SPECTRAL DATA FOR L^2 COHOMOLOGY

Xiaofei Jin

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Supervisor of Dissertation

Tony Pantev, Class of 1939 Professor in Mathematics

Graduate Group Chairperson

Wolfgang Ziller, Francis J. Carey Term Professor in Mathematics

Dissertation Committee:
Tony Pantev, Professor in Mathematics
Ron Donagi, Professor in Mathematics
Jonathan Block, Professor in Mathematics

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ABSTRACT

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Xiaofei Jin

Tony Pantev

We study the spectral data for the higher direct images of a parabolic Higgs bundle along a map between a surface and a curve with both vertical and horizontal parabolic divisors. We describe the cohomology of a parabolic Higgs bundle on a curve in terms of its spectral data. We also calculate the integral kernel that reproduces the spectral data for the higher direct images of a parabolic Higgs bundle on the surface. This research is inspired by and extends the works of Simpson [21] and Donagi-Pantev-Simpson [7].

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Chapter 1

Introduction

A Higgs bundle (E, φ) on a smooth complex projective variety X is an algebraic vector bundle E equipped with an algebraic Higgs field φ, i.e. an $\mathcal{O}_X$-linear map $\varphi : E \to E \otimes \Omega^1_X$ such that $\varphi \wedge \varphi = 0$. A flat bundle (F, ∇) on X is an algebraic vector bundle F equipped with an algebraic flat connection ∇, i.e. an $\mathbb{C}$-linear map $\nabla : F \to F \otimes \Omega^1_X$ satisfying the Leibniz rule and such that $\nabla^2 = 0$. Via the Riemann-Hilbert correspondence specifying a flat bundle on X is the same thing as specifying a local system L on X, i.e. a locally constant sheaf of $\mathbb{C}$-vector spaces on X or equivalently a representation of the fundamental group $\pi_1(X)$.

The Non Abelian Hodge Correspondence (NAHC) on X says that semistable Higgs bundles with vanishing Chern classes and flat bundles are equivalent data [3, 23]. The equivalence in the NAHC is mediated by a richer object called a harmonic bundle which determines both a Higgs bundle and a flat bundle. In fact, a harmonic bundle determines a whole family of λ-connections parametrized by $\lambda \in \mathbb{C}$. The Higgs field corresponds to the $\lambda = 0$ value of the parameter while the flat connection of corresponds to the $\lambda = 1$ value.

The notion of a harmonic bundle was introduced by Corlette and Simpson [3, 23, 25] as a gauge theoretic extension of the classical notion of a variation of Hodge structures. Recall [23] that a complex variation of Hodge structures on X is a C^∞ complex vector bundle V equipped with a decomposition $V = \bigoplus_{p+q=k} V^{p,q}$, a non-degenerate Hermitian pairing (polarization), and a flat connection $D : V \to A^1(V)$, satisfying the Griffiths transversality property – D is of degree one with respect to the total gradings on V and $A^1(V)$. In other words D decomposes into four pieces:

$$D = \partial + \overline{\partial} + \theta + \overline{\theta} : V^{p,q} \to A^{1,0}(V^{p,q}) \oplus A^{0,1}(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1}) \oplus A^{0,1}(V^{p+1,q-1}).$$

A complex variation of Hodge structures is a generalization of the notion of an integral variation of Hodge structures, which was first introduced and studied by Griffiths in his classical work [8]. A special and very interesting class of such variations of Hodge structures are the variations of geometric origin that arise from smooth projective families of varieties. Indeed if $Z \to X$ is a smooth projective morphism, then the family of vector spaces $L_x = \mathrm{H}^k(Z_x, \mathbb{C})$ yields a local system L on X. The corresponding C^∞ vector bundle $V = L \otimes_{\mathbb{C}} \mathcal{C}^\infty_X$ comes equipped with a natural flat connection - the Gauss-Manin connection - corresponding to the locally constant structure of L. Finally the Hodge decomposition of L_x with pieces $L_x^{p,q} = \mathrm{H}^{p,q}(Z_x)$ induces the desired decomposition $V = \bigoplus_{p+q=k} V^{p,q}$ of V.

From a variation of Hodge structure, we obtain a system of Hodge bundles, that is a direct sum $E = \bigoplus_{p+q=k} E^{p,q}$ of holomorphic vector bundles with a holomorphic endomorphism $\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_X$ satisfying $\theta \wedge \theta = 0$. In fact, $E^{p,q}$ is the bundle $V^{p,q}$ equipped with the holomorphic structure given by $\overline{\partial}$, and θ is the corresponding piece in the decomposition of D. By Griffiths' infinitesimal period relations [8] it is known that we can in fact recover the variation of Hodge structures from the system of Hodge bundles. The systems of Hodge bundles which arise from irreducible variations of Hodge structures are necessarily stable and have vanishing Chern classes. Using Yang-Mills-Higgs theory Simpson reinterpreted the period relations as the existence of a special metric - the so called harmonic metric - on the system of Hodge bundles. He showed that given a stable system of Hodge bundles with vanishing Chern classes, we can recover the irreducible complex variation of Hodge structures as the Chern connection for this special metric. Therefore, there is an equivalence between the category of irreducible complex variations of Hodge structures and stable systems of Hodge bundles with vanishing Chern classes. The NAHC extends this equivalence to arbitrary flat bundles and general semistable Higgs bundles with vanishing Chern classes. In fact one can use the extended NAHC to characterize the flat bundles underlying complex variations of Hodge structures. By a theorem of Simpson [23] they are exactly the ones corresponding to Higgs bundles which are fixed by the natural $\mathbb{C}^\times$-action: $(E, \varphi) \mapsto (E, t\varphi)$ for $t \in \mathbb{C}^\times$.

The natural and meaningful cohomology theory of variations of Hodge structures or more general harmonic bundles arises from deriving the functor of global sections that are L^2-summable in the norm corresponding to the harmonic metric. The resulting cohomology theory is hard to compute directly but one can gain access to it through the NAHC. Indeed, the NAHC identifies [23] the L^2 cohomology of a harmonic bundle with the de Rham cohomology of the associated flat bundle and also with the Dolbeault cohomology of the associated Higgs bundle. De Rham and Dolbeault cohomology theories admit respectively a topological and an algebraic model that make them much more tractable. Finally, the general spectral correspondence [11, 6] can be used to recast the Dolbeault cohomology theory of Higgs bundles as a purely sheaf theoretic cohomology of the associated spectral data [21].

The spectral description provides a powerful geometric tool for understanding and computing cohomology and direct images of harmonic, flat, or Higgs bundles when dealing with smooth and compact varieties. The theory becomes much more complicated in the non-compact or singular setting. For smooth quasi-projective varieties and flat or Higgs bundles with tame ramification at infinity the NAHC was developed in the works of Simpson [20] and Mochizuki [13, 14, 15, 16]. The algebraic de Rham and Dolbeault models for the corresponding L^2-cohomology of harmonic metrics with singularities was constructed in the recent work [7] of Donagi-Pantev-Simpson. So for computational purposes the main missing ingredient of this theory is an appropriate version of the spectral correspondence that will account for the blow-up behavior of L^2-cohomology classes at infinity.

In this thesis, we consider the question of how to compute cohomology and push-forwards of Higgs bundles in terms of spectral data in the presence of singularities of the harmonic metrics. More precisely, let $f : X \to Y$ be a map from a smooth projective surface to a curve. Suppose that f is smooth away from some simple normal crossings divisor D_V of X. Let $(\underline{E}, \varphi)$ be a tame parabolic Higgs bundle that has parabolic structures along D_V as well as some horizontal divisor D_H. Suppose each component of D_H is etale over Y. Put $D = D_V + D_H$ and $Q = f(D_V)$. In this situation, the "correct" higher direct images of $(\underline{E}, \varphi)$ can be described algebraically using the parabolic L^2 Dolbeault complex in [7]. By "correct" we mean that these higher direct images correspond under the tamely ramified NAHC to the higher direct images of the corresponding filtered local system as stated in Theorem 2.3.1. The analytic motivation of the L^2 cohomology comes from estimates for the norms of sections or holomorphic 1-forms under a metric which is asymptotically the Poincaré metric near punctures so that meromorphic sections along fibers X_y for $y \in Y - Q$ are in L^2.

On the other hand, a Higgs bundle can be encoded in its spectral sheaf by the spectral correspondence [11, 6]. We discuss a parabolic refinement of the spectral construction in which the spectral datum is a parabolic coherent sheaf on the total space of $\Omega^1_X(\log D)$ and we proceed to describe the higher images of $(\underline{E}, \varphi)$ in terms of this spectral data.

The structure of this thesis is as follows. In Chapter 2, we review some basic notions of the theory of Higgs bundles including the spectral construction, the concept of a tame parabolic Higgs bundle, the non-compact NAHC which was given in [13], and the standard root stack construction through which a parabolic Higgs bundle can be converted into a meromorphic Higgs bundle on an orbifold.

In Chapter 3, we study the case when f is smooth and (E, φ) is just a holomorphic Higgs bundle. In this case, we review a result of Simpson's from [21]. Basically, this says that the higher direct images for (E, φ) are the hyperdirect images of the relative Dolbeault complex (3.1.1) and the Higgs fields of the higher direct images are defined by the coboundary maps for hyperdirect images of a certain short exact sequence of complexes consisting of relative and absolute Dolbeault complexes. After this, in Section 3.2, we find the spectral sheaves for the higher direct images of (E, φ) in terms of the spectral sheaf for (E, φ). It is this result that we want to generalize to the situation where we have a parabolic Higgs bundle with parabolic structure specified along both vertical and horizontal divisors.

In Chapter 4, we define the parabolic L^2 Dolbeault complex and state the main theorem in [7]. In Chapter 5, we study our main problem of how to describe the cohomology and higher direct images of a parabolic Higgs bundle by using its spectral data.

Chapter 2

Basic notions on Higgs bundles

2.1 Higgs bundles and their spectral covers

Let X be a smooth projective variety.

Definition 2.1.1. A Higgs bundle (E, φ) on X is a pair consisting of a holomorphic vector bundle E and a $\mathcal{O}_X$-linear map $\varphi : E \to E \otimes \Omega^1_X$ such that $\varphi \wedge \varphi = 0$.

For a vector bundle K over X, there is a more general notion of a K-valued Higgs bundle, which is a pair $(E, \varphi : E \to E \otimes K)$ satisfying $\varphi \wedge \varphi = 0$. A morphism $a : (E_1, \varphi_1) \to (E_2, \varphi_2)$ between two K-valued Higgs bundles is a map of coherent sheaves which intertwines the respective Higgs fields, that is a map $a : E_1 \to E_2$ for which $\varphi_2 \circ a = a \circ \varphi_1$.

The category of Higgs bundles has internal tensor products and internal Homs. Given two Higgs bundles (E_1, φ_1) and (E_2, φ_2) on X, we can define their tensor product as the Higgs bundle:

$$(E_1, \varphi_1) \otimes (E_2, \varphi_2) := (E_1 \otimes E_2, \varphi_1 \otimes \mathrm{id}_{E_2} + \mathrm{id}_{E_1} \otimes \varphi_2).$$

Similarly we can define their Hom Higgs bundle as:

$$\underline{\mathrm{Hom}}((E_1, \varphi_1), (E_2, \varphi_2)) := (\underline{\mathrm{Hom}}(E_1, E_2), h \mapsto -h \circ \varphi_1 + \varphi_2 \circ h). \tag{2.1.1}$$

Furthermore, suppose $f : X \to Y$ is a map of smooth projective varieties. Given a Higgs bundle (E, φ) on Y, the pullback of (E, φ) is defined by

$$f^*(E, \varphi) := (f^*E, (\mathrm{id}_{f^*E} \otimes df^*) \circ f^*\varphi),$$

where $df^* : f^*\Omega^1_Y \to \Omega^1_X$ is the codifferential of f.

