}|\varpi_v|^s + \psi(\varpi_v)|\varpi_v|^{2s-(k-1)})^{-1} \\ &= \prod_v (1-\lambda_v|\varpi_v|^{s-\frac{k-1}{2}} + \psi(\varpi_v)|\varpi_v|^{2(s-\frac{(k-1)}{2})})^{-1} \\ &= \prod_v (1-\lambda_v|\varpi_v|^{s'} + \psi(\varpi_v)|\varpi_v|^{2s'})^{-1}, \end{aligned}$$

where $s' = s - \frac{k-1}{2}$, hence one obtains

$$L(s,f) = L(s - \frac{k-1}{2}, \pi_f).$$

Thus $L(s,f)$ yields a functional equation under $s \to k-s$.

2.5.2. *Rankin-Selberg L-functions associated to ϕ and χ.* We only discuss the Rankin-Selberg L-function associated to a newform ϕ and χ. For more general Rankin-Selberg convolution associated to automorphic representations, see Jacquet [13] or Zhang [21].

The Rankin-Selberg L-function $L(s,\phi,\chi)$ is defined by an Euler product over primes of F:

$$L(s,\phi,\chi) = \prod_v L_v(s,\phi,\chi),$$

where the factors is of degree ≤ 4 in $|\varpi_v|^s$. The local factors can be defined explicitly as follows.

For a finite place v, let write

$$L_v(s,\phi) = (1-\alpha_1|\varpi_v|^s)^{-1}(1-\alpha_2|\varpi_v|^s)^{-1},$$

$$\prod_{w|v} L_w(s,\chi_w) = (1-\beta_1|\varpi_v|^s)^{-1}(1-\beta_2|\varpi_v|^s)^{-1},$$

then

$$L_v(s,\phi,\chi) = \prod_{i,j}(1-\alpha_i\beta_j|\varpi|^s)^{-1}.$$

Here for a place w of K, the local factor $L(s,\chi_w)$ is defined as follows:

$$L(s,\chi_w) = \begin{cases} (1-\chi_w(\varpi_w)|\varpi_w|^s)^{-1}, & \text{if } w \nmid c\cdot\infty; \\ G_2(s), & \text{if } w|\infty; \\ 1, & \text{if } w|c(\chi). \end{cases}$$

Here $G_2(s) := 2(2\pi)^{-s}\Gamma(s)$.

If v is an infinite place of F, one may write

$$L_v(s,\phi) = G_1(s+\sigma_1)G_1(s+\sigma_2)$$

$$L_v(s,\chi) = G_1(s+\tau_1)G_1(s+\tau_2).$$

The the local L-factor $L_v(s,\phi,\chi)$ is defined in the following

$$\begin{aligned} L_v(s,\chi,\chi) &= \prod_{i,j} G_1(s+\sigma_i+\tau_j) \\ &= \begin{cases} G_2(s+\frac{k_v-1}{2})^2, & \text{if } k_v \geq 2; \\ G_2(s+it_v)G_2(s-it_v), & \text{if } k_v = 0. \end{cases} \end{aligned}$$

Where t_v is the parameter associated to ϕ at place v where the weight is 0 and $G_2(s) = G_1(s)G_1(s+1)$.

2.6. **Unitary similitudes.** Gross[6] proposes a programme unifying both special values of $L(1/2,\phi,\chi)$ and $L'(1/2,\phi,\chi)$ which refines the work of Waldspurger. One of his key observations is that the group $\mathrm{GL}_2 \times T/\Delta\mathbb{G}_m(F)$ is isomorphic to the group of F-rational points of a group of unitary similitudes GU. We show in this subsection this isomorphism.

2.6.1. **Local theory.** We begin with the local theory. Let F be a local field, and let K be an etale quadratic extension of F. Let $e \mapsto \overline{e}$ be the nontrivial involution of K fixing F. There are two cases:

(1) K is a field, then $e \mapsto \overline{e}$ is the nontrivial element in $\mathrm{Gal}(K/F)$.

(2) K is a split F-algebra. Then K is isomorphic to $F[x]/(x^2-x) \simeq F+F$. There are two orthogonal idempotents e_1 and e_2 in K, with $e_1+e_2=1$, and $\overline{e_1}=e_2$.

By local class field theory, there is a unique character $\omega : F^\times \longrightarrow \{\pm 1\}$ whose kernel is the norm group $\mathrm{N}K^\times = \{e\overline{e} : e \in K^\times\} \subset F^\times$.

Let π be an irreducible (complex) representation of $\mathrm{GL}_2(F)$, with central character $\omega : F^\times \longrightarrow \mathbb{C}^\times$. We assume that π is generic, or equivalently, that π is infinite-dimensional.

Let S be the two-dimensional torus $\mathrm{Res}_{K/F}\mathbb{G}_m$, and let χ be an irreducible complex representation of the group $S(F)=K^\times$. Since K has rank 2 over F, we have an embedding of the groups:

$$S(F) \simeq \mathrm{Aut}_K(K) \longrightarrow \mathrm{GL}_2(F) \simeq \mathrm{Aut}_F(E).$$

We will consider the tensor product $\pi \otimes \chi$ as an irreducible representation of the group $\mathrm{GL}_2(F) \times S(F)$, and wish to restrict this representation to the diagonally embedded subgroup $S(F)$. The central local problem is to compute the space of coinvariants $\mathrm{Hom}_{S(F)}(\pi \otimes \chi, \mathbb{C})$. If this is non-zero, we must have

$$\omega \cdot \chi|_{F^\times} = 1 \qquad (*)$$

as a character of $F^\times$. From now on, we assume that $(*)$ holds. Then $\pi \otimes \chi$ is an irreducible representation of the group $G(F)$, with

$$G = (\mathrm{GL}_2 \times S)/\Delta\mathbb{G}_m,$$

and we wish to restrict it to the subgroup $T(F)$, where T is the diagonally embedded one-dimensional torus $S/\mathbb{G}_m$.The group G defined above is a group of unitary similitudes. We explain this for general situation. Note that any quadratic K/F can be embedded into $M_2(F)$ via $\mathrm{End}_K(K) \subset \mathrm{End}_F(K)$, but this is not true for quaternion algebra.

2.6.2. **Unitary Similitudes.** Let B be a quaternion algebra over F with a fixed embedding $K \subset B$. We will show that $B^\times \times K^\times/\Delta F^\times$ is a group of unitary similitudes.

Proposition 2.7.2.1 *Let F be a local field and K an etale quadratic algebra over F, then there is a natural one to one correspondence between the following two sets*

(1) *Quaternion algebras B with an inclusion $K \subset B$.*
(2) *Non-degenerate unitary space (V, ϕ) of dimension 2 over K, with a vector v satisfying $\phi(v, v) = 1$.*

Proof: On the one hand, assume that $K \subset B$ as above. The inclusion defines a graded algebra structure on B: $B = B_+ + B_-$ with

$$B_+ = K$$

$$B_- = \{b \in B : be = \bar{e}b, \text{ for all } e \in K\}$$

Both B_+ and B_- are free K-modules of rank 1. Note that all elements in B_- have trace 0. The following pairing

$$\phi : B \times B \longrightarrow K; \quad (b_1, b_2) \mapsto (b_1\overline{b_2})_+$$

is a non-degenerate Hermitian form on the free K module B of rank 2. The group $GU(B,\phi)$ of unitary similitudes has F-valued points isomorphic to $B^\times \times K^\times/\Delta F^\times$. Recall that by definition

$$GU(B,\phi) = \Big\{ g \in \mathrm{GL}(B) \mid \phi(gv, gw) = \lambda(g)\phi(v,w), \text{ for some } \lambda(g) \in F^\times \Big\},$$

where λ is called a similitude factor and is a $F^\times$-valued character of $GU(B,\phi)$.

To give a specific isomorphism, we define an action of $B^\times \times K^\times$ on B by

$$(b,e)x = e^{-1}xb, \qquad \text{for all } (b,e) \in B^\times \times K^\times, \quad x \in B.$$

Then $\Delta F^\times$ acts trivially on B, and the similitude factor for ϕ is $\mathrm{N}b/\mathrm{N}e$ in $F^\times$. It is easy to see that we have obtained an injective homomorphism

$$B^\times \times K^\times / F^\times \longrightarrow GU(B,\phi).$$

Now we show it is surjective. Note that

$$\mathrm{N}g(1) = \phi(g(1), g(1)) = \lambda(g)\phi(1,1) = \lambda(g) \in F^\times,$$

we have that $b := g(1) \in B^\times$. It is easy to check that $h := r_{b^{-1}} \circ g \in SU(B,\phi) \subset GU(B,\phi)$ and we only need to show that h has form of $h(v) = e^{-1}ve$. Let's now compute h:

(1) $(hv)_+ = \phi(hv, h1) = \phi(v,1) = v_+$,

(2) Choose any element $u \in B_-$, we have $(hu)_+ = u_+ = 0$, so $hu \in B_-$, then $hu = ue_0$, for some $e_0 \in K = B_+$. For any $v \in B$, we have

$$hv\overline{e_0 u} = \phi(hv, hu) = \phi(v,u) = (v\overline{u})_+ = v_-\overline{u},$$

thus $(hv)_- = v_-\overline{e_0}^{-1} = e_0^{-1}v_-$, and $hu = ue_0 = u\overline{e_0}^{-1}$. Then $\mathrm{N}e_0 = 1$, and therefore by Hilbert 90, there exists an element $e \in K$ such that $e_0 = e\overline{e}^{-1}$.

Thus $hv = v_+ + e^{-1}\overline{e}v_- = e^{-1}ve$.

On the other hand, if (V, ϕ) is a non-degenerate unitary space of dimension 2 over K, with a vector $v \in V$ satisfying $\phi(v, v) = 1$. We give V the structure of a quaternion algebra over F, with an inclusion $K \subset B$. Indeed

$$V = K \cdot v + (K \cdot v)^\perp$$

and we define multiplication by

$$(ev + u)(e'v + u') = (ee' - \phi(u, u'))v + (eu' + \overline{e}'u).$$

The group $GU(V, \phi)(F)$ is then isomorphic to $B^\times \times K^\times/\Delta F^\times$, with B the quaternion algebra so defined. $\square$

3. Special value formula of level ND

We shall explain briefly the proof of the formula of level ND due to Hui Xue [20] in this chapter. Using Rankin-Selberg method, Gross and Zagier represent the L-function $L(s,\phi,\chi)$ as the inner product of ϕ with $\theta_\chi E$, where θ_χ is the theta series associated to χ and E certain Eisenstein series. Since $\theta_\chi E$ is of level ND, taking trace of $\theta_\chi E$ from ND to N, they obtain a form of level N:

$$\Phi_s(g) = \mathrm{Tr}_{ND/N}\overline{(d_F^{\frac{1}{2}-s}\theta_\chi(g)E(g))} = \sum_{\gamma\in U_0(N)/U_o(ND)} \overline{(d_F^{\frac{1}{2}-s}\theta_\chi(g)E(g))},$$

with the property that

$$L(s,\phi,\chi) = (\phi,\Phi_s)_{U_0(N)} = (\phi, \mathrm{pr}(\Phi_s)),$$

where $\mathrm{pr}(\Phi_s)$ is the projection of Φ_s into the space $\pi(\phi)$. Hence $L'(s,\phi,\chi)$ is the inner product of ϕ with Φ_s'. The form Φ_s' is not holomorphic. So one needs to get the holomorphic projection Ψ_s of Φ_s'. One has

$$L'(s,\phi,\chi) = (\phi,\Psi_s) = (\phi,\mathrm{pr}(\Psi_s)),$$

where $\mathrm{pr}(\Psi_s)$ is the projection of Ψ_s into the space of $\pi(\phi)$. By newform theory, One has $\mathrm{pr}(\Psi_s) = \lambda\phi$. Thus

$$\lambda = \frac{L(\frac{1}{2},\phi,\chi)}{||\phi||^2}.$$

On the other hand, let x be the CM-point on the modular curve $X_0(N)$. One can show that the form Φ whose Fourier coefficient is given by

$$\widehat{\Phi}(a) = |a| < x, \mathrm{T}_a x >,$$

is actually a cusp form of level N. The inner product of ϕ and Φ gives $(x_\phi, x_\phi)||\phi||^2$. They show that $\Phi - \Psi$ is an old form by computing the Fourier coefficient at a for $N|a$. Thus follows the formula

$$L'(\frac{1}{2}, \chi, \chi) = \frac{2^{g+1}}{\sqrt{N(D)}}||\phi||^2||x_\chi||^2.$$