A Higgs bundle on X can be thought of as a vector bundle E together with an action of the symmetric algebra of the tangent bundle $\mathrm{Sym}^\bullet T_X$. Note that the one-form φ gives an action of the symmetric algebra rather than of the full tensor algebra because of the condition $\varphi \wedge \varphi = 0$, which says that the endomorphisms $\varphi_\xi = \varphi \lrcorner \xi : E \to E$ corresponding to different tangent vectors $\xi \in T_X$ commute.

The projection from the cotangent space $\pi : T^*_X \to X$ is an affine map. Note that

$$T^*_X = \underline{\mathrm{Spec}}(\mathrm{Sym}^\bullet T_X), \quad \pi_* \mathcal{O}_{T^*_X} = \mathrm{Sym}^\bullet T_X.$$

Thus we see that a coherent sheaf $\mathcal{E}$ on T^*_X is the same thing as a coherent sheaf $F = \pi_*\mathcal{E}$ on X together with an action of $\mathrm{Sym}^\bullet T_X$ on F. In particular a Higgs bundle (E, φ) can be viewed as a coherent sheaf $\mathcal{E}$ on T^*_X.

The coherent sheaf $\mathcal{E}$ can be described explicitly as follows. Given a Higgs bundle (E, φ) on X, for each $x \in X$ we can look at the spectrum of the linear map φ_x. Let λ be the tautological section of $\pi^*\Omega^1_X$. We define $\mathcal{E}$ to be the cokernel of the map $\lrcorner(\mathrm{id}_{\pi^*E} \otimes \lambda - \pi^*\varphi)$:

$$\pi^*E \otimes \pi^*T_X \xrightarrow{\lrcorner(\mathrm{id}_{\pi^*E} \otimes \lambda - \pi^*\varphi)} \pi^*E \longrightarrow \mathcal{E} \longrightarrow 0. \tag{2.1.2}$$

This $\mathcal{E}$ is called the **spectral sheaf** for the Higgs bundle (E, φ). The spectral sheaf $\mathcal{E}$ is a coherent sheaf on T^*_X which is finite and flat over $\mathcal{O}_X$. The **spectral cover** for (E, φ) is defined to be the subscheme $\mathrm{Supp}(\mathcal{E}) \subset T^*_X$, and the map from $\mathrm{Supp}(\mathcal{E})$ to X is proper. Explicitly, $\mathrm{Supp}(\mathcal{E})$ is given as the zero scheme of the section

$$\det(\lambda \cdot \mathrm{id} - \pi^*\varphi) \in \mathrm{Sym}^r \pi^*\Omega^1_X.$$

We can recover the Higgs data (E, φ) from $\mathcal{E}$ by simply setting $E = \pi_*\mathcal{E}$ and $\varphi = \pi_*(- \otimes \lambda)$. Therefore, we have set up an equivalence of categories

$$\left(\begin{array}{l} \text{coherent sheaves on } T^*_X, \\ \text{finite and flat over } X \end{array} \right) \Longleftrightarrow \left(\text{ Higgs bundles on } X \right).$$

This equivalence is called the **spectral correspondence**. This correspondence also works for any K-valued Higgs bundle. More generally, if (E, φ) is a coherent Higgs sheaf rather than a Higgs bundle, then in the spectral correspondence above it will correspond to a coherent sheaf on T^*_X that is finite but not necessarily flat over X.

Remark 2.1.2. We want to mention another reason why the condition "$\varphi \wedge \varphi = 0$" in Definition 2.1.1 is needed. Consider a K-valued Higgs bundle (E, φ). If $K|_U \simeq \mathbb{C}^n \otimes \mathcal{O}_U$ is a local trivialization of K over an open subset $U \subset X$, then we see that φ consists of n endomorphisms of E:

$$\varphi|_U = (\varphi_1, \cdots, \varphi_n), \text{ with } \varphi_i \in \Gamma(U, \operatorname{End} E).$$

We may look at the spectrum of each φ_i, but the collection of spectral covers we get in this way makes little sense since these φ_i may not share any common eigenvectors. Hence we need all φ_i to have a common spectrum, which is equivalent to requiring that $[\varphi_i, \varphi_j] = 0$ for all i, j. This is the condition $\varphi \wedge \varphi = 0$.

2.2 Parabolic Higgs bundles

Let $D \subset X$ be a simple normal crossings divisor. We recall the notion of a parabolic vector bundle with parabolic structure along D.

Definition 2.2.1. A **parabolic bundle** $\underline{E}$ on (X, D) consists of a family of vector bundles $\underline{E} = \{E_\beta\}$ on X labeled by a collection of real numbers $\beta = \{\beta_i\}$, one for each irreducible component D_i of D, which satisfies the following conditions:

- If $\beta \leq \beta'$, then E_β is a subsheaf of $E_{\beta'}$,
- E_β jumps only at discrete levels, that is if $\epsilon = \{\epsilon_i\}$ is a collection of small positive numbers, then $E_{\beta+\epsilon} = E_\beta$,

- $E_{\beta+\delta_i} = E_\beta(D_i)$, where δ_i is Kronecker's delta.

If X is a smooth projective curve, then $D = \{p_i\}$ is a finite set of points, and a parabolic bundle $\underline{E}$ on (X, D) is equivalent to a vector bundle E on X equipped with a flag of subspaces in the fiber E_{p_i} for each $p_i \in D$:

$$0 = E^0_{p_i} \subset E^1_{p_i} \subset \cdots E^{\ell_i - 1}_{p_i} \subset E^{\ell_i}_{p_i} = E_{p_i},$$

together with a collection of real numbers - **parabolic weights** assigned to each subspace in the flag:

$$0 = \alpha_{i0} < \alpha_{i1} < \cdots < \alpha_{i(\ell_i - 1)} < \alpha_{i\ell_i} = 1.$$

We denote $\dim(E^{j+1}_{p_i}) - \dim(E^j_{p_i})$ by m_{ij} and define the parabolic degree of $\underline{E}$ by

$$\operatorname{par}\deg \underline{E} = \deg E + \sum_{i=1}^{k} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} m_{ij}.$$

The parabolic slope of $\underline{E}$ is defined by

$$\mu(\underline{E}) = \frac{\operatorname{par}\deg \underline{E}}{\operatorname{rank} E}.$$

We can thus speak of stable (respectively semistable) parabolic bundles by requiring that for every proper subbundle F of E with the induced parabolic structure, we have

$$\mu(\underline{F}) < (\text{respectively} \leq)\ \mu(\underline{E}). \tag{2.2.1}$$

A Higgs field $\underline{\varphi}$ (respectively a flat connection $\underline{\nabla}$) on a parabolic bundle $\underline{E}$: $\underline{\varphi} = \{\varphi_\beta\}$ (respectively $\underline{\nabla} = \{\nabla_\beta\}$) is a family where φ_β (respectively ∇_β) is a

Higgs field (respectively a flat connection) on E_β so that for $\beta \leq \beta'$ we the fields φ_β and $\varphi_{\beta'}$ (respectively the connections ∇_β and $\nabla_{\beta'}$) are compatible under the inclusion $E_\beta \subset E_{\beta'}$. This compatibility condition forces the restrictions of all φ_β to $X - D$ to be equal to each other (respectively the restrictions of all ∇_β to $X - D$ to be equal to each other). Thus any of the fields φ_β (respectively ∇_β) determines the whole family $\underline{\varphi}$ (respectively $\underline{\nabla}$). To simplify notation we will write just φ or ∇ for one of the members of the family and this will be understood as specifying the Higgs field or flat connection on the parabolic bundle. From the nature of the compatibility condition it is clear that the notions of a parabolic Higgs field and flat connection also make sense if we allow their coefficients to be meromorphic, with some poles along D.

In other words, a parabolic Higgs bundle can be viewed as a refinement of a meromorphic one, i.e. a meromorphic Higgs bundle with poles along a simple normal crossings divisor, which is endowed with a parabolic structure preserved by the the Higgs field. Throughout this thesis we are only concerned for parabolic Higgs bundles with logarithmic Higgs fields. These are defined as follows:

Definition 2.2.2. A **logarithmic parabolic Higgs bundle** $(\underline{E}, \varphi)$ on (X, D) is a pair consisting of a parabolic bundle $\underline{E}$ and a Higgs field φ such that φ preserves the parabolic structure in the sense that for each parabolic level β, we have

$$\varphi : E_\beta \to E_\beta \otimes \Omega^1_X(\log D).$$

For a parabolic Higgs bundle $(\underline{E}, \varphi)$ on a curve, we also have a stability condition, that is, $\underline{E}$ is stable (respectively semistable) if for every subbundle F preserved by the Higgs field φ, we have the condition in (2.2.1).

2.3 Non Abelian Hodge Correspondence

Higgs bundles are closely related to flat bundles through the Non Abelian Hodge Correspondence (NAHC) [10, 3, 23, 25], which establishes an equivalence between the two categories on X

$$\begin{pmatrix} \text{semisimple flat} \\ \text{bundles on } X \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{polystable Higgs bundles on } X \text{ with} \\ \text{vanishing Chern classes } c_1 = c_2 = 0 \end{pmatrix}$$

given by Hitchin's equations [10, 23].

A noncompact version of the NAHC on $X - D$, where X is compact and $D \subset X$ is a simple normal crossings divisor, was established in T. Mochizuki [13]. This setup involves a flat bundle (F°, ∇) and a Higgs bundle (E°, φ) that are defined on $X - D$ and carry order of growth filtrations along the components of D. More precisely, in the logarithmic setting, $(\underline{F}, \nabla)$ is a polystable parabolic flat bundle with vanishing parabolic Chern classes, and $(\underline{E}, \varphi)$ is a parabolic Higgs bundle, consisting of a locally abelian parabolic vector bundle $\underline{E}$ with vanishing parabolic Chern classes, together with a Higgs field φ that is logarithmic with respect to the parabolic structure along D.

Recall that a **harmonic bundle** over $X - D$ consists of the data $(\mathcal{L}, \boldsymbol{D'}, \boldsymbol{D''}, h)$

where $\mathcal{L}$ is a $\mathcal{C}^\infty$ bundle over $X - D$ equipped with a hermitian metric h and operators

$$\boldsymbol{D'} = \partial + \overline{\varphi}, \quad \boldsymbol{D''} = \overline{\partial} + \varphi : \mathcal{L} \to \mathcal{A}^1(\mathcal{L})$$

such that ∂, φ are of type $(1,0)$ and $\overline{\partial}$, $\overline{\varphi}$ are of type $(0,1)$. Put $\boldsymbol{D} = \boldsymbol{D'} + \boldsymbol{D''}$. These are subject to the following conditions:

1. $\partial + \overline{\partial}$ is an h-unitary connection;
2. $\varphi + \overline{\varphi}$ is h-self-adjoint;
3. $(\boldsymbol{D''})^2 = 0$ so that $E^\circ = (\mathcal{L}, \overline{\partial})$ is a holomorphic bundle and $\varphi : E^\circ \to E^\circ \otimes \Omega^1_{X-D}$ is a holomorphic Higgs field;
4. $\boldsymbol{D}^2 = 0$, so that $L = \mathcal{L}^{\boldsymbol{D}}$ is a local system, or equivalently (F°, ∇) is a flat bundle, where F° is the holomorphic bundle $F^\circ = (\mathcal{L}, \overline{\partial} + \overline{\varphi})$ and $\nabla : F^\circ \to F^\circ \otimes \Omega^1_{X-D}$ is the holomorphic connection given by $\nabla = \partial + \varphi$.

Consider a smooth point $p \in D_i$ of one of the divisor components, and let z_i a coordinate function defining D_i near p. Let $\{r(t)\}_{t\in(0,1)}$ be a ray emanating from p, with $|z_i(r(t))| = t$. If $\{u(t) \in \mathcal{L}_{r(t)}\}$ is a flat section of the local system L over the ray, we can look at the growth rate of $\left\|u(t)\right\|_{h(r(t))}$ with respect to the harmonic metric h. We say that u has **polynomial growth** (respectively **sub-polynomial growth**) along the ray, if for some (respectively all) $b > 0$ we have

$$\left\|u(t)\right\|_{h(r(t))} \leq Ct^{-b}.$$

The harmonic bundle is said to be **tame** if all its flat sections have polynomial growth along rays. It is said to be **tame with trivial filtrations** if flat sections have sub-polynomial growth along rays.

Suppose $(\mathcal{L}, \boldsymbol{D'}, \boldsymbol{D''}, h)$ is a tame harmonic bundle. Using the order of growth, we obtain a collection of filtrations on the restrictions of the local system L to punctured neighborhoods of each of the divisor components. Let $j : X - D \hookrightarrow X$. If $\eta = \{\eta_i\}$ is a parabolic level then L_η is the subsheaf of $j_* L$ consisting of sections that have growth rate $\leq Ct^{-\eta_i - \epsilon}$ for any $\epsilon > 0$, along rays going towards smooth points of D_i. The collection $\underline{L} = \{L_\eta\}$ is the filtered local system associated to $(\mathcal{L}, \boldsymbol{D'}, \boldsymbol{D''}, h)$. Note that the condition of trivial filtration is equivalent to $L_0 = j_* L$ and $L_\eta = j_! L$ for $\eta_i < 0$. Similarly, the Higgs bundle (E°, φ) extends to a parabolic sheaf $(\underline{E}, \varphi)$.

The following summarizes some of T. Mochizuki's main results in [13]:

Theorem 2.3.1 (Non-compact tame non-abelian Hodge correspondence)**.** *Let $(\mathcal{L}, \boldsymbol{D'}, \boldsymbol{D''}, h)$ be a tame harmonic bundle on $X - D$ and let $\underline{L}$, $(\underline{F}, \nabla)$, and $(\underline{E}, \varphi)$ be the associated filtered local system, parabolic flat bundle, and parabolic Higgs bundle respectively. Then:*

- *The filtered local system $\underline{L}$ is locally abelian, that is, it is locally an extension of rank 1 filtered local systems.*

- *The parabolic sheaves $\underline{E}$ and $\underline{F}$ are both locally abelian parabolic bundles, that is, they are locally direct sums of parabolic line bundles.*

- *The Higgs field φ and the flat connection ∇ are both logarithmic.*
- *The filtered local system, the parabolic logarithmic flat bundle, and the parabolic logarithmic Higgs bundle, are all polystable objects with vanishing Chern classes.*
- *Furthermore, any polystable filtered local system or parabolic logarithmic Higgs bundle with vanishing Chern classes comes from a unique harmonic bundle.*

This sets up one to one correspondences among the four kinds of objects.

Throughout this thesis, we will always work under the following standing assumption:

Definition 2.3.2 (**Nilpotence Assumption**)**.** For every logarithmic parabolic Higgs bundle $(\underline{E}, \varphi)$, we assume the residue of φ along each parabolic divisor component has zero eigenvalues.

Under the NAHC this assumption corresponds to the condition that the filtered local system $\underline{L}$ has a trivial filtration and has local monodromies whose eigenvalues are in $S^1 \subset \mathbb{C}^\times$ [7].

2.4 The root stack construction

Throughout this thesis we assume that all the parabolic weights are rational numbers. So all the weights are multiples of $1/\ell$ for some fixed $\ell \in \mathbb{Z}_{>0}$. It is a classic result that such parabolic Higgs bundles on a smooth curve can be identified with ordinary Higgs bundles on an orbicurve. In this section we review this construction [1], [2], [4], [9].

Given a curve C and a marked point p, we first construct an orbicurve $\widetilde{C}$ as follows. Let $U = C \setminus p$. Let Δ_p denote a small analytic (or formal) disk centered at p and let $\Delta_p^\circ = \Delta_p \times_C U$ be the corresponding punctured disk. Take $\phi : \widetilde{\Delta_p} \to \Delta_p$ to be the $\ell : 1$ cover given by $z \mapsto z^\ell$. The group $\boldsymbol{\mu}_\ell$ of ℓ-th roots of unity acts naturally on $\widetilde{\Delta_p}$ sending $z \mapsto \omega z$, where $\omega = \exp(2\pi\sqrt{-1}/\ell)$. The quotient stack $[\widetilde{\Delta_p}/\boldsymbol{\mu}_\ell]$ can then be glued to U using the morphism ϕ to identify the open substack $[\widetilde{\Delta_p}^\circ/\boldsymbol{\mu}_\ell]$ with the punctured disk Δ_p°. This yields a smooth Deligne-Mumford stack $\widetilde{C}$ equipped with a map $\mu : \widetilde{C} \to C$ which identifies C as its coarse moduli space. We denote the orbifold point in $\widetilde{C}$ by $\widetilde{p}$.

Suppose that we have a holomorphic Higgs bundle on $\widetilde{C}$, that is a vector bundle $\widetilde{E}$ equipped with a Higgs field $\widetilde{\varphi} : \widetilde{E} \to \widetilde{E} \otimes \Omega^1_{\widetilde{C}}$. This data determines a parabolic Higgs bundle on C as follows. Observe that $\mu^*\mathcal{O}_C(p)$ corresponds to the rank one free $\mathbb{C}[[z]]$-module generated by $z^{-\ell}$. Hence, there is a ℓ-th root line bundle $\widetilde{L}$ on $\widetilde{C}$ which corresponds to the module generated by z^{-1}. Now let $E = \mu_*\widetilde{E}$ and

$$F_i = \mu_*(\widetilde{E} \otimes \widetilde{L}^i)$$

for $1 \leq i \leq \ell - 1$. By the base change theorem, all the direct images are locally free and the sheaves F_i form a filtration

$$E \subset F_1 \subset \cdots \subset F_{\ell-1} \subset E(p). \tag{2.4.1}$$

More explicitly, the morphism $\mu : \widetilde{C} \to C$ is locally of the form $w = z^\ell$, where w is a local coordinate on C centered at p. Note that $\widetilde{E}$ is locally isomorphic to a sum of line bundles of the form $\bigoplus_{j=1}^r \mathcal{O}(n_j)$, corresponding to the $\mathbb{C}[[z]]$-module $\bigoplus_{j=1}^r z^{-n_j}\mathbb{C}[[z]]$ with the natural $\boldsymbol{\mu}_\ell$ action. We can decompose it into a sum of submodules $\bigoplus_{k=0}^{\ell-1} M_k$, such that for each k, the action of $\boldsymbol{\mu}_\ell$ on M_k is given by the multiplication by ω^k. Then the $\mathbb{C}[[w]]$-module F_i is the $\boldsymbol{\mu}_\ell$-fixed part of $\bigoplus_{k=0}^{\ell-1} z^{-i} M_k$, that is $F_i = z^{-i} M_i$.

The filtration (2.4.1) defines a flag

$$E_i = \ker(E_p \to E/F_i(-p)) \tag{2.4.2}$$

in the fiber E_p, and we thus obtain a parabolic bundle $\underline{E}$ on C. According to [18] assigning $\widetilde{E}$ to the parabolic bundle $\underline{E}$ yields an equivalence of groupoids. Furthermore, the degree of $\widetilde{E}$ as an orbibundle is equal to the parabolic degree of $\underline{E}$ if we assign E_i the weight $\alpha_i = i/\ell$.

Next, we note that

$$\Omega^1_{\widetilde{C}} \simeq \widetilde{L}^{\ell-1} \otimes_{\widetilde{C}} \mu^*\Omega^1_C \simeq \widetilde{L}^{-1} \otimes_{\widetilde{C}} \mu^*\Omega^1_C(p). \tag{2.4.3}$$

Hence $\varphi = \mu_* \widetilde{\varphi} : E \to E \otimes_C \Omega^1_C(p)$ is a Higgs field on E. Moreover, (2.4.3) implies that

$$\varphi(F_i) \subset F_{i-1} \otimes_C \Omega^1_C(p).$$

Such Higgs bundles are called **strongly parabolic** Higgs bundles with respect to the flag (2.4.2).

In general we may consider a meromorphic Higgs bundle $(\widetilde{E}, \widetilde{\varphi})$ with a logarithmic pole at the point $\widetilde{p}$, that is

$$\varphi : \widetilde{E} \to \widetilde{E} \otimes \Omega^1_{\widetilde{C}}(\widetilde{p}).$$

Then in the same way the root stack construction tells us that $(\widetilde{E}, \widetilde{\varphi})$ corresponds to an ordinary (not necessarily strongly) parabolic Higgs bundle on C with respect to the flag (2.4.2). We summarize this correspondence in the following proposition.

Proposition 2.4.1. *The root stack construction yields a one-to-one correspondence between meromorphic Higgs bundles on the orbicurve $\widetilde{C}$ with a logarithmic pole at $\widetilde{p}$ and parabolic Higgs bundles on C with respect to a flag $\{E_i\}_{1 \leq i \leq \ell}$ at p. In particular, holomorphic Higgs bundles on $\widetilde{C}$ correspond to strongly parabolic Higgs bundles on C.*

Chapter 3

Direct images of Higgs bundles

In this chapter we consider projective semistable morphisms $f : X \to Y$ between smooth varieties and study the f-pushforward formulas for parabolic Higgs bundles in various situations. We successively analyze the cases of holomorphic Higgs bundles, logarithmic Higgs bundles with trivial parabolic structures, tame strongly parabolic Higgs bundles, and general tame parabolic Higgs bundles with nilpotent residues.

In the case that the map is smooth and the Higgs bundle is holomorphic without any parabolic structure, i.e. there are no vertical or horizontal divisors, Simpson gave out an algebraic definition of the higher direct images in [21]. He also described how to pushforward a Higgs bundle in terms of its spectral data. In the following, we review Simpson's results which will give us a heuristic guide of what we need to do in the general situation.

3.1 Direct images of a holomorphic Higgs bundle under a smooth map

Suppose that X and Y are smooth projective varieties and $f : X \to Y$ is a smooth projective morphism. Suppose that (E, φ) is a Higgs bundle on X. We define its higher direct images $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E)$ as follows. Let $\Omega^\bullet_{X/Y}(E)$ be the complex of sheaves (the **relative Dolbeault complex** of E on X/Y)

$$\cdots \xrightarrow{\wedge\varphi} \Omega^i_{X/Y} \otimes E \xrightarrow{\wedge\varphi} \Omega^{i+1}_{X/Y} \otimes E \xrightarrow{\wedge\varphi} \cdots, \tag{3.1.1}$$

where $\Omega^i_{X/Y}$ is the sheaf of relative differentials.