Generalizing the formula to Hilbert modular forms of weight $(2, \cdots, 2)$, one encounters the difficulty that the trace $\mathrm{Tr}_{ND/N}\theta_\chi E$ is very hard to compute when χ is ramified as well as other geometric technical difficulties, instead Shou-Wu Zhang works directly on level ND. So naturally one would expect to have some other form to replace the role of ϕ in the level ND. Zhang uses quasi-newform $\phi^\#$, see the definition below, to replace ϕ. By developing a notion of geometric pairing, he computes the geometric pairing $(\mathrm{T}_a\eta, \eta)$ of ϕ and a special CM-cycle η. The computation shows that there is a close relationship between local Fourier coefficient of $\Phi_{\frac{1}{2}}$ and local geometric pairing $(\mathrm{T}_a\eta, \eta)$, which he calls the local Gross-Zagier formula. The special value formula of level ND follows from this local Gross-Zagier formula.

3.1. **Kernel function and quasi-newform.** We have explained in last chapter that the L-function $L(s, \phi, \chi)$ can be represented as inner product of ϕ with $\theta_\chi E$, i.e.,

$$\begin{aligned} L(s, \phi, \chi) &= |\delta|^{s-\frac{1}{2}} \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\mathrm{GL}_2(\mathbb{A})} \phi(g)\theta_\chi(g)E(s, g)dg \\ &= |\delta|^{s-\frac{1}{2}}(\phi, \overline{\theta_\chi E}). \end{aligned}$$

In order to get a more symmetric form, Zhang applies Atkin-Lehner operator to $\theta_\chi E$. Let S be the set of finite places ramified in K. Recall that for each subset T of S, the *Atkin-Lehner* operator is an element h_T in $\mathrm{GL}_2(\mathbb{A})$ whose v-th component is 1 for $v \notin T$ and $\begin{pmatrix} 0 & 1 \\ -t_v & 0 \end{pmatrix}$, where t_v has the same order as $c(\epsilon)$ such that $\epsilon_v(t_v) = 1$, for $v \in T$. One can show that

$$L(s, \phi, \chi) = \frac{\gamma_T(s)}{\mathrm{vol}(U_0(ND))} \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \phi(g)\theta(gh_T^{-1})E(s, gh_T^{-1})dg$$

with $\gamma(s)$ certain exponential function of s. Thus if we define

$$\Theta(s, g) = 2^{-|S|}|\delta|^{s-\frac{1}{2}} \sum_{T \subset S} \gamma_T(s)\theta_\chi(gh_T^{-1})E(s, gh_T^{-1})$$

and call it the kernel function. Then one has

$$L(s, \phi, \chi) = (\phi, \overline{\Theta})_{U_0(ND)}.$$

The kernel function has functional equation

$$\Theta(s, g) = \epsilon(s, \chi)\Theta(1 - s, g)$$

from which the functional equation of $L(s, \phi, \chi)$

$$L(1 - s, \phi, \chi) = (-1)^{\#\Sigma}\mathrm{N}_{F/\mathbb{Q}}(ND)^{1-2s}L(s, \phi, \chi)$$

follows. Note that the kernel function $\Theta(s, g)$ is of level ND. What Gross and Zagier do is that they take trace of $\Theta(s, g)$ from ND to N to have a form of level N. Because of the technical difficulty-the trace is very massy if χ is ramified, so instead of taking trace, Zhang works directly on level ND. In this case, one needs to find an analogue of ϕ for the level ND. The analogue is called *quasi-newform* associated to χ. It is defined as follows. Let $\mathrm{pr}(\overline{\Theta})$ be the projection of the kernel function Θ into the space $\pi(\phi)$. The quasi-newform $\phi_s^{\#}$ is the projection of ϕ into the line spanned by $\mathrm{pr}(\Theta)$, i.e., $\phi_s^{\#}$ is the unique nonzero form in the space $\pi(\phi)$ of level ND satisfying the following identities

$$(\phi_s^{\#}, \phi_a) = \nu^*(a)(\phi_s^{\#}, \phi_s^{\#}), a|D,$$

where $\nu^*(a) = \prod_{v \in S} \frac{|a|_v^{s-\frac{1}{2}} + |a|_v^{\frac{1}{2}-s}}{2} \cdot \begin{cases} \nu(a), & \text{if } a|c(\epsilon); \\ 0, & \text{otherwise.} \end{cases}$

and $\phi_a := \rho\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\phi.$

Note that the kernel function $\Theta(s, g)$ is non-holomorphic if $k > 1$. Thus one needs to consider the holomorphic projection of $\Theta(s, g)$. We still denote it by $\Theta(s, g)$. Let's write the Fourier expansion of the kernel function $\Theta(s, g)$ as follows:

$$\Theta(s, g) = C(s, g) + \sum_{\alpha \in F^\times} W\left(s, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Since $\Theta(s, g)$ is a linear combination of the form

$$\Theta(s, g) = \sum_i \theta_i(g) E_i(g)$$

by definition. The constant and Whittaker function of $\Theta(s, g)$ can be expressed in terms of Fourier expansions of θ_i and E_i.

Let

$$\theta_i(g) = \sum_{\xi \in F} W_{\theta_i}(\xi, g), \quad E_i(g) = \sum_{\xi \in F} W_{E_i}(\xi, g),$$

be Fourier expansions of θ_i and E_i. Then

$$C(s,g) = \sum_{\xi \in F} C(s,\xi,g),$$

$$W(s,g) = \sum_{\xi \in F} W(s,\xi,g),$$

where

$$C(s,\xi,g) = \sum_i W_{\theta_i}(-\xi,g) W_{E_i}(\xi,g),$$

and

$$W(s,\xi,g) = \sum_i W_{\theta_i}(1-\xi,g) W_{E_i}(\xi,g).$$

Furthermore, one can decompose $W(s,\xi,g)$ into

$$W(s,\xi,g) = \otimes_v W_v(s,\xi_v,g_v).$$

The local Fourier coefficients $W_v(\frac{1}{2},\xi,g)$ is related to certain local height pairing of some special CM cycle via local Gross-Zagier formula.

3.2. **Geometric pairing and local Gross-Zagier formula.** Let B be the quaternion algebra over F ramified at exactly in Σ. We denote by G the algebraic group $B^\times/F^\times$, and by T the torus given by $K^\times/F^\times$ embedded in G. The set

$$C := T(F)\backslash G(\mathbb{A}_f)$$

is called the set of *CM* points. For any open compact subgroup $U \subset G(\mathbb{A}_f)$, we also denote by C_U the set $T(F)\backslash G(\mathbb{A}_f)/U$. Gross defines an intersection pairing, for some fixed maximal order R of B,

$$(,) : C_U \times C_U \to \mathbb{R},$$

such that given two points $P, P' \in C_U$,

$$(P, P') = \begin{cases} 0, & \text{if } \pi(P) \neq \pi(P'); \\ \#(R_P^{\times}), & \text{if } \pi(P) = \pi(P'). \end{cases}$$

Here $\pi : C_U \to G(F)\backslash G(\mathbb{A}_f)/U$, and R_P is the oriented order of B corresponding to the point $P \in C_U$, i.e., $R_P := B \cap P\widehat{R}P^{-1}$. A vast generalization is introduced by Zhang via his *geometric pairing*. Using this geometric pairing, Zhang proves a local version of Gross-Zagier formula which relates the local Fourier coefficients of the kernel function to the local geometric pairing of some special CM-cycle. This local Gross-Zagier formula is the key to proving special value formula of both $L(\frac{1}{2}, \phi, \chi)$ and $L'(\frac{1}{2}, \phi, \chi)$. We briefly review Zhang's theory of geometric pairing.

Let m be a real-valued locally constant function on $G(\mathbb{A}_f)$. In Zhang's definition, he requires the function m first defined on $G(F)$ and invariant under $T(F)$ such that $m(\gamma) = m(\gamma^{-1})$, and then extend it to a function m on $G(\mathbb{A}_f)$ by requiring

$$m(\gamma, g_f) = \begin{cases} m(\gamma), & \text{if } g_f = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now the kernel function

$$k(x, y) = \sum_{\gamma \in G(F)} m(x^{-1}\gamma y)$$

is a function on $C \times C$. Let $S(C)$ denote the set of locally constant functions with compact support and call it the space of *CM-cycles*. Note that C admits a natural action of $T(\mathbb{A}_f)(G(\mathbb{A}_f))$ on the left (right), which induces an action of $T(\mathbb{A}_f)$ on $S(C)$. Since $T(F)\backslash T(\mathbb{A}_f)$ is compact, the set $S(C)$ is decomposed as

$$S(C) = \oplus_{\chi} S(C, \chi)$$

where χ runs through characters of $T(F)\backslash T(\mathbb{A}_f)$. For any given CM-cycles $\alpha, \beta \in S(C)$, Zhang defines

$$\begin{aligned} < \alpha, \beta > &= \int_{C^2} \alpha(x) k(x,y) \overline{\beta}(y) dx dy \\ &= \lim_{U \to 1} \int_{C^2} \alpha(x) k_U(x,y) \overline{\beta}(y) dx dy. \end{aligned}$$

where U runs over open subgroups of $G(\mathbb{A}_f)$ and

$$k_U(x,y) = \mathrm{vol}(U)^{-2} \int_{C^2} k(xu, yv) du dv.$$

It's called the *geometric pairing with multiplicity function m.*

Remark 3.2.1 In particular, let m be the characteristic function of the open compact subgroup U given by a maximal order R of B considered by Gross. For two points $P, P' \in C$, let α and α' be the characteristic function of P and P' respectively. Thus $\alpha, \alpha' \in S(C)$, then one can easily see that

$$< \alpha, \alpha' >= \begin{cases} 0, & \text{if } \pi(P) \neq \pi(P'); \\ \#(R_P^{\times}), & \text{if } \pi(P) = \pi(P'). \end{cases}$$

By identifying $P(P')$ with $\alpha(\alpha')$, one thus recovers Gross' intersection pairing.

It is not hard to see that the geometric pairing

$$< \alpha, \beta >= \sum_{\gamma \in T(F)\backslash G(F)/T(F)} m(\gamma) < \alpha, \beta >_{\gamma},$$

where $<\alpha,\beta>_\gamma = \int_{T_\gamma(F)\backslash G(\mathbb{A}_f)} \alpha(\gamma y)\overline{\beta}(y)dy$ and

$$T_\gamma = \begin{cases} T, & \text{if } \gamma \in N_T; \\ 1, & \text{if } \gamma \notin N_T. \end{cases}$$

with N_T the normalization of T in G. Since $<\alpha,\beta>_\gamma$ only depends on the class of γ in $T(F)\backslash G(F)/T(F)$, one may pass from $T(F)\backslash G(F)/T(F)$ to F by the following embedding

$$\xi : T(F)\backslash G(F)/T(F) \longrightarrow F$$

$$a + b\epsilon \longrightarrow \frac{\mathrm{N}(b\epsilon)}{\mathrm{N}(a+b\epsilon)}$$

One thus defines

$$<\alpha,\beta>_\xi = \begin{cases} <\alpha,\beta>_\gamma, & \text{if } \xi = \xi(\gamma); \\ 0, & \text{else.} \end{cases}$$

Hence

$$<\alpha,\beta> = \sum_{\xi\in F} m(\gamma) <\alpha,\beta>_\xi \ .$$

If both $\alpha = \prod_v \alpha_v$ and $\beta = \prod_v \beta_v$ are decomposable, then $<\alpha,\beta>_\xi$ can be further decomposed as

$$<\alpha,\beta>_\xi = \prod_v <\alpha_v,\beta_v>_\xi$$

with $<\alpha_v,\beta_v>_\xi = \int_{G(F_v)} \alpha_v(\gamma y)\overline{\beta}_v(y)dy$. It is this local geometric pairing of some special CM-cycle that is related to the local Fourier coefficient of the kernel function Φ, which Zhang calls the local Gross-Zagier formula. To state the formula, we need to define the special CM-cycle.

Let A be an order of B such that, locally for each finite place v,

$$A_v = \mathcal{O}_{K,v} + \lambda_v c(\chi_v)\mathcal{O}_{K,v},$$

where $\lambda_v \in B_v$ with the properties that

(1) $\lambda_v x = \bar{x}\lambda_v, \forall x \in K_v$,

(2) $\mathrm{ord}_v(\det\lambda_v) = \mathrm{ord}_v(N)$.

We denote by Δ the subgroup of $G(\mathbb{A}_f)$ such that

$$\Delta = \prod_{v\nmid c(\chi)} A_v^\times F_v^\times / F_v^\times \cdot \prod_{v|c(\chi)} A_v^\times K_v^\times / F_v^\times.$$

Notice that one has a natural isomorphism

$$A_v/\lambda_v c(\chi_v) A_v \cong \mathcal{O}_{K,v}/\lambda_v c(\chi_v)\mathcal{O}_{K,v}.$$

Thus one may extend χ to a character, still denoted by χ, of Δ. Now the special CM-cycle is the character $\eta = \prod_v \eta_v$ with

$$\eta_v : T(F_v)\Delta_v \longrightarrow \mathbb{C}^\times$$

$$tu \longrightarrow \chi_v(t)\chi_v(u).$$

For $a \in \mathbb{A}_f^\times$ an integral idéle prime to ND, the Hecke operator is defined to be

$$T_a\eta = \prod_v T_{a_v}\eta_v$$

$$T_{a_v}\eta_v(x) = \int_{H(G_v)} \eta_v(xg)dg,$$

where $H(G_v) = \{g \in \mathrm{M}_2(\mathcal{O}_v) || \det(g)| = |a_v|\}$ and we choose the measure dg such that $\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}_v)) = 1$. The local Gross-Zagier formula is the following

Proposition 3.2.3 *Let* $g = \begin{pmatrix} a_v\delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, *then*

$$\overline{W_v}(\frac{1}{2}, \xi, g) = |c(\omega_v)|^{\frac{1}{2}} \epsilon(\omega_v, \psi_v)\chi_v(u)|(1-\xi)\xi|_v^{\frac{1}{2}}|a_v|_v \mathrm{vol}(\Delta_v)^{-1} < T_{a_v}\eta_v, \eta_v >_\xi,$$

where u is any trace free element in K_v.

In particular, globally one obtains

Corollary 3.2.4 *Let $<,>$ be the geometric pairing on the CM-cycle with multiplicity m on F such that $m(\xi) = 0$ is ξ is not in the image of the map ξ. Assume that $\delta_v = 1$ for $v|\infty$. Then there exist constants c_1 and c_2 such that for a an integral idéle prime to ND,*

$$\begin{aligned} |c(\omega)|^{\frac{1}{2}}|a| < T_a\eta, \eta >_\Delta &= (c_1 m(0) + c_2 m(1))|a|^{\frac{1}{2}} W_f(g) \\ &\quad + i^g \sum_{\xi \in F - \{0,1\}} |\xi(1-\xi)_\infty^{\frac{1}{2}} \overline{W}_f(\frac{1}{2}, \xi, g) m(\xi). \end{aligned}$$

Now we need to choose the right multiplicity function m. For each archimedean place v, define

$$m(\gamma, g_f) = 2CP_{k-1}(1 - 2\xi(\gamma))W_v(g_v),$$

where $C = \frac{4^{k-1}[(k-1)!]^2}{(2k-1)!}$, $P_{k-1}(g)$ is a function on $G(F_v) \cong \mathrm{SO}_3(\mathbb{R})$ such that

$$\int_{\mathrm{SO}_3(\mathbb{R})} P_{k-1}^2 dg = \frac{1}{2k-1},$$

and W_v is the standard Whittaker function of weight $2k$ at v, i.e., W_v is a function in $\mathcal{W}(\pi_v, \psi_v)$ such that

$$W_v\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 2a^k e^{-2\pi a}, & \text{if } a > 0; \\ 0, & \text{else.} \end{cases}$$

Here we view the multiplicity function parameterized by the continuous parameter g_∞. Now we look at the spectral decomposition of the geometric pairing $< \alpha, \beta > (g_\infty)$ as a Whittaker function on $\mathrm{GL}_2(\mathbb{R})$. For that, it suffices to determine the spectral decomposition of the kernel function $K_U(x, y)$.

Proposition 3.2.5 *As a Whittaker function on* $\mathrm{GL}_2(\mathbb{R})$,

$$K_U(x,y) = C \cdot \sum_i W_i(g_\infty)\phi_i(x)\overline{\phi}_i(y) + C\int_{\mathfrak{M}} W_{\mathfrak{M}}(g_\infty)E_{\mathfrak{M}}(x)\overline{E}_{\mathfrak{M}}(y)d\mathfrak{M}.$$

Where the sum runs over all cuspidal eigenforms ϕ_i of Hecke operators and Laplace operators, and W_i is the Whittaker function of ϕ_i^{new}, ϕ_i^{new} being the newform of weight $(2k, \cdots, 2k)$ in the representation π of $\mathrm{PGL}_2(\mathbb{A})$ corresponding to the representation π_i' of $G(\mathbb{A})$ generated by ϕ_i via Jacquet-Langlands correspondence. In particular, let $\alpha = \beta = \eta$, then the form

$$\Psi = C|c(\omega)|^{\frac{1}{2}} \sum_i \phi_i^{\mathrm{new}} |(\phi, \eta)|^2 + C|c(\omega)|^{\frac{1}{2}} \int_{\mathfrak{M}} E_{\mathfrak{M}}^{\mathrm{new}} |(E_{\mathfrak{M}}, \eta)|^2 d\mathfrak{M}$$

has the same a-th Fourier coefficients with the kernel function Φ for a prime to ND. Thus $\Phi - \Psi$ is an old form. On the other hand, since η has character χ under the action of $T(\mathbb{A})$, so we may assume that ϕ_i has the same character χ under the action of $T(\mathbb{A}_f)$, which turns out to be exactly the toric newform ϕ_χ defined in chapter 1.

The projection of $\Phi - \Psi$ on the space $\pi(\phi)$ is still an old form. By taking the first Fourier coefficient, the desired formula is obtained, i.e.,

$$\phi^{\#}(1)L(\frac{1}{2}, \phi, \chi) = C|c(\omega)|^{\frac{1}{2}}||\phi^{\#}||^2|(\phi_\chi, \eta)|^2.$$

4. Special value formula of level N

In this section, we shall deduce the final formula, i.e., the special value formula of level N from the formula of level ND in last section. We shall express the special value of $L(s,\phi,\chi)$ at $s=\frac{1}{2}$ in terms of certain test form on a Shimura variety evaluated at certain CM-point. Let's explain the CM-divisor P_χ occurring in the formula of level N. Recall that our formula is

$$L(\frac{1}{2},\phi,\chi)=\frac{C}{\sqrt{\mathrm{N}(D)}}||\phi||^2\cdot|(\widetilde{\phi},P_\chi)|^2.$$

To define P_χ, let R be an order of B containing $\mathcal{O}_K$ with reduced discriminant N. One may construct such an order as follows. Choose a maximal order $\mathcal{O}_B$ of B containing $\mathcal{O}_K$ and an ideal $\mathcal{N}$ of $\mathcal{O}_K$ such that

$$\mathrm{N}_{K/F}(\mathcal{N})\cdot\mathrm{disc}_{B/F}=N.$$

Then take $R=\mathcal{O}_K+\mathcal{N}\mathcal{O}_B$. The group

$$U_v=R_v^\times/\mathcal{O}_v^\times$$

defines an open compact subgroup of $G(F_v)$. Let $U=\prod_{v\nmid\infty}U_v$. It is an open compact subgroup of $G(\mathbb{A}_f)$. The Shimura variety defined by G is isomorphic to $G(F)\backslash G(\mathbb{A}_f)/U$, since B is ramified at all archimedean places. Thus it is 0-dimensional. We first define a CM point $P_c\in G(F)\backslash G(\mathbb{A}_f)/U$. Let $i_c\in G(\mathbb{A}_f)$ such that

$$U_T:=i_cUi_c^{-1}\cap T(\mathbb{A}_f)^\times=\widehat{\mathcal{O}}_{c(\chi)^\times}/\widehat{\mathcal{O}_F}^\times.$$

Set $P_c = [i_c] \in G(F)\backslash G(\mathbb{A}_f)/U$. Finally

$$P_\chi = \sum_{t \in T(F)\backslash T(\mathbb{A}_f)/U_T} \chi(t)[ti_c].$$

As we see from last section that the formula of level ND has an extra term $\phi^\#$ involved which is an obstruction to arithmetic applications. To deduce the formula of level N from the formula of level ND. One rewrites the formula of level ND in the following way:

$$\frac{\widehat{\phi^\#}(1)}{\mathrm{N}(c(\chi))}\frac{||\phi||^2}{||\phi^\#||^2}\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2}L(\frac{1}{2}, \chi, \phi) = \frac{C}{\sqrt{\mathrm{N}(D)}}||\phi||^2|(\widetilde{\phi}, P_\chi)|^2.$$

We shall prove that

$$\frac{\widehat{\phi^\#}(1)}{\mathrm{N}(c(\chi))}\frac{||\phi||^2}{||\phi^\#||^2}\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2}L(\frac{1}{2}, \chi, \phi) = C(\chi)\prod_{v|D} Q_v(\lambda_v),$$

where λ_v is the parameter in the local L-factor $L_v(s, \phi) = (1 - \lambda_v|\varpi_v|^s + |\varpi_v|^{2s})^{-1}$, and $C(\chi)$ is a constant depending only on χ and for each $v|D$, Q_v is a rational function in $\mathbb{C}(t)$ depending only on χ_v which takes value 1 at $t = 0$ and is regular for $t \leq |\varpi_v|^{\frac{1}{2}} + |\varpi_v|^{\frac{-1}{2}}$. The idea is that we view ϕ as a form varying in the space $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$, a space consisting of discrete part (cusp forms) and continuous part (continuous sum of Eisenstein series). The form

$$\frac{\widehat{\phi^\#}(1)}{\mathrm{N}(c(\chi))}\frac{||\phi||^2}{||\phi^\#||^2}\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2}L(\frac{1}{2}, \chi, \phi)$$

can be viewed as a formula associated to each form in the space $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$. The above shows exactly that

$$\frac{\widehat{\phi^\#}(1)}{\mathrm{N}(c(\chi))}\frac{||\phi||^2}{||\phi^\#||^2}\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2}L(\frac{1}{2}, \chi, \phi)$$

is independent of the choices of ϕ. Thus one can use Eisenstein series to determine $C(\chi)\prod_{v|D} Q_v(\lambda_v)$. To that end, a similar special value formula of Rankin-Selberg L-function associated to an Eisenstein series and χ will be deduced to explicitly determine the constant $C(\chi)\prod_{v|D} Q_v(\lambda_v)$. The proof consists of the following two steps:

(1) compare two formulae of level N and ND and show they are equal up to universal constants;

(2) obtain a special value formula of level D for Rankin-Selberg L-function $L(s, E, \chi)$ associated to an Eisenstein series E and χ and explicitly compute the universal constants, thus prove the final formula.