Now define

$$\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) = \mathbb{R}^i f_*(\Omega^\bullet_{X/Y}(E)). \tag{3.1.2}$$

These are coherent sheaves on Y. We give them structures of Higgs sheaves in the following way. Let $\Omega^\bullet_X(E)$ denote the Dolbeault complex of E on X, with differentials given by the operator φ. Let $I^1 = I^1\Omega^\bullet_X(E)$ be the subcomplex of $\Omega^\bullet_X(E)$ that is the image of $f^*\Omega^1_Y \otimes \Omega^\bullet_X(E)$ and let $I^2 = I^2\Omega^\bullet_X(E)$ be the image of $f^*\Omega^1_Y \otimes I^1\Omega^\bullet_X(E)$. Note that the relative Dolbeault complex is the quotient

$$\Omega^\bullet_{X/Y}(E) = \Omega^\bullet_X(E)/I^1,$$

and we have an isomorphism

$$f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}(E) \simeq I^1\Omega^\bullet_X(E)/I^2.$$

Hence we get an exact sequence of complexes

$$0 \to f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}(E)[-1] \to \Omega^\bullet_X(E)/I^2 \to \Omega^\bullet_{X/Y}(E) \to 0. \tag{3.1.3}$$

The hyperdirect image of the complex on the left is

$$\mathbb{R}^{i+1}f_*(f^*\Omega^1_Y \otimes \Omega^\bullet_{X/Y}(E)[-1]) \simeq \Omega^1_Y \otimes \mathbb{R}^i f_*(\Omega^\bullet_{X/Y}(E)).$$

So the coboundary map for the hyperdirect images gives a morphism

$$\theta : \mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) \to \mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) \otimes \Omega^1_Y. \tag{3.1.4}$$

We can check that $\theta \wedge \theta = 0$ (Propostion 3.3.2). If the $\mathrm{H}^i_{\mathrm{DOL}}(X_y, E)$ have the same dimensions for all $y \in Y$, then by the base-change theorem, the direct-image sheaves $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E)$ are locally free with $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E)_y = \mathrm{H}^i_{\mathrm{DOL}}(X_y, E)$. In this case, the direct images are Higgs bundles [21].

3.2 Direct images via spectral data

In the previous section we defined the higher direct image functor $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, -)$ from the category of coherent Higgs sheaves $\mathbf{CohHiggs}(X)$ to $\mathbf{CohHiggs}(Y)$. Composing this functor with the spectral correspondence:

$$\begin{tikzcd}
\mathbf{CohHiggs}(X) \arrow[r, "\simeq", "\text{SC on } X"'] \arrow[d, "\mathrm{H}^i_{\mathrm{DOL}}(X/Y,-)"] & \mathbf{Coh}_f(T^*_X) \arrow[d, "\Phi_i"] \\
\mathbf{CohHiggs}(Y) \arrow[r, "\simeq", "\text{SC on } Y"'] & \mathbf{Coh}_f(T^*_Y)
\end{tikzcd} \tag{3.2.1}$$

we obtain a functor Φ_i from $\mathbf{Coh}_f(T^*_X)$ to $\mathbf{Coh}_f(T^*_Y)$. Here, we use $\mathbf{Coh}_f(T^*_X)$ to denote the category of coherent sheaves over T^*_X that are finite over X.

Let $d = \dim X - \dim Y$ and $Z = f^*T^*_Y$ be the pull back of T^*_Y by f. In [21], Simpson showed that Φ_i is given by a Fourier-Mukai type transform with the kernel equal to the complex $\omega_{Z/T^*_X}[-d]$, where ω_{Z/T^*_X} is the restriction of $\pi^*_X\Omega^d_{X/Y}$ to Z and we form the complex by placing the sheaf ω_{Z/T^*_X} in cohomological degree d.

$$\begin{tikzcd}
 & Z \arrow[r, "g"] \arrow[d] \arrow[dl, hook', "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & T^*_Y \arrow[d, "\pi_Y"] \\
T^*_X \arrow[r, "\pi_X"] & X \arrow[r, "f"] & Y
\end{tikzcd} \tag{3.2.2}$$

Our aim in this thesis is to extend this result to the situation where f is not necessarily smooth but only semistable, and where we are pushing forward tame parabolic Higgs bundles with parabolic structure along both vertical and horizontal divisors as will be described in Section 4.1. In this section, we will first complete the proof of Simpson's result and then generalize it to the situation where we have a logarithmic Higgs bundle with poles along some vertical divisors.

Theorem 3.2.1 (C. Simpson [21])**.** *Let $\mathcal{E}$ be the spectral sheaf for (E, φ). Define*

$$\mathcal{F}_i := \mathbb{R}^i g_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_{Z/T^*_Y}[-d]). \tag{3.2.3}$$

Then the coherent sheaf $\mathcal{F}_i$ is the spectral sheaf for the Higgs sheaf $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E)$.

Proof. The tautological section of $\pi_X^*\Omega_X^1$ maps to a section of $\pi_X^*\Omega_{X/Y}^1$, which will be again denoted by λ. Consider the complex $\Lambda^\bullet$ defined by setting

$$\Lambda^i = \pi_X^*\Omega_{X/Y}^i$$

with the differential given by the wedge product with λ. Using the fact that $\pi_*\mathcal{E} = E$ and the projection formula, we have

$$\Omega^\bullet(X/Y, E) = \pi_{X*}(\mathcal{E} \otimes \Lambda^\bullet).$$

Note that each Λ^i is a vector bundle and hence is torsion free. By construction the complex $\Lambda^\bullet$ is exact everywhere except at the last term $\pi_X^*\Omega_{X/Y}^d$ where its cokernel is ω_{Z/T_Y^*}. It follows that $\Lambda^\bullet$ is quasi-isomorphic to $\omega_{Z/T_Y^*}[-d]$. Therefore, we can now represent the Dolbeault cohomology (3.1.2) as

$$\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) = \mathbb{R}^i(f \circ \pi_X)_*(\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_{Z/T_Y^*}[-d]). \tag{3.2.4}$$

Since $\omega_{Z/T_Y^*}[-d]$ is supported on the closed subvariety $Z = f^*T_Y^*$ and $f \circ \pi_X|_Z = \pi_Y \circ g$, we conclude that

$$\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) = \mathbb{R}^i(\pi_Y \circ g)_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_{Z/T_Y^*}[-d]). \tag{3.2.5}$$

Finally, since π_Y is affine, the direct image π_{Y*} is equal to the derived direct image and therefore we have $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E) = \pi_{Y*}\mathcal{F}_i$. □

To show that $\mathcal{F}_i$ is the spectral sheaf for $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E)$, it suffices to prove the following.

Proposition 3.2.2. *The map θ in (3.1.4) is equal to the pushforward of*

$$\mathcal{F}_i \xrightarrow{\otimes\sigma} \mathcal{F}_i \otimes \pi_Y^* \Omega_Y^1, \tag{3.2.6}$$

where we denote the tautological section of $\pi_Y^*\Omega_Y^1$ by σ.

Proof. For each $i \geq 0$ we have the short exact sequence of locally free sheaves on X:

$$0 \to f^*\Omega_Y^1 \otimes \Omega_{X/Y}^{i-1} \to \Omega_X^i / I^2 \to \Omega_{X/Y}^i \to 0,$$

Note that the pullback π_X^* is exact. Furthermore, since $\pi_X^*\Omega_{X/Y}^i$ is locally free, $\mathrm{Tor}^1(\mathcal{E}, \pi_X^*\Omega_{X/Y}^i) = 0$ and hence taking the tensor product with $\mathcal{E}$ is also exact. We thus obtain the following short exact sequence of complexes on T_X^*:

$$0 \to (f \circ \pi_X)^*\Omega_Y^1 \otimes \mathcal{E} \otimes \pi_X^*\Omega_{X/Y}^\bullet[-1] \to \mathcal{E} \otimes \pi_X^*(\Omega_X^\bullet/I^2) \to \mathcal{E} \otimes \pi_X^*\Omega_{X/Y}^\bullet \to 0, \tag{3.2.7}$$

where the differential in each complex is given by the wedge product with λ. Note that the coboundary map for the cohomology of (3.2.7) is given by the tensor product with λ.

If we apply π_{X*} to (3.2.7) then we will obtain (3.1.3). From the identity $f \circ \pi_X|_Z = \pi_Y \circ g$ it follows that the hyperdirect images under $\mathbb{R}f_* \circ \pi_{X*}$ in (3.2.7) are equal to the hyperdirect images under $\pi_{Y*} \circ \mathbb{R}g_*$. Therefore we see that the Higgs field θ is the pushforward of the coboundary map:

$$\begin{array}{ccc} \mathbb{R}^i g_* i^*(\mathcal{E} \otimes \pi_X^*\Omega_{X/Y}^\bullet) & \xrightarrow{\delta} & \mathbb{R}^{i+1} g_* i^*((f \circ \pi_X)^*\Omega_Y^1 \otimes \mathcal{E} \otimes \pi_X^*\Omega_{X/Y}^\bullet[-1]) \\ \Vert & & \Vert \\ \mathcal{F}_i & & \mathcal{F}_i \otimes \pi_Y^*\Omega_Y^1. \end{array}$$

To see that the coboundary map δ is equal to the tensor product with the tautological section σ, note that we have a spectral sequence with $E_2^{k,i-k}$ term $\mathbb{R}^k g_*(i^*(\mathcal{H}^{i-k}(\mathcal{E} \otimes \pi_X^*\Omega^\bullet_{X/Y}))$, converging to the hyperdirect image $\mathcal{F}_i$. On the E_2 level, the map δ is given by

$$\mathbb{R}^k g_* i^*(\mathcal{H}^{i-k}(\mathcal{E} \otimes \pi_X^*\Omega^\bullet_{X/Y})) \xrightarrow{g_* i^*(\otimes\lambda)} \mathbb{R}^k g_* i^*(\mathcal{H}^{i+1-k}((f \circ \pi_X)^*\Omega^1_Y \otimes \mathcal{E} \otimes \pi_X^*\Omega^\bullet_{X/Y}[-1])),$$

which is induced from the coboundary map for the cohomology of (3.2.7). We see that $g_* i^*(\otimes\lambda)$ is in fact equal to the tensor product with σ. $\square$

Remark 3.2.3. Note that the cohomology sheaves of the complex $\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_{Z/T^*_Y}[-d]$ are supported on $R = \mathrm{Supp}(\mathcal{E}) \cap Z$, which is a closed subset of the spectral cover $\mathrm{Supp}(\mathcal{E})$. In particular, $g(R)$ is closed and proper over Y. We see that $\mathcal{F}_i$ is in fact supported on the closed subset $g(R)$ (See [21] for details).