4.1. **Universal constants.** In this subsection, we shall prove that both formulae of level N and ND are equal up to some universal constants. We have

Proposition 4.1.1 *For each $v|D$, there exists a rational function $Q_v \in \mathbb{C}(t)$, depending only on χ_v, $Q_v(0) = 1$, and being regular for $t \leq |\varpi_v|^{\frac{1}{2}} + |\varpi_v|^{-\frac{1}{2}}$, such that*

$$\frac{\widehat{\phi^{\#}}(1)}{\mathrm{N}(c(\chi))} \frac{||\phi||^2}{||\phi^{\#}||^2} \frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2} = C(\chi) \prod_{v|D} Q_v(\lambda_v),$$

where $C(\chi)$ is a constant depending only on χ and λ_v is the parameter in the local L-factor

$$L(s, \phi) = (1 - \lambda_v |\varpi_v|^s + |\varpi_v|^{2s})^{-1}.$$

Proof. We first reduce it to local case, then show everything locally. We need the local version of quasi-newform. Before that, Let's fix a hermitian form on the Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. By the discussion in Chapter 1, one sees that the irreducible cuspidal representation $\pi = \otimes\pi_v$ is unitary. Thus π_v is unitary for each v. We choose a hermitian form $(,)$ on $\mathcal{W}(\pi_v, \psi_v)$ for each v, such that $||W_v||^2 = 1$, for almost all v, where W_v is the normalized newform of $\mathcal{W}(\pi_v, \psi_v)$. Hence the global hermitian form is proportional to L^2-norm. For each v, the Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ has the normalized newform W_v of level $N_v := \varpi_v^{\mathrm{ord}_v(N)}$ and the quasi-newform $W_v^{\#}$ with respect to χ_v is the form of level$N_vD_v = \varpi_v^{\mathrm{ord}_v(N)+\mathrm{ord}_v(D)}$ satisfying

$$(W_v^{\#}, W_{v,i}) = \nu^i(W_v^{\#}, W_v^{\#}), \quad i = 0, 1, \cdots, \mathrm{ord}_v(D),$$

where

$$\nu = \begin{cases} 0, & \text{if } v|c(\omega); \\ \chi_v(\varpi_{K,v}), & \text{if } v \nmid c(\omega). \end{cases}$$

In particular, if $v \nmid D$, so $W_v^{\#}$ is of level $\varpi_v^{\mathrm{ord}_v(N)}$ and the relation above implies that $W_v^{\#} = W_v$. The uniqueness of the global quasi-newform $\phi^{\#}$ implies that $\phi^{\#} = \otimes_v W_v^{\#}$. Now we divide the proof into three steps showing that each term has the property in proposition.

(1) $\frac{||\phi||^2}{||\phi^{\#}||^2}$. From the above, one sees that

$$\frac{||\phi||^2}{||\phi^{\#}||^2} = \frac{\prod_v (W_v, W_v)}{\prod_v (W_v^{\#}, W_v^{\#})} = \frac{\prod_v (W_v, W_v)}{\prod_v (W_v^{\#}, W_v^{\#})}.$$

We now show that there exists, for each $v|D$, a rational function $Q_{1,v}(t) \in \mathbb{C}(t)$, depending only on χ_v, $Q_{1,v}(0) = 1$, and $Q_{1,v}(t)$ is regular for $t \leq |\varpi_v|^{\frac{1}{2}} + |\varpi_v|^{-\frac{1}{2}}$ such that

$$\frac{(W_v, W_v)}{(W_v^{\#}, W_v^{\#})} = C_{1,v}(\chi_v) Q_{1,v}(\lambda_v).$$

Write $W_v^\# = \sum_{i=0}^{\mathrm{ord}_v(D)} \alpha_{v,i} W_{v,i}$. The definition of quasi-newform can be translated into the system of following equations

$$\sum_{i=0}^{\mathrm{ord}_v(D)} \alpha_{v,i} \frac{(W_{v,i-j}, W_v)}{(W_v, W_v)} = \nu_j \sum_{i=0}^{\mathrm{ord}_v(D)} \alpha_{v,i} \frac{(W_{v,i}, W_v)}{(W_v, W_v)}, j = 1, \cdots, \mathrm{ord}_v(D)$$

$$\frac{(W_v\#, W_v)}{(W_v, W_v)} = \frac{(W_v^\#, W_v^\#)}{(W_v, W_v)} = \sum_{i=0}^{\mathrm{ord}_v(D)} \nu^i \alpha_{v,i} \frac{(W_v\#, W_v)}{(W_v, W_v)} \Leftrightarrow \sum_{i=0}^{\mathrm{ord}_v(D)} \nu^i \alpha_{v,i} = 1.$$

Hence each $\alpha_{v,i}$ is a rational function in $\frac{(W_{v,j}, W_v)}{(W_v, W_v)}$ for $j = 0, \cdots, \mathrm{ord}_v(D)$. It remains to show that $\frac{(W_{v,j}, W_v)}{(W_v, W_v)}$ is a polynomial in λ_v. Let $U_v = \mathrm{GL}_2(\mathcal{O}_v)$. Define $H_{v,j}$ be $\mathrm{vol}(H_{v,j})^{-1}$ times the characteristic function of $U_v \begin{pmatrix} \varpi_v^{-j} & 0 \\ 0 & 1 \end{pmatrix} U_v$ and $\mathrm{T}(\varpi_v^{-j}$ to be the Hecke operator corresponding to $H_{v,j}$. One obtains that

$$\begin{aligned} \frac{(W_{v,j}, W_v)}{(W_v, W_v)} &= (W_v, W_v)^{-1} \mathrm{vol}(U_v)^{-1} \int_{U_v} (\pi(u) W_{v,j}, \pi(u) W_v) du \\ &= (W_v, W_v)^{-1} \mathrm{vol}(H_{v,j})^{-1} \int_{H_{v,j}} (\pi(u) W_v, W_v) du \end{aligned}$$

Since $T(\varpi_v^{-j})$ is generated by usual Hecke operators: T_v, R_v and R_v^{-1} which correspond to $U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v$, $U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix}$ and $U_v \begin{pmatrix} \varpi_v^{-1} & 0 \\ 0 & \varpi_v^{-1} \end{pmatrix}$ respectively, See Bump [3]. The action of R_v on W_v is trivial and $\mathrm{T}_v \cdot W_v = \lambda_v |\varpi_v|^{\frac{1-k}{2}} W_v$. Hence $\mathrm{T}(\varpi_v^{-j}) \cdot W_v = Q'_{1,v}(\lambda_v) W_v$, where $Q'_{1,v}(t)$ is a polynomial independent of π_v. In fact, one has explicit relation

$$\mathrm{T}(\varpi_v^j) = \mathrm{T}_v(\varpi_v^{j+1}) + |\varpi_v|^{-1} \mathrm{R}_v \mathrm{T}(\varpi_v^{j-1}).$$

Thus $Q'_{1,v}(0)$ is a constant depending only on χ_v. Finally we may simply take $Q_{1,v}(t) = {Q'_{1,v}}^{-1}(0)Q'_{1,v}(t)$.

(2) $\widehat{\phi}^{\#}(1)$. Applying the definition of $\widehat{\phi^{\#}}(1)$ directly, we have

$$\widehat{\phi}^{\#}(1) = \frac{W_{\phi^{\#}}\begin{pmatrix} y_\infty \delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}}{W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= \frac{\prod_v W_v^{\#}\begin{pmatrix} y_\infty \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}}{W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= \frac{\prod_{v\nmid D} W_v^{\#}\begin{pmatrix} y_\infty \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \prod_{v\mid D} W_v^{\#}\begin{pmatrix} y_\infty \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}}{W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= \frac{\prod_{v\nmid D} W_v\begin{pmatrix} y_\infty \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} \prod_{v\mid D} \sum_{i=0}^{\mathrm{ord}_v(D)} \alpha_{v,i} W_{v,i}\begin{pmatrix} y_\infty \delta_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}}{W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}}.$$

Where the last equality is obtained, since for any $a|D$, $\phi_a := \pi \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi$ is an old form, therefore $\phi_a(1) = 0$.

(3) $\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2}$. To show that the above quantity has the desired property, one needs to analyze both $\widetilde{\phi}$ and ϕ_χ. First notice that the point P_χ corresponds to the following CM-cycle $\xi : T(\mathbb{A}_f)i_cU, ti_cu \mapsto \chi(t)$. Thus one sees that $(\widetilde{\phi}, P_\chi) = (\widetilde{\phi}, \xi)$, where the right hand side of the equality is regarded as the pairing between a form on $G(F)\backslash G(\mathbb{A})$ and a CM-cycle. So the idea is to compare both CM-cycles ξ and η, more precisely, there exists $h : U \to \mathbb{C}^\times$, such that

$$\rho(h)\eta = \zeta,$$

and then we replace η by ζ in the decomposition of the geometric pairing $\Phi(\zeta, \zeta)$ to deduce the desired property. Let ξ be the function on $T(\mathbb{A}_f)i_c$,

$$\xi : T(\mathbb{A}_f)i_cU \to \mathbb{C}^\times$$

$$ti_cu \mapsto \chi(t), \forall t \in T(\mathbb{A}_f), u \in U.$$

We first show that $(\widetilde{\phi}, P_c)_\chi = (\widetilde{\phi}, \xi)$. We have

$$\begin{aligned}(\widetilde{\phi}, \xi) =& \mathrm{vol}(U)^{-1} \int_{T(F)\backslash G(\mathbb{A}_f)} \overline{\xi}(x)\widetilde{\phi}(x)dx \\ =& \mathrm{vol}(U)^{-1} \int_{T(F)\backslash T(\mathbb{A}_f)i_cU} \overline{\xi}(x)\widetilde{\phi}(x)dx \\ =& \int_{T(F)\backslash T(\mathbb{A}_f)/U_T} \chi^{-1}(t)\widetilde{\phi}(tP_c) \\ =& (\widetilde{\phi}, P_c)_\chi.\end{aligned}$$

Now we compare the CM-cycles ξ and η. Let h be the characteristic function on Ui_c^{-1}. We claim that

$$\rho(h)\eta = \zeta.$$

This is a local problem. For any finite place v, let h_v be $\mathrm{vol}(U_v)$ times the characteristic function on $U_v i_{c,v}^{-1}$. We want to verify that $\rho(h_v)\eta_v = \zeta_v$.

If B is ramified at v, then $v|N$, thus $v \nmid c(\chi)$. We shall determine $i_{c,v}$. Recall that $i_{c,v}$ is an element of $G(F_v)$ such that

$$i_{c,v} U_v i_{c,v}^{-1} \cap T(F_v) = \mathcal{O}_{c_v}^{\times}/\mathcal{O}_v^{\times}.$$

Since $v \nmid c(\chi)$, thus $\mathcal{O}_{c_v} = \mathcal{O}_v$. Therefore we may take $i_{c,v} = I$. Hence if

$$\begin{aligned}
\rho(h_v)\eta_v(g) &= \frac{1}{\mathrm{vol}(U_v)} \int_{G(F_v)} \eta_v(gx) h_v dx \\
&= \frac{1}{\mathrm{vol}(U_v)} \int_{U_v} \eta_v(gui_{c,v}) du \\
&= \frac{1}{\mathrm{vol}(U_v)} \int_{U_v} \eta_v(gu) du \neq 0,
\end{aligned}$$

then $gu \in T(F_v)\Delta_v$, hence $g \in T(F_v)\Delta_v = T(F_v)U_v$, since $\Delta_v = U_v$, for $v \nmid c(\chi)$, i.e., $\rho(h_v)\eta_v$ is supported on $T(F_v)U_v$. For any $g = tu \in T(F_v)U_v$, one has

$$\begin{aligned}
\rho(h_v)\eta_v(g) &= \frac{1}{\mathrm{vol}(U_v)} \int_{U_v} \eta_v(gx) dx \\
&= \frac{1}{\mathrm{vol}(U_v)} \int_{U_v} \eta_v(tux) dx \\
&= \frac{1}{\mathrm{vol}(U_v)} \chi_v(t) \int_{U_v} \chi_v(x) dx \\
&= \chi_v(t),
\end{aligned}$$

i.e., $\rho(h_v)\eta_v = \xi_v$.

If B splits at v. Let $U'_v = \mathcal{O}_{c_v} + c_v\lambda_v\mathcal{O}_{K,v}$. If $\rho(h_v)\eta_v(g) = \int_{U_v}\eta_v(gxi_{c,v})dx \neq 0$. Then one has $gxi_{c,v} \in T(F_v)U'_v$, i.e., $g \in T(F_v)\Delta_v i_{c,v}$. Using the lemma in Zhang̃citeZh2, we see that $i_{c,v}U_v = U'_v i_{c,v}$. Therefore $g \in T(F_v)i_{c,v}^{-1}U_v$. Now for any $g = ti_{c,v}^{-1}u \in T(F_v)i_{c,v}^{-1}U_v$,

$$\begin{aligned}\rho(h_v)\eta_v(g) &= \frac{1}{\mathrm{vol}(U'_v)}\int_{U_v}\eta_v(ti_{c,v}^{-1}uxi_{c,v})dx\\ &= \frac{1}{\mathrm{vol}(U'_v)}\int_{U_v}\eta_v(ti_{c,v}^{-1}xi_{c,v})dx\\ &= \int_{U'_v}\eta_v(tu)dx = \chi_v(t),\end{aligned}$$

i.e., $\rho(h_v)\eta_v = \zeta_v$.

We prove the final step now. Recall that we have defined a geometric pairing $\Psi(\alpha,\beta)$ associated to two CM-cycles α and β and have obtained the spectral decomposition of $\Psi(\alpha,\beta)(g_\infty)$ with g_∞ a continuous parameter varying in $\mathrm{GL}_2(F_\infty)$. We have

$$\begin{aligned}\Psi(\alpha,\beta) =& C|c(\omega)|^{\frac{1}{2}}\sum_i \phi_i^{\mathrm{new}}(\phi_i,\overline{\alpha})_\Delta(\overline{\phi_i},\beta)_\Delta\\ &+ C|c(\omega)|^{\frac{1}{2}}\int_{\mathcal{M}} E_{\mathcal{M}}^{\mathrm{new}}(E_{\mathcal{M}},\overline{\alpha})(\overline{E_{\mathcal{M}}},\beta)d\mathcal{M}.\end{aligned}$$

In particular, let $\alpha = \beta = \xi$, we see that

$$\begin{aligned}\Psi(\xi,\xi) =& C|c(\omega)|^{\frac{1}{2}}\sum_i \phi_i^{\mathrm{new}}|(\phi_i,\overline{\xi})_\Delta|^2\\ &+ C|c(\omega)|^{\frac{1}{2}}\int_{\mathcal{M}} E_{\mathcal{M}}^{\mathrm{new}}|(E_{\mathcal{M}},\overline{\xi})|^2.\end{aligned}$$

Since ξ is fixed by U, we may assume that ϕ_i is fixed by U as well. For ϕ_i such that the representation generated by ϕ_i corresponds to ϕ via Jacquet-Langlands correspondence, this ϕ_i must be test form $\widetilde{\phi}$. Now write $H(\xi,\xi) = \sum \phi_i^{\mathrm{new}} |(\phi_i,\xi)_\Delta|^2 + \text{continuous} \quad \text{spectrum}$. Using the relation

$$|a| < T_a\alpha, \beta > (g_\infty) = CW_{H(\alpha,\beta)}\left(g_\infty \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and simple fact that $< \alpha, \rho(h)\beta >=< \rho(h^\vee)\alpha, \beta >$, it follows that

$$H(\xi,\xi) = H(\rho(h^\vee * h)\eta, \eta).$$

By the decomposition $S(T(F)\backslash G(\mathbb{A}_f)) = \oplus_\tau S(T(F)\backslash G(\mathbb{A}_f), \tau)$, one has

$$h^\vee * h = \sum_\tau h_\tau,$$

where τ runs through all characters of $T(\mathbb{A}_f)$. If $\tau \neq \chi$, then it is easy to see that

$$< \rho(h_\tau)\eta, \eta >= 0,$$

since η has character χ under the action of $T(\mathbb{A}_f)$. Therefore, we obtain

$$\begin{aligned} H(\rho(h^\vee * h)\eta, \eta) &= H(\rho(h_\chi)\eta, \eta) \\ &= \sum_i \phi_i^{\mathrm{new}} (\phi_i, \overline{\rho(h_\chi)\eta})(\overline{\phi_i}, \eta) + \text{continuous} \quad \text{spectrum} \\ &= \sum_i \phi_i^{\mathrm{new}} (\rho(h_\chi)\phi_i, \overline{\eta})(\overline{\phi_i}, \eta) + \text{continuous} \quad \text{spectrum}. \end{aligned}$$

Once again, observing that η has character χ under the action of $T(\mathbb{A})$. We may assume that ϕ_i has same character χ under the action of $T(\mathbb{A})$. Thus ϕ_i

is exactly the toric newform ϕ_χ, if $\phi_i^{\mathrm{new}} = \phi$. Now we claim that

$$\rho(h_\chi)\phi_i = \prod_{v|D} Q_{2,v}(\lambda_v)\phi_i,$$

where $Q_{2,v}$ is certain rational function.

To that end, notice that for $v \nmid D = c(\chi)^2 c(\epsilon)$, one has $\mathcal{O}_{c_v} = \mathcal{O}_{K,v}$, hence we may take $i_{c,v} = I$, then $h^\vee * h$ is the characteristic function of U_v, which may be regarded as the identity element in the algebra $\mathcal{H}_{U_v}$ of bi-U_v invariant functions on $G(F_v)$. Hence the action is trivial.

If $v|D$, it is easy to show that

$$h_v^\vee * h_v \in \mathcal{H}_{U_v}.$$

Thus

$$\rho(h_\chi)\phi_i = \prod_{v|D} Q_{2,v}(\lambda_v)\phi_i,$$

here we use the explicit construction that $\pi_v = \pi_v'$ if B is ramified at v. Therefore finally we obtain that

$$\frac{|(\widetilde{\phi}, P_\chi)|^2}{|(\phi_\chi, \eta)|^2} = C \prod_{v|D} Q_v(\lambda_v).$$

□

4.2. **Determination of universal constants.** We shall use the continuous spectrum to compute the universal constants. First we deduce a similar special value formula for the L-function $L(s, E, \chi)$ associated to a continuous family E

and character χ. The explicit construction of E allows us to compute the universal constants and finally show that the product of these universal constants is 1. Hence the formula of level N is proved.

4.2.1. *Special value formula of* $L(s, E, \chi)$. we now deduce a formula for $L(s, E, \chi)$ associated to an form $E \in L^2_{cont}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F))$ and χ in terms of quasi-newform $E^{\#}$. The idea follows exactly as before. For a fixed character $\mu : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, we shall use the two forms Φ and Ψ constructed in [Zh1]:

(1) the form Φ is the holomorphic projection of $\overline{\Theta}_{\frac{1}{2}}$;
(2) the form Ψ comes from the spectral decomposition of certain geometric pairing.

The difference Φ - Ψ is an old form via local Gross-Zagier formula. So is the projection of Φ - Ψ on the space $\mathrm{Eis}(\mu)$, a subspace of $L^2_{cont}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F))$ chosen appropriately, from which the desired formula follows.

Let's fix a character μ of $\mathbb{A}_F^\times/F^\times$ such that μ^2 is not of form $|\cdot|^t$, for some $0 \neq t \in \mathbb{R}$. Let $\mathrm{Eis}(\mu)$ be the space of forms E in $L^2_{cont}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}))$ so that

$$E(g) = \int_{-\infty}^{\infty} E_t(g)dt,$$

with $E_t(g) := E(it, g)$ certain Eisenstein series in $\pi_t := \pi(\mu|\cdot|^{it}, \mu^{-1}|\cdot|^{-it}), \forall t \in \mathbb{R}$. For two such forms E_1 and E_2, one has inner product

$$(E_1, E_2) = \int_{-\infty}^{\infty} (E_{1,t}, E_{2,t})_t dt,$$

where $(,)_t$ is some Hermitian form on the space $\pi(\mu|\cdot|^{it}, \mu^{-1}|\cdot|^{-it})$.