Next, we want to consider a logarithmic meromorphic Higgs bundle and a not necessarily smooth map $f : X \to Y$. For the time being we only allow to have vertical polar divisors. We assume that f is smooth away from a simple normal crossings divisor D_V and $D_V = f^{-1}(Q)$. Let (E, φ) be a meromorphic Higgs bundle with the Higgs field $\varphi : E \to E \otimes \Omega^1_X(\log D_V)$. The construction of relative Dolbeault complex (3.2.1) applies to the meromorphic case to yield a logarithmic relative

Dolbeault complex $\Omega^\bullet_{X/Y}(E, \log D)$:

$$\cdots \xrightarrow{\wedge\varphi} \Omega^i_{X/Y}(\log D_V) \otimes E \xrightarrow{\wedge\varphi} \Omega^{i+1}_{X/Y}(\log D_V) \otimes E \xrightarrow{\wedge\varphi} \cdots, \tag{3.2.8}$$

where

$$\Omega^i_{X/Y}(\log D_V) := \frac{\Omega^i_X(\log D_V)}{f^*(\Omega^1_Y(\log Q)) \otimes \Omega^{i-1}_X(\log D_V)}.$$

We now define the higher direct images of (E, φ) to be

$$\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E, \log D_V) := \mathbb{R}^i f_*(\Omega^\bullet_{X/Y}(E, \log D_V)),$$

with the Higgs fields induced from appropriate coboundary maps as in (3.1.4). However, the direct images defined in this way are not always "correct" in the sense that they do not always coincide with the higher direct images of the local system corresponding to the Higgs bundle as stated in the non-abelian Hodge correspondence Theorem 2.3.1. In the case where X is a surface and Y is a curve, we will see in next chapter that the correct direct images can be described algebraically using the parabolic L^2 Dolbeault complex. It turns out that in this case $\mathrm{H}^i_{\mathrm{DOL}}(X/Y, E, \log D_V)$ does represent the correct higher direct image if we view meromorphic Higgs bundles as parabolic ones with trivial parabolic structures.

We can now describe the higher direct images using the spectral data. Similar to the case of a smooth map f, let Z be the pull back of $\mathrm{tot}(\Omega^1_Y(\log Q))$ by f and ω_Z be the restriction of $\pi_X^*\Omega^1_{X/Y}(\log D_V)$ to Z.

$$\begin{tikzcd}
 & Z \arrow[r, "g"] \arrow[dl, hook', "i"'] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{tot}(\Omega^1_Y(\log Q)) \arrow[d, "\pi_Y"] \\
\mathrm{tot}(\Omega^1_X(\log D_V)) \arrow[r, "\pi_X"] & X \arrow[r, "f"] & Y
\end{tikzcd} \tag{3.2.9}$$

It is easy to prove the following result.

Proposition 3.2.4. *Let $\mathcal{E}$ be the spectral sheaf for (E, φ). Then the spectral sheaf for the i-th higher direct image of (E, φ) is*

$$\mathcal{F}_i := \mathbb{R}^i g_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_Z[-1]).$$

Chapter 4

The L^2 parabolic Dolbeault complex

In this chapter, we review the algebraic description given in [7] of the higher direct images of a parabolic Higgs bundle along a map that have both vertical and horizontal parabolic divisors.

4.1 Some preparation

4.1.1 The underlying geometry

Suppose we are given a smooth projective surface X with a morphism $f : X \to Y$ to a smooth projective curve Y. Suppose we are given a simple normal crossings divisor $D \subset X$. Additionally suppose $Q \subset Y$ is a reduced divisor, i.e. Q is just a finite collection of points q_i. We assume that the divisor D has the following structure:

- D decomposes as
$$D = D_V + D_H$$
into two simple normal crossings divisors meeting transversally, called the **vertical** and **horizontal** divisors respectively.

- $D_V = f^{-1}(Q)$ as a divisor, so the fibers of f over the points q_i are reduced with simple normal crossings.

- D_H is etale over Y, so it is a disjoint union of smooth component, not intersecting each other but possibly intersecting D_V.

We use D_k to refer to any component irreducible component of these divisors. Each D_k is smooth and irreducible:

$$D_V = \sum_{i=1}^{n_v} D_{v(i)}, \quad D_H = \sum_{j=1}^{n_h} D_{h(j)}.$$

We assume that f is smooth away from D_V, so the only singular fibers are among the fibers $f^{-1}(q_l)$. It follows from our etaleness assumption that D_H is entirely contained in the smooth locus of f.

4.1.2 The monodromy weight filtration

Before we explain the definition of the L^2 parabolic Dolbeault complex, we need to recall a basic construction from linear algebra. Let V be a finite dimensional complex vector space, $N : V \to V$ be a nilpotent linear operator. The **monodromy weight filtration** of N is the unique increasing filtration $W = W_\bullet(N)$ of V:

$$0 \subset W_{-m}(N) \subset W_{-m+1}(N) \subset \cdots \subset W_{m-1}(N) \subset W_m(N) = V$$

with the properties

- $N(W_i(N)) \subset W_{i-2}(N)$;
- the map $N^\ell : \mathrm{gr}_\ell^W \to \mathrm{gr}_{-\ell}^W$ is an isomorphism for all $\ell \geq 0$.

Explicitly, the mondromy weight filtration $W_\bullet(N)$ is defined as follows. Choose a Jordan basis for the nilpotent endomorphism N and assign integer weights to the basis vectors so that N lowers weights by 2, and so that the weights of each Jordan block are arranged symmetrically about 0. Note that even though the Jordan basis is not unique, the monodromy weight filtration will be uniquely determined since $W_k(N)$ is the span of the basis vectors of weights less than or equal to k.

4.2 Local study of parabolic Higgs bundles

Let $(\underline{E}, \varphi)$ be a parabolic Higgs bundle satisfying the Nilpotence Assumption (Defintion 2.3.2). Along a component D_k of the parabolic divisor D, our $\underline{E}$ determines a family $\mathrm{Gr}_{k,b}(\underline{E})$, indexed by real b, of parabolic vector bundles on D_k. Their parabolic structure is along the divisor $D_{\cap k} = \overline{(D - D_k)} \cap D_k$ where D_k meets the other components. These come with endomorphisms $N_{k,b}$ induced by the residues of the Higgs field φ. In this section we discuss the monodromy weight filtration $W(\mathrm{Gr}_{k,b}(\underline{E})) = W(N_{k,b})$.

To understand the parabolic structure on $\mathrm{Gr}_{k,b}(\underline{E})$ focus on a points $p \in D_{\cap k}$. Note that for each such point there is an index $j \neq k$ such that $p \in D_j \cap D_k$. Given a vector α of parabolic levels for these indices, then we obtain a bundle $E^{(D_k)}_{\alpha,b}$ in a tubular neighborhood of D_k. Near a point $p \in D_j \cap D_k$, the divisor D_j contains a piece transverse to D_k at p and we can use the parabolic structure of $\underline{E}$ with level α_p for such a piece of D_j, and level b along D_k.

Define

$$\mathrm{Gr}_{k,b}(\underline{E})_\alpha := E^{(D_k)}_{\alpha,b} / E^{(D_k)}_{\alpha,b-\epsilon}.$$

Assuming that the original bundle was locally abelian, then $\mathrm{Gr}_{k,b}(\underline{E})$ will be a locally abelian parabolic bundle on D_k with respect to the divisor $D_{\cap k}$.

The Higgs field

$$\varphi : E_\beta \to E_\beta \otimes \Omega^1_X(\log D)$$

induces a map

$$N_{k,b} := \mathrm{res}_b\, \varphi : \mathrm{Gr}_{k,b}(\underline{E}) \to \mathrm{Gr}_{k,b}(\underline{E}).$$

This is a map of parabolic bundles on D_k since φ respects the parabolic structure of $\underline{E}$. Our Nilpotence Assumption tells us that $N_{k,b}$ is nilpotent.

Using the basic fact that if $p, p' \in D_k^* := D_k - D_{\cap k}$, then $N_{k,b}(p)$ and $N_{k,b}(p')$ are conjugate as nilpotent endomorphisms of a vector space, we can prove

Proposition 4.2.1 (Donagi-Pantev-Simpson [7]). *Over D_k^* for any real number b, there is a weight filtration $W(N_{k,b})$ of the vector bundle $\mathrm{Gr}_{k,b}(\underline{E})$ with respect to the $N_{k,b}$ such that the restriction of this filtration to any point p is the weight filtration of $N_{k,b}(p)$.*

According to Lemma 3.4 in [7], this property extends to the normal crossings points too. As a result, we obtain weight filtrations $W(\mathrm{Gr}_{k,b}(\underline{E})) := W(N_{k,b})$ of the parabolic vector bundles $\mathrm{Gr}_{k,b}(\underline{E})$ over D_k, with parabolic structure along $D_{\cap k}$. These are filtrations by strict parabolic subbundles.

4.3 The L^2 parabolic Dolbeault complex

We proceed to define a complex on X as follows. For any divisor component D_k, for any parabolic level multi-index β, put

$$\mathrm{Gr}_{k,\beta_k}(E_\beta) := \mathrm{Gr}_{k,\beta_k}(\underline{E})_{\beta(\cap k)},$$

where $\beta(\cap k)$ consists of the coordinates of β for the components of $D_{\cap k}$. We have explicitly that

$$\mathrm{Gr}_{k,\beta_k}(E_\beta) = E_\beta / E_{\beta - \epsilon\delta_k}.$$

In the previous section we defined the weight filtration $W(\mathrm{Gr}_{k,\beta_k}(\underline{E}))$ of the parabolic bundle $\mathrm{Gr}_{k,\beta_k}(\underline{E})$ on D_k. By assigning parabolic levels $\beta(\cap k)$ on $D_{\cap k}$ this gives a weight filtration of the bundle $\mathrm{Gr}_{k,\beta_k}(E_\beta)$, and we call that $W(\mathrm{Gr}_{k,\beta_k}(E_\beta))$.

Denote by

$$W(k, E_\beta) \subset E_\beta$$

the pullback of the weight filtration $W(\mathrm{Gr}_{k,\beta_k}(E_\beta))$ over D_k, to a filtration on E_β by locally free subsheaves, via the map

$$E_\beta \to \mathrm{Gr}_{k,\beta_k}(E_\beta).$$

Let $W(H, E_\beta)$ denote the weight filtration obtained by using $W(h(j), E_\beta)$ along each horizontal component $D_{h(j)}$. More precisely, we use the weight filtration as we have defined in the previous section on the parabolic bundle $\mathrm{Gr}_{k,\beta_k}(\underline{E})$, and take the resulting weight filtration on the piece $\mathrm{Gr}_{k,\beta_k}(E_\beta) = \mathrm{Gr}_{k,\beta_k}(\underline{E})_{\beta(\cap k)}$ of this parabolic bundle.

For any real number a, let $\alpha(a)$ denote the parabolic level for the divisor D determined by using parabolic level a along the vertical components and parabolic level 0 along the horizontal components. We then obtain the levels of the horizontal weight filtrations

$$W_\ell(H, E_{\alpha(a)}) \subset E_{\alpha(a)}.$$

Note that since along the horizontal divisor components $D_{h(j)}$ we have $\alpha(a)_{h(j)} = 0$ so the horizontal weight filtrations come from filtrations on the parabolic level zero graded pieces $\mathrm{Gr}_{h(j),0}(E_{\alpha(a)})$.

Definition 4.3.1 (Donagi-Pantev-Simpson [7])**.** The relative L^2 parabolic Dolbeault complex is

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)}) := \left[\ W_0(H, E_{\alpha(a)}) \xrightarrow{\varphi} W_{-2}(H, E_{\alpha(a)}) \otimes_{\mathcal{O}_X} \Omega^1_{X/Y}(\log D)\ \right],$$

where $W_0(H, E_{\alpha(a)})$ sits at degree 0, and $\Omega^1_{X/Y}(\log D) = \Omega^1_X(\log D)/f^*\Omega^1_Y(\log Q)$ is the sheaf of relative logarithmic one forms along the fibers of f.