For any form $E(g) = \int_{-\infty}^{\infty} E_t(g)dt \in \mathrm{Eis}(\mu)$, and a continuous function φ on $\mathbb{R}$, one obtains another form E_φ twisted by φ by

$$E_\varphi(g) = \int_{-\infty}^{\infty} \varphi(t)E_t(g)dt.$$

In particular, let E_t^{new} be the newform of the representation $\pi(\mu|\cdot|^{it}, \mu^{-1}|\cdot|^{-it})$. One has

$$E_\varphi^{\mathrm{new}}(g) = \int_{-\infty}^{\infty} \varphi(t)E_t^{\mathrm{new}}(g)dt.$$

We now compute the inner product $(E_\varphi^{\mathrm{new}}, \Phi_s)$. Assume that χ is not of the form $\iota \cdot \mathrm{N}_{K/F}$, thus $\theta = \theta_\chi$ is a cusp form and the kernel function Φ_s is square-integrable since its constant term has exponential decay. Hence we have

$$\begin{aligned}
(E_\varphi^{\mathrm{new}}, \Phi_s) = (E_\varphi^{\mathrm{new}}, \overline{\Theta}_s) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} E_\varphi^{\mathrm{new}} \Theta_s(g)dg \\
&= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \int_{-\infty}^{\infty} \varphi(t)E_\varphi^{\mathrm{new}}(it, g)\Theta_s(g)dtdg \\
&= \int_{-\infty}^{\infty} \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(t)E_\varphi^{\mathrm{new}}(it, g)\Theta_s(g)dgdt \\
&= \int_{-\infty}^{\infty} \varphi(t)L(s, \pi_t, \chi)dt.
\end{aligned}$$

We need to compute $L_v(s, \pi_t, \chi)$. One has

$$\begin{aligned}
L_v(s, \pi_t, \chi) &= \prod_{w|v}(1 - \mu(\varpi_v)\chi(\varpi_w)|\varpi_v|^{s+it})^{-1} \cdot \prod_{w|v}(1 - \mu^{-1}(\varpi_v)\chi(\varpi_w)|\varpi_v|^{s-it})^{-1} \\
&= L_v(s + it, \mu_K \otimes \chi) \cdot L_v(s - it, \mu_K^{-1} \otimes \chi),
\end{aligned}$$

here, $\mu_K = \mu \cdot \mathrm{N}_{K/F}$. Therefore one obtains

$$(E_\varphi^{\mathrm{new}}, \Phi_s) = \int_{-\infty}^{\infty} \varphi(t)L(s + it, \mu_K \otimes \chi) \cdot L(s - it, \mu_K^{-1} \otimes \chi).$$

In particular, for $s = \frac{1}{2}$, applying the fact that χ is of finite order as well as $\chi|_{\mathbb{A}^\times} \equiv 1$, one has

$$\begin{aligned} L(\frac{1}{2} - it, \mu_K \otimes \chi) &= \prod_w L_w(\frac{1}{2} - it, \mu_K \otimes \chi) \\ &= \prod_w (1 - \mu_K^{-1}(\varpi_w)\chi(\varpi_w)|\varpi_w|^{\frac{1}{2}-it})^{-1} \\ &= \overline{\prod_w (1 - \mu_K(\varpi_w^\sigma)\chi(\varpi_w^\sigma)|\varpi_w^\sigma|^{\frac{1}{2}+it})^{-1}} \\ &= \overline{L(\frac{1}{2} + it, \mu_K \otimes \chi)}, \end{aligned}$$

Here σ is the nontrivial automorphism in $\mathrm{Gal}(K/F)$. Hence

$$(E_\varphi^{\mathrm{new}}, \Phi) = \int_{-\infty}^{\infty} \varphi(t)|L(\frac{1}{2} + it, \mu_K \otimes \chi)|^2 dt.$$

Now we need to compute the projection of both Φ and Ψ on the space $\mathrm{Eis}(\mu)$. Let's start with the projection of Φ on $\mathrm{Eis}(\mu)$ first.

Recall that the projection of Φ on $\mathrm{Eis}(\mu)$ is the unique form $pr(\Phi) \in \mathrm{Eis}(\mu)$, satisfying

$$(E, pr(\Phi)) = (E, \Phi), \forall E \in \mathrm{Eis}(\mu).$$

And the quasi-newform $E_t^{\#}$ of π_t with respect to χ is the form satisfying

$$(E_t^{\#}, E_{t,a}^{\mathrm{new}})_t = \nu(a)(E_t^{\#}, E_t^{\#})_t, \forall a|D,$$

where $E_{t,a}^{\mathrm{new}} = \rho \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} E_t^{\mathrm{new}}$. We have

Lemma 4.2.1.1 *The projection* $pr(\Phi)$ *of* Φ *on the space* $\mathrm{Eis}(\mu)$ *is* $E_{\varphi}^{\#}$ *with*

$$\varphi(t) = \frac{|L(\frac{1}{2}+it, \mu_K \otimes \chi)|^2}{||E_t^{\#}||_t^2}.$$

Proof. We first compute $(E, \Phi), \forall E = \int_{-\infty}^{\infty} E_t dt \in \mathrm{Eis}(\mu)$. One has

$$\begin{aligned}(E, \Phi) = (E, \overline{\Theta}_{\frac{1}{2}}) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \int_{-\infty}^{\infty} E_t(g)\Theta_{\frac{1}{2}}(g) dt dg \\ &= \int_{-\infty}^{\infty} E_t(g)\Theta_{\frac{1}{2}}(g) dg dt.\end{aligned}$$

Thus the linear map $E_t \mapsto \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} E_t(g)\Theta_{\frac{1}{2}}(g) dg$ is well defined, and by Riesz representation theorem, there exists $E_t^{\Theta} \in \pi_t$ such that

$$|L(\frac{1}{2}+it, \mu_K \otimes \chi)|^2 = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} E_t(g)\Theta_{\frac{1}{2}}(g) dg = (E_t, E_t^{\Theta}).$$

One easily sees that

$$E_t^{\Theta} = \frac{|L(\frac{1}{2}+it, \mu_K \otimes \chi)|^2}{||E_t^{\#}||_t^2} E_t^{\#}.$$

Thus

$$(E, \Phi) = \int_{-\infty}^{\infty} \varphi(t)(E_t, E_t^{\#})_t dt = (E, E_{\varphi}^{\#}),$$

i.e., $E_t^{\#} = pr(\Phi)$. $\square$

The projection $pr(\Psi)$ of Ψ on $\mathrm{Eis}(\mu)$ is easier. It's the continuous contribution for $\mathrm{Eis}(\mu)$ in Ψ. One has

$$pr(\Psi) = 2^{2g} |c(\omega)|^{\frac{1}{2}} E_{\psi}^{\mathrm{new}}$$

with $\psi(t) = |(E_{t,\chi}, \eta)_\Delta|^2$. By local Gross-Zagier formula, $\Phi - \Psi$ is an old form. Thus $pr(\Phi) - pr(\Psi)$ is also an old form. Hence

$$\varphi(t) E_t^{\#} - 2^{2g} |c(\omega)|^{\frac{1}{2}} \psi(t) E_t^{\mathrm{new}}$$

is an old form too. Therefore we obtain

Proposition 4.2.1.2 *Assume that χ is not of the form $\iota \cdot \mathrm{N}_{K/F}$ with ι a character of $\mathbb{A}^\times / F^\times$. Then*

$$\widehat{E_t^{\#}}(1) |L(\frac{1}{2} + it, \mu_K \otimes \chi)|^2 = 2^{2g} |c(\varepsilon)|^{\frac{1}{2}} ||E_t^{\#}||_t^2 |(E_{t,\chi}^{\mathrm{new}}, \eta)_\Delta|^2. \tag{4.1}$$

Rewrite the formula (4.1) in the following way:

$$\frac{\widehat{E_t^{\#}}(1)}{\mathrm{N}(c(\chi))} \cdot \frac{||E_t^{\mathrm{new}}||_t^2}{||E_t^{\#}||_t^2} \cdot \frac{|(\widetilde{E_t^{\mathrm{new}}}, P_\chi)|^2}{|(E_{t,\chi}^{\mathrm{new}}, \eta)_\Delta|^2} = \frac{2^2 g}{\sqrt{\mathrm{N}(D)}} \frac{||E_t^{\mathrm{new}}||_t^2 |(\widetilde{E_t^{\mathrm{new}}}, P_\chi)|^2}{|L(\frac{1}{2} + it, \mu_K \otimes \chi)|^2}.$$

Then the universality of the functions Q_v implies

Proposition 4.2.1.3 *Assume that χ is not of form $\iota \cdot \mathrm{N}_{K/F}$ with ι a character of $\mathbb{A}^\times / F^\times$, let*

$$\lambda_v(t) = \mu_v(\varpi_v) |\varpi_v|^{it} + \mu_v^{-1}(\varpi_v) |\varpi_v|^{-it},$$

and $E_t^* := ||E_t^{\mathrm{new}}||_t \widetilde{E_t^{\mathrm{new}}}$. Then

$$C(\chi) \prod_{v|D} Q_v(\lambda_v(t)) = \frac{2^{2g}}{\sqrt{\mathrm{N}(D)}} \left| \frac{(E_t^*, P_\chi)}{L(\frac{1}{2} + it, \mu_K \otimes \chi)} \right|^2.$$

4.2.2. *Determination of universal constants.* We shall finally compute the universal constants. We need to compute the pairing $|(E_t^*, P_\chi)|^2$. By the universality of the constants, we may assume that the character μ is unramified, thus the quaternion algebra B splits everywhere, i.e., $B = \mathrm{M}_2(F)$. We first simplify the form E_t^*.

Lemma 4.2.2.1 *Let $j = j_f \otimes \prod_{v|\infty} j_v \in G(\mathbb{A})$ such that $j_f \in G(\mathbb{A}_f)$ with $j_f^{-1} U(N,K) j_f = \mathrm{GL}_2(\widehat{\mathrm{O}}_F)$, and for $v|\infty$, $j_v \in G(F_v)$ with*

$$j_v(\mathrm{SO}_2(F_v)/\left\{\pm I\right\}) j_v^{-1} = T(F_v).$$

Then we have

$$E_t^* = \rho(j) E_t^{\mathrm{new}}.$$

Proof. Recall that the test form $\widetilde{E_t^{\mathrm{new}}}$ is a form fixed by U and normalized such that $||E_t^{\mathrm{new}}||^2 = 1$. We show first that $\rho(j)E_t^{\mathrm{new}}$ is fixed by $U(N,K)$. For any $h \in U(N,K)$,

$$\begin{aligned}\rho(h)(\rho(j)E_t^{\mathrm{new}})(g) &= E_t^{\mathrm{new}}(ghj)\\ &= E_t^{\mathrm{new}}(gjj^{-1}hj)\\ &= \rho(j)E_t^{\mathrm{new}}(g).\end{aligned}$$

Since the dimension of the forms fixed by $U(N,K)$ is one, so

$$\frac{\rho(j)E_t^{\mathrm{new}}}{||\rho(j)E_t^{\mathrm{new}}||^2} = \pm\widetilde{E_t^{\mathrm{new}}}.$$

We may assume that

$$\frac{\rho(j)E_t^{\mathrm{new}}}{||\rho(j)E_t^{\mathrm{new}}||^2} = \widetilde{E_t^{\mathrm{new}}},$$

i.e.,

$$\rho(j)E_t^{\mathrm{new}} = ||\rho(j)E_t^{\mathrm{new}}||^2\widetilde{E_t^{\mathrm{new}}} = ||E_t^{\mathrm{new}}||^2\widetilde{E_t^{\mathrm{new}}},$$

since the representation π is hermitian and we only need to compute $|(E_t^*, P_\chi)|^2$.

□

Proposition 4.2.2.2 *Assume that the character μ is unramified and χ is not of the form $\mu_K := \mu \circ \mathrm{N_{K/F}}$, then*

$$(E_t^*, P_\chi) = 2^{-g}\mu(\delta^{-1}\sqrt{\frac{4\lambda}{D}})|4\lambda/D|^{\frac{1}{4}(1+2it)}2^{it}L(\frac{1}{2}, \overline{\chi}\cdot\mu_K),$$

where λ is a trace-free element of K.

Proof. Recall that $P_\chi = \sum_{\sigma\in\mathrm{Gal}(H_K/K)} \chi^{-1}(\sigma)P_c^\sigma$. We have

$$\begin{aligned}(E_t^*, P_\chi) &= \int_{T(F)\backslash T(\mathbb{A})} \chi^{-1}(s)E_t^*(sP_c)ds \\ &= \int_{T(F)\backslash T(\mathbb{A})} \chi^{-1}(s)E_t^{\mathrm{new}}(sP_cj)ds.\end{aligned}$$

here we choose the Haar measure $ds = \otimes ds_v$ such that if v is non-archimedean, then $\mathrm{vol}(\mathcal{O}_{c_v}) = 1$; if v is archimedean, then $\mathrm{vol}(T(F_v)) = 1$. Recall that

$$E_t^{\mathrm{new}}(g) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \mu^{-1}(\delta) f_{\Omega,t}(\gamma s P_c j),$$

where

$$f_{\Omega,t}(g) = \mu_1(\det g)|\det g|^{t+\frac{1}{2}} \int_{\mathbb{A}^\times} \Omega[(0,x)g]\mu_1\mu_2^{-1}(x)|x|^{1+2t} d^\times t$$

with Ω a Schwartz-Bruhat function on $\mathbb{A}^2$ and μ_i being character of $\mathbb{A}^\times$, $i = 1, 2$. In particular, we choose the Schwartz-Bruhat function $\Omega = \otimes \Omega_v$, such that for v non-archimedean, Ω_v is the characteristic function of the set $\mathcal{O}_v^2$, for v archimedean, $\Omega_v = e^{-\pi(x^2+y^2)}$. Therefore, we have

$$\begin{aligned}(E_t^*, P_\chi) = {} & \mu(\det(P_c j))|\det(P_c j)|^{\frac{1}{2}+it} \int_{T(\mathbb{A})} \int_{\mathbb{A}^\times} \chi^{-1}(s)\mu(\det s)|\det s|^{\frac{1}{2}+it} \\ & \cdot \Omega((0,x)sP_c j)\mu^2(x)|x|^{1+2it} d^\times x ds \\ = {} & \mu(\det(P_c j))|\det(P_c j)|^{\frac{1}{2}+it} Z(\frac{1}{2}, \mu \cdot |\cdot|^{it}, \Omega).\end{aligned}$$

So it suffices to compute the local zeta function $Z_v(s, \mu \cdot |\cdot|^{it}, \Omega)$ as well as $\det(P_c j)$. We start with the local zeta function.

Since the function Ω_v is supported on $\mathcal{O}_v \oplus \mathcal{O}_v$ for v non-archimedean, one easily sees that

$$Z(\frac{1}{2}, \mu \cdot |\cdot|^{it}, \Omega) = \int_{\mathcal{O}_{c_v} - 0} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it} d^\times x.$$

I. If v is non-archimedean place.

(1) If $v|c(\chi)$, then $\mathcal{O}_{c_v} = \mathcal{O}_v$. Thus

$$\begin{aligned} Z(\frac{1}{2}, \chi^{-1}\mu_{K_v}|\cdot|^{it}, \Omega) &= \int_{\mathcal{O}_v - 0} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it} d^\times x \\ &= L(\frac{1}{2}, \chi^{-1}\mu_{K_v}|\cdot|^{it}) \end{aligned}$$

(2) If $v \nmid c(\chi)$, write $\mathcal{O}_{c_v} - 0 = \bigcup_{n=0}^{\infty} \mathcal{O}_{c_v,n}$, where $\mathcal{O}_{c_v,n}$ is the set of elements of order n in $\mathcal{O}_{c_v}$.

i) If v is inert in K, then K_v is a field. Let's write $\mathcal{O}_{K_v} = \mathcal{O}_v + \mathcal{O}_v\lambda$. We have

$$\int_{\mathcal{O}_{c_v} - 0} \chi^{-1}\mu_K(x)|x|^{\frac{1}{2}+it} d^\times x = \sum_{n=0}^{\infty} \int_{\mathcal{O}_{c_v,n}} \chi^{-1}\mu_K(x)|x|^{\frac{1}{2}+it} d^\times x.$$

Case 1. $n = 0$, then $\mathcal{O}_{C_v,0} = \mathcal{O}_{c_v}^\times$. We obtain

$$\int_{\mathcal{O}_{c_v,0}} \chi^{-1}(x)\mu_K(x)|x|^{\frac{1}{2}+it} = \mathrm{vol}(\mathcal{O}_v^\times) = 1.$$

Case 2. $0 < n < \mathrm{ord}_v(c(\chi))$, It's easy to see that

$$\mathcal{O}_{c_v,n} = \varpi_v^n \mathcal{O}_v^\times + c_v \mathcal{O}_{K_v} = \varpi_v^n(\mathcal{O}_v^\times + c_v \varpi_v^{-n} \mathcal{O}_{K_v}).$$

Hence

$$\int_{\mathcal{O}_{c_v,n}} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it} d^\times x = 0,$$

since χ restricted to $\mathcal{O}_v^\times + c_v\varpi^{-n}\mathcal{O}_{K,v}$ is not identically one.

Case 3. $n \geq \mathrm{ord}_v(c_v)$. One sees easily that $\mathcal{O}_{c_v,n} = \varpi_v^n \mathcal{O}_{K_v}^\times$. Now applying $\chi\big|_{\mathcal{O}_{K_v}^\times} \equiv 1$ and μ being unramified, we have

$$\int_{\mathcal{O}_{c_v,n}} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it} d^\times x = 0.$$

Hence $Z(\frac{1}{2}, \chi^{-1}\mu_{K_v}|\cdot|^{it}, \Omega_v) = 1$, for $v|c(\chi)$ and v is inert in K.

ii) If v splits in K. Then $K_v = F_v \oplus F_v$, thus $\mathcal{O}_{K_v} = \mathcal{O}_v \oplus \mathcal{O}_v$.

Case 1. $n = 0$, similarly one has

$$\int_{\mathcal{O}_{c_v}-0} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it}d^\times x = \mathrm{vol}(\mathcal{O}_{c_v}^\times) = 1.$$

Case 2. $0 < n < \mathrm{ord}_v(c(\chi))$, then $\mathcal{O}_{c_v,n} = \varpi_v^n\mathcal{O}_v^\times + c_v\mathcal{O}_{K_v} = \varpi_v^n(\mathcal{O}_v^\times + c_v\varpi_v^{-n}\mathcal{O}_{K_v})$. Hence

$$\int_{\mathcal{O}_{c_v}-0} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it}d^\times x = 0,$$

since $\chi\big|_{\mathcal{O}_v^\times + c_v\varpi_v^{-n}\mathcal{O}_{K_v}} \not\equiv 1$.

Case 3. $n \geq \mathrm{ord}_v(c(\chi))$. The fact that $\mathcal{O}_{c_v,n} = \varpi_v^n\mathcal{O}_{K_v}^\times$, and $\chi\big|_{\mathcal{O}_{K_v}^\times} \not\equiv 1$ implies that

$$\int_{\mathcal{O}_{c_v,n}} \chi^{-1}(x)\mu_{K_v}(x)|x|^{\frac{1}{2}+it}d^\times x = 0.$$

Again we obtain that

$$Z(\frac{1}{2}, \chi^{-1}\mu_{K_v}|\cdot|^{it}, \Omega_v) = 1.$$

II. If v is archimedean place.

Recall that $j_v \in \mathrm{GL}_2(\mathbb{R})$ such that $j_v(\mathrm{SO}_2(\mathbb{R})/\{\pm I\})j_v^{-1} = T(\mathbb{R})$. Writing $K = F + F\sqrt{\lambda}$, one may simply take $j_v = \begin{pmatrix} |\lambda|_v^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$. Using the polar coordinate (r,θ), the measure on $\mathbb{C}^\times$ induced from the standard measure on $\mathbb{R}^\times$ and the measure on $\mathbb{C}^\times/\mathbb{R}^\times$ such that the volume of $\mathbb{C}^\times/\mathbb{R}^\times$ is one has the express $\frac{drd\theta}{\pi r}$.

Then the function $\Omega_v((0,x)sj_v)$ is of the form $e^{-\pi r^2}$. Therefore we get

$$\begin{aligned} Z(\frac{1}{2},\chi^{-1}\mu_{K_v}|\cdot|^{it}) &= \int_0^{2\pi}\int_0^{\infty}\mu(r)|r|^{1+2it}e^{-\pi r^2}\frac{drd\theta}{\pi r} \\ &= 2\int_0^{\infty}\mu(r)|r|^{1+2it}e^{-\pi r^2}\frac{dr}{r} \\ &= \frac{1}{2}\mu(\pi)^{-1}\pi^{-\frac{1}{2}(1+2it)}\Gamma(\frac{1+2it}{2}). \end{aligned}$$

Finally We compute the $\det(j_fP_c)$ as well as $\det(j_v)$ for v archimedean. By the definition of j_f and P_c, one sees easily that

$$\mathcal{O}_v+\mathcal{O}_v\sqrt{\lambda}=\mathcal{O}_{c_v}j_fP_c.$$

Hence taking discriminant both sides, one obtains

$$4\lambda = D\cdot\det(j_fP_c).$$

For $\det(j_v)$, one has $\det(j_v)=|\lambda|_v^{\frac{1}{2}}$ by the specific form we have chosen.

Summing up, We have the following formula:

$$(E_t^*,P_\chi)=\mu\left(\delta^{-1}\sqrt{(\frac{4\lambda}{D})}\right)|\frac{4\lambda}{D}|^{\frac{1}{4}(1+2it)}2^{-g}\cdot 2^{it}L(\frac{1}{2},\chi^{-1}\cdot\mu_K\cdot|\cdot|^{it}).$$

□

We now compute the universal constants using the formula just obtained. In particular we choose $\mu(x)=|x|^{is}$. Observe that

$$|L(\frac{1}{2}+it+is,\chi^{-1})|^2=|L(\frac{1}{2}+it+is,\chi)|^2,$$

since $L(\frac{1}{2}+it+is,\chi^{-1}) = L(\frac{1}{2}-it-is,\chi^{-1})$ and $|L(\frac{1}{2}-it-is,\chi^{-1})| = |L(\frac{1}{2}+it+is,\chi)|$ by the functional equation of $L(\frac{1}{2}+it+is,\chi^{-1})$. Applying the formula in Proposition 4.2.1.3, we have

$$C(\chi)\prod_{v|D} Q_v(\lambda_v(t)) = 1.$$

Observe that each $Q_v(\lambda(t))$ is a rational function in p^{nt}. One can show that

$$\prod_{v|D_p} Q_v(\lambda(t)) = \text{const},$$

where D_p is the set of places v dividing D and lying over p. We claim that the constant is one. Thus end the proof of special value formula of level N.

To that end, it's known ([1]) that there exists a character χ' of finite order of $\mathbb{A}_K^\times/K^\times\mathbb{A}^\times$ such that χ' satisfies all properties that χ has except that χ' is unramified at w lying over p but ramified at all other places $w|c(\chi)$. Applying the above argument to this χ', one obtains that

$$\prod_{v|D_p} Q_v(\lambda(t)) = 1.$$

Hence we have

Theorem 4.2.2.3 *The constant $C(\chi) = 1$ and the polynomial $Q_v(\lambda(t)) = 1$, for $v|D$.*

5. **Appendix. Continuous spectrum of $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$**

We follow the notations in Gelbart and Jacquet [6]. Let Z be the center of GL_2 and

$$P=\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, A=\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, N=\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

If ω is a (unitary) character of $\mathbb{A}^\times/F^\times$, we denote by $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ the space of the functions φ on $\mathrm{GL}_2(\mathbb{A})$ such that

$$\varphi(\gamma z g)=\omega(z)\varphi(g), \forall\gamma\in\mathrm{GL}_2(F), z\in Z(\mathbb{A}),$$

and

$$\int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}|\varphi(g)|^2dg<\infty.$$

If, in addition,

$$\int_{N(F)\backslash N(\mathbb{A})}\varphi(ng)\equiv 0, \forall g\in\mathrm{GL}_2(\mathbb{A}),$$

then we say that φ is a *cuspidal* and the subspace of cuspidal functions is denoted by $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$. Let $\rho_\omega(\rho_{\omega,0})$ be the natural representation of $\mathrm{GL}_2(\mathbb{A})$ in $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)(L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega))$ via the right translation.