The following theorem explains that the higher direct images of the relative L^2 parabolic Dolbeault complex yields the "correct" pushforward of the parabolic Higgs bundle on X in the sense that they correspond via the NAHC to the pushforwards of the corresponding local system.

Theorem 4.3.2 (Donagi-Pantev-Simpson [7])**.** *Let*

$$F^i_a := \mathbb{R}^i f_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)})).$$

1. *The F^i_a are locally free, and fit together as a varies into a parabolic bundle $\underline{F}^i$.*
2. *Formation of the higher direct images is compatible with base-change, in other words $F^i_a(y)$ is the cohomology of the fiber over $y \in Y$.*
3. *The parabolic bundle $\underline{F}^i$ has a Higgs field θ given by the usual Gauss-Manin construction (see below), making it into a tame parabolic Higgs bundle.*
4. *This parabolic Higgs bundle on (Y, Q) is the one associated to the middle perversity higher direct image (of degree $i = 0, 1, 2$) of the local system underlying our original harmonic bundle.*

5. *More specifically, over $Y - Q$, the bundle $\underline{F}^i$ has a harmonic metric given by the L^2 metric on cohomology classes in the fibers, and the parabolic Higgs structure is the one associated to this harmonic metric.*

To give the definition of the Higgs field for the parabolic bundle $\underline{F}^i$ in the above theorem, we use the analogous construction as we did in Section 3.1. More precisely, we need to construct the absolute Dolbeault complex $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)})$ on X, which is presumably a subcomplex of the Dolbeault complex of $E_{\alpha(a)}$:

$$E_{\alpha(a)} \xrightarrow{\wedge\varphi} E_{\alpha(a)} \otimes \Omega^1_X(\log D) \xrightarrow{\wedge\varphi} E_{\alpha(a)} \otimes \Omega^2_X(\log D).$$

We use the same notation as in [7] to denote $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)})$ by

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)}) = \left[\begin{array}{c} W_{-2,0}(H, E_{\alpha(a)}) \\ \downarrow{\scriptstyle \wedge\varphi} \\ W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^1_X(\log D)) \\ \downarrow{\scriptstyle \wedge\varphi} \\ W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D)) \end{array}\right] \tag{4.3.1}$$

We fit the absolute Dolbeault complex into the short exact sequence of complexes

$$\begin{array}{c} 0 \\ \downarrow \\ \mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)})[-1] \otimes f^*\Omega^1_Y(\log Q) \\ \downarrow \\ \mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)})/I^2(E_{\alpha(a)}) \\ \downarrow \\ \mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)}) \\ \downarrow \\ 0. \end{array} \tag{4.3.2}$$

Here, we again define the subcomplexes $I^k(E_{\alpha(a)})$ formally as below:

$$\begin{aligned} I^0(E_{\alpha(a)}) &= \mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)}), \\ I^{k+1}(E_{\alpha(a)}) &= \text{image of } I^k(E_{\alpha(a)})[-1] \otimes f^*\Omega^1_Y(\log Q) \text{ in } \mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)}). \end{aligned}$$

From (4.3.2) we see immediately that we must have

$$W_{-2,0}(H, E_{\alpha(a)}) = W_0(H, E_{\alpha(a)}).$$

Consequently, $W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^1_X(\log D))$ must fit into the short exact sequence:

$$\begin{array}{c} 0 \\ \downarrow \\ W_0(H, E_{\alpha(a)}) \otimes f^*\Omega^1_Y(\log Q) \\ \downarrow \\ W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^1_X(\log D)) \\ \downarrow \\ W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \\ \downarrow \\ 0. \end{array}$$

To find $W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^1_X(\log D))$, we use the short exact sequence on X:

$$0 \longrightarrow f^*\Omega^1_Y(\log Q) \longrightarrow \Omega^1_X(\log D) \longrightarrow \Omega^1_{X/Y}(\log D) \longrightarrow 0.$$

and take the tensor product with $W_0(H, E_{\alpha(a)})$ to get

$$\begin{array}{c} 0 \\ \big\downarrow \\ W_0(H, E_{\alpha(a)}) \otimes f^*\Omega^1_Y(\log Q) \\ \big\downarrow \\ W_0(H, E_{\alpha(a)}) \otimes \Omega^1_X(\log D) \\ \big\downarrow \\ W_0(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \\ \big\downarrow \\ 0. \end{array} \tag{4.3.3}$$

Now we define $W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^1_X(\log D))$ to be the preimage of $W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D)$ in $W_0(H, E_{\alpha(a)}) \otimes \Omega^1_X(\log D)$ by the natural inclusion

$$W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \subset W_0(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D).$$

For $W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D))$, since $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)})$ has no degree 2 term, we see that $W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D))$ must fit into the short exact sequence:

$$\begin{array}{c} 0 \\ \big\downarrow \\ \text{image of } W_0(H, E_{\alpha(a)}) \otimes f^*\Omega^1_Y(\log Q) \otimes f^*\Omega^1_Y(\log Q) \text{ in } W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D)) \\ \big\downarrow \\ W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D)) \\ \big\downarrow \\ W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^1_{X/Y}(\log D) \otimes f^*\Omega^1_Y(\log Q) \\ \big\downarrow \\ 0. \end{array} \tag{4.3.4}$$

Note that the first term in (4.3.4) must be 0, and we have

$$\Omega^1_{X/Y}(\log D) \otimes f^*\Omega^1_Y(\log Q) \simeq \Omega^2_X(\log D).$$

Therefore, we conclude that

$$W_{-2,0}(H, E_{\alpha(a)} \otimes \Omega^2_X(\log D)) = W_{-2}(H, E_{\alpha(a)}) \otimes \Omega^2_X(\log D).$$

Now, we have obtained the absolute Dolbeault complex $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X, E_{\alpha(a)})$ and by analogy the push forward of (4.3.2) by f yields the Higgs field of $\underline{F}^i$:

$$\begin{tikzcd}
\mathbb{R}^i f_* \mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)}) \arrow[r, "\theta"] \arrow[d, equal] & \mathbb{R}^{i+1} f_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, E_{\alpha(a)})[-1] \otimes f^*\Omega^1_Y(\log Q)) \arrow[d, equal] \\
F^i_a & F^i_a \otimes \Omega^1_Y(\log Q).
\end{tikzcd}$$

To help understand the L^2 parabolic Dolbeault complex, we briefly look at the analytic motivation behind its definition (See [7] for details).

Consider an open fiber $X^\circ_y = (X - D)_y$ for $y \in Y - Q$. Give it a metric that is asymptotically the Poincaré metric near puncture points. We are interested in the L^2 cohomology of the harmonic bundle $(\mathcal{L}, \boldsymbol{D'}, \boldsymbol{D''}, h)$ restricted to this fiber. This is the cohomology of the complex of the complex of forms with coefficients in $\mathcal{L}$ that are L^2, and whose derivative is L^2.

The main fact is that the L^2 Dolbeault cohomology is isomorphic to the hypercohomology of the complex on X_y consisting of holomorphic forms whose restriction to X°_y in in L^2 and whose differential is also in L^2. A standard estimate from [20] implies that a holomorphic section of $E|_{X^\circ_y}$ (resp. $E|_{X^\circ_y} \otimes \Omega^1_{X^\circ_y}$) can be in L^2

only if it extends to a section of the 0-component of the parabolic structure E_0 ($E_0 \otimes \Omega^1_{X_y}(\log D_y)$).Furthermore, if it is in E_a (respectively $E_a \otimes \Omega^1_{X_y}(\log D_y)$) for $a < 0$ then it is automatically L^2.

Suppose e is a section of $(E_0)_y$ near a point p on the horizontal divisor component D_k and denote by $\mathrm{gr}_{k,0}(e)$ its projection in $\mathrm{Gr}_{k,0}(\underline{E})_y$. Suppose this projection is in W_ℓ but not $W_{\ell-1}$. Then, denoting by z a coordinate on X_y vanishing at p, the norm of e is asymptotically

$$|e| \sim |\log |z||^{\ell/2}.$$

Calculation with the Poincare metric for the norms of sections tells us that e is in L^2 if and only if $\ell \leq 0$. Similarly, a section $e\frac{dz}{z}$ of $E_0 \otimes \Omega^1_{X_y}(\log D_y)$ is in L^2 if and only if

$$\mathrm{gr}_{k,0}(e) \in W_{-2}\,\mathrm{Gr}_{k,0}(\underline{E}).$$

Chapter 5

Direct images of Higgs bundles revisited

In this chapter, we analyze the spectral data for the pushforward of a parabolic Higgs bundle $(\underline{E}, \varphi)$ on X that has both vertical and horizontal parabolic divisors. Our strategy is to first use the root stack construction to remove the parabolic structure by converting $(\underline{E}, \varphi)$ to a meromorphic Higgs bundle $(\widetilde{E}, \widetilde{\varphi})$ on the orbifold $\widetilde{X}$. The Higgs data $(\widetilde{E}, \widetilde{\varphi})$ is then encoded in a coherent sheaf (the spectral sheaf $\widetilde{\mathcal{E}}$) over the total space of the logarithmic cotangent bundle of $\widetilde{X}$. After that we show how to describe the spectral sheaves for the higher direct images of $(\underline{E}, \varphi)$ in terms of $\widetilde{\mathcal{E}}$.

To start, we first study the case that $f : X \to Y$ is smooth and we only have horizontal parabolic divisors. To do this, we look at the cohomology of $(\underline{E}, \varphi)$ along each fiber of f. Note that the fiber of f is a smooth curve, and the restriction of $(\underline{E}, \varphi)$ to each fiber will be a parabolic Higgs bundle with several poles at the intersections with those horizontal divisors. To reduce complexity in the notation we will focus on the case when there is only one pole p in the fiber, that is when the horizontal divisor is a section. The general case easily reduces to this one by a base change.

5.1 Spectral data on the orbicurve

Suppose that we fix a parabolic Higgs bundle $(\underline{E}, \varphi)$ on a smooth curve C and the parabolic structure is given at $p \in C$. We will calculate algebraically the L^2 cohomology of $(\underline{E}, \varphi)$ and describe it using the spectral sheaf. Denote the root stack of (C, p) by $(\widetilde{C}, \widetilde{p})$. We will write $\mu : \widetilde{C} \to C$ and $\widetilde{\pi} : \mathrm{tot}(\Omega^1_{\widetilde{C}}) \to \widetilde{C}$ for the natural projections, and we will denote the composite map $\mu \circ \widetilde{\pi}$ by π. Thus we have the basic diagram:

$$\begin{tikzcd} \mathrm{tot}(\Omega^1_{\widetilde{C}}(\widetilde{p})) \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & \widetilde{C} \arrow[d, "\mu"] \\ & C \end{tikzcd} \tag{5.1.1}$$

We start with a simple case where $(\underline{E}, \varphi)$ is strongly parabolic.

Proposition 5.1.1. *Suppose $(\underline{E}, \varphi)$ is strongly parabolic. Then the parabolic L^2 Dolbeault complex of $(\underline{E}, \varphi)$ is isomorphic to the complex*

$$\pi_* \widetilde{\mathcal{E}} \xrightarrow{\pi_*(\otimes \lambda)} \pi_*(\widetilde{\mathcal{E}} \otimes \widetilde{\pi}^* \Omega^1_{\widetilde{C}}).$$

Proof. Since $(\underline{E}, \varphi)$ is strongly parabolic, the residue map on $\mathrm{Gr}_{p,0}(\underline{E})$ is trivial. Hence we find that $W_0E_0 = E_0$ and $W_{-2}E_0 = E_{-\epsilon}$. By our root stack construction, we have $E_0 = \mu_*(\widetilde{E})$ and

$$\begin{aligned} E_{-\epsilon} \otimes \Omega^1_C(p) &= F_{\ell-1} \otimes \Omega^1_C \\ &= \mu_*(\widetilde{E} \otimes \widetilde{L}^{\ell-1}) \otimes \Omega^1_C \\ &= \mu_*(\widetilde{E} \otimes \widetilde{L}^{\ell-1} \otimes \mu^*\Omega^1_C) \\ &= \mu_*(\widetilde{E} \otimes \Omega^1_{\widetilde{C}}). \end{aligned}$$

On the other hand, since $\widetilde{\pi}_*\widetilde{\mathcal{E}} = \widetilde{E}$, we have $\pi_*\widetilde{\mathcal{E}} = \mu_*(\widetilde{\pi}_*\widetilde{\mathcal{E}}) = \mu_*\widetilde{E}$ and using the projection formula we have

$$\pi_*(\widetilde{\mathcal{E}} \otimes \widetilde{\pi}^*\Omega^1_{\widetilde{C}}) = \mu_*(\widetilde{\pi}_*(\widetilde{\mathcal{E}} \otimes \widetilde{\pi}^*\Omega^1_{\widetilde{C}})) = \mu_*(\widetilde{\pi}_*\widetilde{\mathcal{E}} \otimes \Omega^1_{\widetilde{C}}) = \mu_*(\widetilde{E} \otimes \Omega_{\widetilde{C}}).$$

This proves the proposition. □

Since all the maps in (5.1.1) have no higher direct images, this proposition tells us that

$$\begin{aligned} \mathbb{R}\Gamma(C, \mathrm{DOL}^{\mathrm{par}}_{L^2}(C, \underline{E})) &= \mathbb{R}\Gamma(\widetilde{C}, \widetilde{E} \to \widetilde{E} \otimes \Omega^1_{\widetilde{C}}) && (5.1.2) \\ &= \mathbb{R}\Gamma(\mathrm{tot}(\Omega^1_{\widetilde{C}}), \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}} \otimes \widetilde{\pi}^*\Omega^1_{\widetilde{C}}) && (5.1.3) \end{aligned}$$

Next, we consider the case where $(\underline{E}, \varphi)$ is tame parabolic with nilpotent residue but is not necessarily strongly parabolic. In this case the root stack construction converts $(\underline{E}, \varphi)$ into a meromorphic (non-parabolic) Higgs bundle $(\widetilde{E}, \widetilde{\varphi})$ with nilpotent residue on $\widetilde{C}$. To get the correct cohomology we again need to take into account the weight filtration of $\widetilde{E}$ by thinking of it as a parabolic Higgs bundle with the trivial parabolic structure at $\widetilde{p}$.

Lemma 5.1.2. *The parabolic L^2 Dolbeault complex of $(\underline{E}, \varphi)$ is isomorphic to the complex*

$$\mu_*(W_0\widetilde{E}) \xrightarrow{\mu_*\widetilde{\varphi}} \mu_*(W_{-2}\widetilde{E} \otimes \Omega^1_{\widetilde{C}}(\widetilde{p})). \tag{5.1.4}$$

Proof. Recall from Section 2.4 that locally around $\widetilde{p}$, $\widetilde{E}$ decomposes into $\bigoplus_{j=0}^{\ell-1} M_j$ and we have for $0 \leq i \leq \ell - 1$

$$F_i = z^{-i}M_i,$$

where F_i is the parabolic filtration of $(\underline{E}, \varphi)$ in (2.4.1), and z is a local coordinate of the covering space of $\widetilde{C}$. In particular, locally we have $E_0 = M_0$ and $E_{-\epsilon} = F_{\ell-1}(-p) = zM_{\ell-1}$. Since the parabolic structure of $\widetilde{E}$ is trivial, we have $\widetilde{E}_{-\epsilon} = \widetilde{E}(-\widetilde{p}) = \bigoplus_{j=0}^{\ell-1} zM_j$. Hence, the $\boldsymbol{\mu}_\ell$ fixed part of $\widetilde{E}_{-\epsilon}$ is $zM_{\ell-1}$.

To finish the proof, we note that $\mathrm{res}_{\widetilde{p}}\,\widetilde{\varphi}$ preserves the decomposition $\bigoplus_{j=0}^{\ell-1} M_j$. Therefore, the weight filtration of $\widetilde{E}$ is the direct sum of the weight filtrations of each M_j. Since μ_* returns only $\boldsymbol{\mu}_\ell$ fixed sections, we see that

$$\mu_*(W_0\widetilde{E}) = W_0E_0, \quad \mu_*(W_{-2}\widetilde{E}) = W_{-2}E_0.$$

The proposition follows from using the projection formula on the right hand side of (5.1.4). $\square$

We need to interpret the complex (5.1.4) in terms of the spectral data of the meromorphic Higgs bundle $(\widetilde{E}, \widetilde{\varphi})$. The key is to identify a geometric filtration $F_\bullet$ on the spectral sheaf $\widetilde{\mathcal{E}}$ of $\widetilde{E}$ which induces the monodromy weight filtration $W_\bullet$ on $\widetilde{E}$ by $\widetilde{\pi}_*$. Since this problem in fact has nothing to do with the orbicurve structure, we will simplify notation and work under the assumption that (E, φ) is a meromorphic Higgs bundle on C with a pole at p and has a nilpotent residue. For simplicity, we assume further that the spectral cover $S = \mathrm{Supp}(\mathcal{E})$ is smooth and the residue of φ is regular at p. This is to say, $\mathrm{res}_p\, \varphi$ is nilpotent and has only one Jordan block, and the projection from S to C is totally ramified at p of degree $r = \mathrm{rank}\ E$.

Let $q \in S$ be the point that maps down to p. Recall that the spectral sheaf satisfies $\pi_*\mathcal{E} = E$. Choose a pair of local coordinates (x, y) for $\mathrm{tot}(\Omega^1_C(p))$ around q by taking x to be the local coordinate along the zero section and y along the vertical direction such that S is locally given by the equation $x = y^r$. At p we see that

$$\begin{aligned} E_p &= \pi_*\mathcal{E} \otimes \mathcal{O}_C/m_p \\ &= \pi_*(\mathcal{E} \otimes \pi^*(\mathcal{O}_C/m_p)) \\ &= \pi_*(\mathcal{E}|_{\pi^{-1}(0)}). \end{aligned}$$

$\mathcal{E}$ is locally represented by a finitely generated $\mathbb{C}[x, y]$-module M, which is supported over the curve $x = y^r$. Hence, we have $\pi_*(\mathcal{E}|_{\pi^{-1}(0)}) \simeq M/xM$. The residue of the Higgs field φ at p is therefore equal to

$$M/xM \xrightarrow{\times y} M/xM.$$

Since $\mathcal{E}$ is flat over C, we can take M to be a free $\mathbb{C}[x]$-module of rank r. By our assumption that $\operatorname{res}_p \varphi$ is regular, we may choose a set of $\mathbb{C}[x]$-basis $\{e_1, \cdots, e_r\}$ for M with the property that the multiplication by y maps e_i to e_{i+1}. It follows that the monodromy weight filtration of M/xM is equal to the filtration

$$W_i(M/xM) := \begin{cases} y^{\frac{r-1}{2}-\lfloor \frac{i}{2} \rfloor} M/xM & \text{if } r \text{ is odd;} \\ y^{\frac{r}{2}-\lfloor \frac{i+1}{2} \rfloor} M/xM & \text{if } r \text{ is even.} \end{cases}$$

Therefore, we have proved

Proposition 5.1.3. *Let $W_\bullet$ be the filtration on $\mathcal{E}$ defined by taking the preimage of $W_i(M/xM)$ in the quotient map $M \to M/xM$. Then the complex (5.1.4) is isomorphic to the following complex:*

$$\pi_*(W_0\mathcal{E}) \xrightarrow{\pi_*(\otimes\lambda)} \pi_*(W_{-2}\mathcal{E} \otimes \pi^*\Omega^1_C(p)). \tag{5.1.5}$$

Put $t = (r-1)/2$ if r is odd and $t = r/2$ if r is even. Let R be the intersection of the spectral cover S with the zero section Σ of $\operatorname{tot}(\Omega^1_C(p))$.

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, hook, "j"] & S \arrow[d, hook, "\iota"] \arrow[dr, "\rho"] & \\ \Sigma \arrow[r, hook, "i"] & \operatorname{tot}(\Omega^1_C(p)) \arrow[r, "\pi"] & C \end{tikzcd}$$

Let $\mathcal{F} = \iota^*\mathcal{E}$. Then we have $\iota_*\mathcal{F} = \mathcal{E}$. From our previous discussion we have seen that

$$W_0\mathcal{E} = \iota_*(\mathcal{F} \otimes \mathcal{O}_S(-tq))$$

and

$$W_{-2}\mathcal{E} = \iota_*(\mathcal{F} \otimes \mathcal{O}_S(-(t+1)q)).$$

Since we will use the derived projection formula in the following calculation, we state the formula before we proceed.

Theorem 5.1.4 (Projection formula)**.** *For any $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{G}^\bullet \in \mathcal{D}^b(Y)$, we have*

$$\mathbb{R}f_!\mathcal{F}^\bullet \overset{\mathbb{L}}{\otimes} \mathcal{G}^\bullet \simeq \mathbb{R}f_!(\mathcal{F}^\bullet \overset{\mathbb{L}}{\otimes} f^*\mathcal{G}^\bullet),$$

where $f_!$ is the direct image with compact supports functor, i.e.

$$f_!\mathcal{F}(U) := \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \text{ such that } f|_{\mathrm{Supp}(s)} : \mathrm{Supp}(s) \to U \text{ is proper }\}.$$

In the case that f is a closed immersion, we have $\mathbb{R}f_! = f_*$. We can now calculate the cohomology of (5.1.5) as follows:

$$\begin{aligned}
& \mathbb{R}\Gamma\pi_*(W_0\mathcal{E} \xrightarrow{\otimes\lambda} W_{-2}\mathcal{E} \otimes \pi^*\Omega^1_C(p)) \\
= \ & \mathbb{R}\Gamma\rho_*(\mathcal{F} \otimes \mathcal{O}_S(-tq) \xrightarrow{\otimes\lambda} \mathcal{F} \otimes \mathcal{O}_S(-(t+1)q) \otimes \rho^*\Omega^1_C(p)) \\
= \ & \mathbb{R}\Gamma\rho_*(\mathcal{F} \overset{\mathbb{L}}{\otimes} (\mathcal{O}_S(-(t+1)q) \otimes \rho^*\Omega^1_C(p)|_R)[-1]) \\
= \ & \mathbb{R}\Gamma\pi_*(\mathcal{E} \overset{\mathbb{L}}{\otimes} \iota_*(\mathcal{O}_S(-(t+1)q) \otimes \rho^*\Omega^1_C(p)|_R)[-1]) \\
= \ & \mathbb{R}\Gamma(\pi \circ i)_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} j_*(\mathcal{O}_S(-(t+1)q) \otimes \rho^*\Omega^1_C(p)|_R)[-1]) \qquad (5.1.6) \\
= \ & \mathbb{R}\Gamma(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_R[-1]), \qquad (5.1.7)
\end{aligned}$$

where in the last step we denote $j_*(\mathcal{O}_S(-(t+1)q) \otimes \rho^*\Omega^1_C(p)|_R)$ by ω_R.