The representation $\rho_{\omega,0}$ decomposes discretely with finite multiplicity. For details, see Bump[1].

In this section, we would like to give a description of the orthocomplement of $L_0^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ inside $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ equipped with the inner product

$$(\varphi_1, \varphi_2) = \int_{\mathrm{GL}_2(F)Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}dg.$$

The orthocomplement can be described in terms of P-series, which we now define.

Definition 5.1 *If f is a function in $C^\infty(N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A}))$ such that*

$$f(zg) = \omega(z)f(g), z \in Z(\mathbb{A}). \tag{5.1.1}$$

Then the series

$$F(g) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} f(\gamma g)$$

is called a P-series.

One can show ([6], P. 197) that if the function f is compactly supported mod $N(\mathbb{A})Z(\mathbb{A})P(F)$, then the P-series is convergent. So in the rest of the notes, we assume that function f is compact mod $N(\mathbb{A})Z(\mathbb{A})P(F)$.

We shall prove that the space of P-series is a dense subset of the space

$$L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F), \omega)^\perp.$$

We need to compute the inner product of a P-series and a cuspidal function φ. Let φ be a function in $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)$, then

$$\begin{aligned}(\varphi, F) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g)\overline{F(g)}dg \\ &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g) \sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)} \overline{f(\gamma g)}dg \\ &= \int_{Z(\mathbb{A})P(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g)\overline{f(g)}dg \\ &= \int_{N(\mathbb{A})Z(\mathbb{A})P(F)\backslash\mathrm{GL}_2(\mathbb{A})} dg \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\overline{f(g)}dn.\end{aligned}$$

Notice the $\int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn$ is the constant term of φ. Thus if φ is a cuspidal form, then $(\varphi, F) = 0$. Therefore $F \in L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)^{\perp}$. Conversely let φ be any form in $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)$, if

$$(\varphi, F) = \int_{N(\mathbb{A})Z(\mathbb{A})P(F)\backslash\mathrm{GL}_2(\mathbb{A})} dg \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\overline{f(g)}dn = 0$$

for any P-series with compact support mod $N(\mathbb{A})Z(\mathbb{A})P(F)$, then it is easy to see that the constant term $\int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)dn$ of φ is 0, i.e., φ is cuspidal. Thus P-series form a dense subset of $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)^{\perp}$. For our purposes, we want to express P-series as continuous sums of Eisenstein series. To define Eisenstein series, let's introduce a Hilbert space $\mathbf{H}$(s). Let F_∞^+ be the set of ideles whose finite components are all 1 and whose infinite components all equal some positive number (independent of infinite place) and $\mathbb{A}_{F,1}^\times$ be the ideles of norm 1. One has $\mathbb{A}_F^\times/F^\times \cong \mathbb{A}_{F,1}^\times/F^\times \times F_\infty^+$.

Definition 5.3 ***H***(s)*is the space of functions* $\phi \in C^{\infty}(\mathrm{GL}_2(\mathbb{A}))$ *such that*

$$\phi\left(\left(\begin{array}{cc}\alpha a u & x \\ 0 & \beta a v\end{array}\right)\right)=\omega(a)\left|\frac{u}{v}\right|^{s+\frac{1}{2}} \phi(g),$$

and

$$\int_K \int_{F^{\times} \backslash \mathbb{A}_{F,1}^{\times}}|\phi|^2\left(\left(\begin{array}{cc}a & 0 \\ 0 & 1\end{array}\right) K\right) d a d k<\infty,$$

where $\alpha, \beta \in F^{\times}, a \in \mathbb{A}^{\times}, u, v \in F_{\infty}^{+}$.

The group $\mathrm{GL}_2(\mathbb{A})$ operates on **H**(s) via the right translation and the resulting representation is denoted by π_s. The representation π_s is unitary if s is purely imaginary. One may view **H**(s) as a trivial fibre bundle of base $\mathbb{C}$. For any open subset U of $\mathbb{C}$, the sections are functions $\phi(g, s)$ on $\mathrm{GL}_2(\mathbb{A}) \times U$ such that

$$\phi\left(\left(\begin{array}{cc}\alpha a u & x \\ 0 & \beta a v\end{array}\right) g\right)=\omega(a)\left|\frac{u}{v}\right|^{s+\frac{1}{2}} \phi(g, s).$$

Now we can define the Eisenstein series associated to a section of the trivial fibre bundle **H**(s).

Definition 5.4. *For a section* ϕ *of the trivial fibre bundle* ***H***(s)*, the series*

$$E(\phi(s), g)=\sum_{\gamma \in P(F) \backslash \mathrm{GL}_2(F)} \phi(s, \gamma g)$$

are called Eisenstein series.

This series converges only for $\mathrm{Re}(s)>\frac{1}{2}$. It can be shown (čiteG-J, § 5) that the Eisenstein series $E(\phi(s), g)$ can be analytically continued to the region for which $\mathrm{Re}(s) \geq 0$.

To explain the relationship between P-series and Eisenstein series, we now define the Fourier-Laplace transform of a function. For a function f satisfying (5.1.1), one defines the Fourier-Laplace transform of f by

$$\widehat{f}(g,s) = \int_{F_\infty^+} f\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right) |t|^{-s-\frac{1}{2}} d^\times t.$$

By our assumption that f is compactly supported mod $N(\mathbb{A})Z(\mathbb{A})P(F)$. then $\widehat{f}$ defines a section of $\mathbf{H}$(s). Fourier inversion implies that

$$f(g) = \frac{1}{2\pi i}\int_{x-i\infty}^{x-i\infty} \widehat{f}(g,s)ds,$$

for any x. Hence the P-series

$$\begin{aligned} F(g) &= \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} f(\gamma g) \\ &= \frac{1}{2\pi i}\sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \int_{-i\infty}^{+i\infty} \widehat{f}(g,s)ds \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \widehat{f}(g,s)ds \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} E(\widehat{f}(s),g)ds. \end{aligned}$$

Here we use the analytic continuation of $E(\phi(s),g)$ to shift the integral to the imaginary axis. In other words, any P-series are "continuous sums" of Eisenstein series.

In the rest of the section, we shall briefly describe the relationship between the space $L^2_{cont}(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}),\omega)$, a subspace of $L^2(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F),\omega)^\perp$, and a subspace of sections of $\mathbf{H}$(s). Let's define $L^2_{cont}(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}),\omega)$ first.

Let's denote by $L^2_{sp}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ the space spanned by characters χ with $\chi^2=\omega$. The space $L^2_{cont}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ is the orthocomplement of $L^2_{sp}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ in $L^2(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F),\omega)^{\perp}$. Now let $\mathcal{L}$ denote the Hilbert space of square-integrable sections a over $i\mathbb{R}$ satisfying

$$M(-it)a(-it)=a(it),$$

where $M(s)$ is certain linear operator from **H**(s) to **H**(-s) which is originally defined for $\mathrm{Re}(s)>\frac{1}{2}$ and can be analytically continued to the entire plane. And let π denote the representation of $\mathrm{GL}_2(\mathbb{A})$ on $\mathcal{L}$, equipped with the inner product

$$(a_1,a_2)=\frac{1}{\pi}\int_{-\infty}^{\infty}(a_1(it),a_2(it))dt=\frac{2}{\pi}\int_0^{\infty}(a_1(it),a_2(it))dt,$$

given by

$$\pi(g)a(it)=\pi_{it}(g)a(it).$$

Then one can show ([6], § 4) that $L^2_{cont}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),\omega)$ is isomorphic to $\mathcal{L}$. The isomorphism can be explicitly determined. In fact, if $F(g)=\sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)} f(\gamma g)$, then

$$a(it)=\frac{1}{2}[\widehat{f}(it)+M(-it)\widehat{f}(-it)].$$

References

[1] E. Artin, J. Tate, *Class Field Theory*, W.A. Benjamin, Inc. 1967.

[2] M. Bertolini, H. Darmon, *Heegner points on Mumford-Tate curves*, Invent. Math., 126(1996) pp. 413-456.

[3] D. Bump, *Automorphic forms and automorphic representations*, Camb. Univ. Press, 1998.

[4] W. Casselman, *On some results of Atkin-Lehner*, Math. Ann. 201 (1973), pp. 301-314.

[5] S. Gelbart, *Automorphic forms on Adele groups*, Princeton Univ. Press, (1975).

[6] S. Gelbart, H. Jacquet, *Forms of* GL(2) *from the analytic point of view*, Prop. Sympos. Pure Math. (Corvallis), 33, Part I, Amer. Math. Soc., Providence, RI, (1979) pp. 213-251.

[7] B. Gross, *Heegner points on* $X_0(N)$, Modular forms, R.A. Rankin, Ed, Ellis Horwood, 1984, pp. 87-105.

[8] B. Gross, *Heights and special values of L-series*, Number Theory (H. Kisilevsky and J. Labute, eds) CMS Conference Proceedings, Vol 7, Amer. Math. Soc., 1987, pp. 115-189.

[9] B. Gross, *Heegner points and Representation theory*, Heegner points and Rankin L-series, Camb. Univ. Press, 2003, pp.

[10] B. Gross, D. Prasad, *Test vectors for linear forms*, Math. Ann., 291:2 (1991), pp. 343-355.

[11] B. Gross, D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math., 84(1986), pp225-320.

[12] H. Hida, *On a p-adic Hecke algebra for* GL_2 *over totally real fields*, Ann. Math. 128, (1988) pp. 295-384.

[13] H. Jacquet, *Automorphic forms on* GL_2 *II*, Lec. Notes Math. 289, Springer-Verlag(1972).

[14] H. Jacquet, R. Langlands, *Automorphic forms on* GL_2, Lecture Notes in Math., vol 114, Springer-Verlag, Berlin and New York, 1970.

[15] H. Shimizu, *Theta series and automorphic forms on* GL_2, J. Math. Soc. Japan 24 (1972) pp. 638-683.

[16] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. Jour. 45 (1978) pp. 637-679.

[17] R. Taylor, *On Galois representations associated to Hilber modular forms*, Invent. Math. 98 (1989) pp. 265-280.

[18] V. Vatsal, *Uniform distribution of Heegner points* Invent. Math. 148(2002) pp. 1-46.

[19] J.L. Waldspurger, *Sur les values de certaines fonctions L automorphes en leur centre de symmétre*, Comsitio Math. 54 (1985), 173-242.

[20] H. Xue, *Central values of twisted Rankin L-functions*, Thesis at Columbia University (2002).

[21] S. Zhang, *Gross-Zagier formula for* GL_2, Asian J. Math. 5, (2001), pp. 183-290.

[22] S. Zhang, *Gross-Zagier formula for* $\mathrm{GL}_2, \mathrm{II}$, Heegner points and Rankin L-series, Camb. Univ. Press, 2003, pp. 191-243.