Remark 5.1.5. Recall that in Theorem 3.2.1, we showed that the the spectral sheaf of the higher image of a holomorphic Higgs bundle (E, φ) is given by $\mathbb{R}g_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_{Z/T^*_Y}[-d])$. In the special case that $X = C$ and $Y = pt$, $f^*T^*_Y$ degenerates to the zero section Σ. We can check that (5.1.6) gives the same result as Theorem 3.2.1. For a meromorphic Higgs bundle on C, (5.1.6) is very similar to (3.2.5).

5.2 The relative case

5.2.1 The case with no vertical divisors

Suppose now $f : X \to Y$ is a smooth map from a surface to a curve. $(\underline{E}, \varphi)$ is a parabolic Higgs bundle with parabolic structure along some horizontal divisor $D_H = \amalg D_i$.

We can apply the root stack construction on X with respect to $(\underline{E}, D_H)$ to obtain a orbisurface $\widetilde{X}$ and a map $\mu : \widetilde{X} \to X$. Note that since D_H is horizontal, along each fiber $X_y = f^{-1}(y)$, $\widetilde{X}|_{X_y}$ is equal to the orbicurve of X_y with respect to the restriction of $(\underline{E}, D_H)$ to X_y. This tells us that passage to the root stack does not alter the horizontal differential forms, that is the differential forms that are pullback from Ω^1_Y. On the other hand, from Section 2.4 it follows that the differentials in the vertical direction satisfy

$$\Omega^1_{\widetilde{X}/Y}(\log \widetilde{D_H}) = \mu^*(\Omega^1_{X/Y}(\log D_H)). \tag{5.2.1}$$

Let $(\widetilde{E}, \widetilde{\varphi})$ be the meromorphic Higgs bundle on $\widetilde{X}$ corresponding to $(\underline{E}, \varphi)$ via the root stack construction. From (5.2.1) and the proof of Lemma 5.1.2, we see that the parabolic L^2 Dolbeault complex for $(\widetilde{E}, \widetilde{D_H})$ is the pushforward of the parabolic L^2 Dolbeault complex for $(\underline{E}, D_H)$:

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, \underline{E}) = \mu_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(\widetilde{X}/Y, \widetilde{E})). \tag{5.2.2}$$

Since μ has no higher direct images, (5.2.2) implies that $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, \underline{E})$ and $\mathrm{DOL}^{\mathrm{par}}_{L^2}(\widetilde{X}/Y, \widetilde{E})$ will have the same hyperdirect images to Y. Furthermore, we can check that the pushforward of Higgs fields are also equal:

Proposition 5.2.1. *The following two short exact sequences of complexes are isomorphic:*

$$\begin{array}{ccc}
(1) & \qquad & (2) \\
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, \underline{E})[-1] \otimes f^*\Omega^1_Y & & \mu_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(\widetilde{X}/Y, \widetilde{E})[-1] \otimes f^*\Omega^1_Y) \\
\downarrow & & \downarrow \\
\mathrm{DOL}^{\mathrm{par}}_{L^2}(X, \underline{E})/I^2(\underline{E})) & & \mu_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(\widetilde{X}, \widetilde{E}))/I^2(\widetilde{E})) \\
\downarrow & & \downarrow \\
\mathrm{DOL}^{\mathrm{par}}_{L^2}(X/Y, \underline{E}) & & \mu_*(\mathrm{DOL}^{\mathrm{par}}_{L^2}(\widetilde{X}/Y, \widetilde{E})) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Proof. Note that since X is a surface $I^2(\underline{E}) = I^2(\widetilde{E}) = 0$. From Section 4.3 we have seen that

$$\mathrm{DOL}^{\mathrm{par}}_{L^2}(X,\underline{E}) = \left[\begin{array}{c} W_0(H,\underline{E}) \\ \big\downarrow{\scriptstyle \wedge\varphi} \\ W_{-2}(H,\underline{E})\otimes\Omega^1_X(\log D_H) + W_0(H,\underline{E})\otimes f^*\Omega^1_Y \\ \big\downarrow{\scriptstyle \wedge\varphi} \\ W_{-2}(H,\underline{E})\otimes\Omega^2_X(\log D_H) \end{array}\right],$$

and $\mathrm{DOL}^{\mathrm{par}}_{L^2}(X,\widetilde{E})$ takes the same form. The claim that (1) and (2) are isomorphic now follows since (5.2.2) and the proof of Lemma (5.1.2) give a term by term isomorphism of these complexes while the Higgs fields coincide generically and intertwine the term by term isomorphisms, and so must be equal everywhere. $\square$

To summarize, we have shown that the higher direct images of the parabolic Higgs bundle $(\underline{E},\varphi)$ are equal to the higher direct images of the meromorphic Higgs bundle $(\widetilde{E},\widetilde{\varphi})$ on the orbisurface $\widetilde{X}$. Next, we shall find the spectral data for the higher direct images using the spectral data of $(\widetilde{E},\widetilde{\varphi})$. For the sake of simplicity, in the following we assume (E,φ) is already a meromorphic Higgs bundle along D_H and thus suppress those cumbersome tilde notations.

It is crucial to note that the monodromy weight filtration $W(H,E)$ is irrelevant to the component of the Higgs field φ with respect to the horizontal differentials. We can extend globally the formula (5.16) which is valid along each fiber of f to X. As in (3.2.2), let Z be the pullback of T^*_Y by f. Put $\rho=\pi_X\circ\iota$, and we consider the following diagram:

$$
\begin{tikzcd}
 & S \arrow[r, hookleftarrow] \arrow[d, hook, "\iota"] & R \arrow[d, hook, "j"] & \\
 & \mathrm{tot}(\Omega^1_X(\log D_H)) \arrow[dr, "\pi_X"'] & Z \arrow[l, hook', "i"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T^*_Y \arrow[d, "\pi_Y"] \\
 & & X \arrow[r, "f"] & Y
\end{tikzcd}
\tag{5.2.3}
$$

Let $\overline{\varphi} : E \to E \otimes \Omega^1_{X/Y}(\log D_H)$ be the relative Higgs field induced from φ. In the following, we assume that the restriction of the $\Omega^1_{X/Y}(\log D_H)$-valued Higgs bundle $(E, \overline{\varphi})$ to each fiber X_y has a smooth spectral curve.

Theorem 5.2.2. *Suppose that the meromorphic Higgs bundle (E, φ) has nilpotent residues along a horizontal divisor D_H and the residue along each divisor component of D_H is regular, i.e. it has only one Jordan block. Let B be the preimage of D_H in the zero section of $\mathrm{tot}(\Omega^1_X(\log D_H))$. Denote the rank of E by r and put $t = \lfloor r/2 \rfloor$. Then the spectral sheaf of the i-th higher direct image of (E, φ) is given by*

$$
\mathbb{R}^i g_*(i^*\mathcal{E} \overset{\mathbb{L}}{\otimes} \omega_R[-1]), \tag{5.2.4}
$$

where $\omega_R = j_*(\mathcal{O}_S(-(t+1)B) \otimes \rho^*\Omega^1_{X/Y}(\log D)|_R)$.

Proof. This follows directly from the calculation in Section 5.1 of the cohomology of parabolic Higgs bundles along each fiber of f. $\square$

5.2.2 The general case

In this section, we consider the situation where we have both type of divisors: $D = D_V + D_H$.

Given a parabolic Higgs bundle $(\underline{E}, \varphi)$ with parabolic structure along D, applying the root stack construction along the horizontal divisor D_H, we again obtain a meromorphic Higgs bundle $(\widetilde{\underline{E}}, \widetilde{\varphi})$ on the orbisurface $\widetilde{X}$ with logarithmic poles along $\widetilde{D_H}$. In addition, $(\widetilde{\underline{E}}, \widetilde{\varphi})$ has a parabolic structure along $\widetilde{D_V}$, which is induced from the original parabolic Higgs bundle $(\underline{E}, \varphi)$. Thus at each parabolic level β along $\widetilde{D_V}$, we have a meromorphic Higgs bundle $(\widetilde{E}_\beta, \widetilde{\varphi})$ on $\widetilde{X}$ with a logarithmic Higgs field $\widetilde{\varphi} : \widetilde{E}_\beta \to \widetilde{E}_\beta \otimes \Omega^1_{\widetilde{X}}(\log \widetilde{D})$.

Note that the spectral sheaf for $(\widetilde{\underline{E}}, \widetilde{\varphi})$ is parabolic, that is to say, the usual spectral sheaf for $(\widetilde{E}_\beta, \widetilde{\varphi})$ at each parabolic level β forms a filtration of sheaves: $\widetilde{\mathcal{E}}_\beta \subseteq \widetilde{\mathcal{E}}_{\beta'}$ if $\beta \leq \beta'$. Furthermore, all these $\widetilde{\mathcal{E}}$ have the same support in $\mathrm{tot}(\Omega^1_{\widetilde{X}}(\log \widetilde{D}))$.

In the following, we will again assume that we work with the Higgs bundle after the root stack construction. Hence, we are given a meromorphic Higgs bundle $(\underline{E}, \varphi)$ on X with parabolic structure along a vertical divisor D_V and the Higgs field $\varphi : E_\beta \to E_\beta \otimes \Omega^1_X(\log D)$. Let S be the spectral cover of E_β for all β and Z be the pullback of $\mathrm{tot}(\Omega^1_Y(\log Q))$ by f.

$$
\begin{tikzcd}
S \arrow[d, hook, "\iota"] & R \arrow[l, hook'] \arrow[d, hook, "j"] & \\
\mathrm{tot}(\Omega^1_X(\log D)) \arrow[dr, "\pi_X"'] & Z \arrow[l, hook', "i"'] \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{tot}(\Omega^1_Y(\log Q)) \arrow[d, "\pi_Y"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
\tag{5.2.5}
$$

Under the same assumption as Theorem 5.2.2, we obtain the following result.

Theorem 5.2.3. *Assume that the residue on E_β along each horizontal divisor component is regular. Let $B \subset \mathrm{tot}(\Omega^1_X(\log D))$ be the preimage of D_H in the zero section of $\mathrm{tot}(\Omega^1_X(\log D))$. Then the spectral sheaf of the i-th higher direct image of $(\underline{E}, \varphi)$ is the parabolic sheaf formed by*

$$\mathbb{R}^i g_*(i^*\mathcal{E}_{\alpha(a)} \overset{\mathbb{L}}{\otimes} \omega_R[-1]), \tag{5.2.6}$$

where $\omega_R = j_*(\mathcal{O}_S(-(t+1)B) \otimes \rho^*\Omega^1_{X/Y}(\log D)|_R)$ and for each real number a, $\alpha(a)$ is the parabolic weight by using weight a along all divisor components of D_V.

Proof. This follows directly from our previous discussion and Theorem 4.3.2. $\square$

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