BAUM-CONNES CONJECTURE, FLAG VARIETIES AND REPRESENTATIONS OF SEMISIMPLE LIE GROUPS

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ABSTRACT

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This thesis consists of three chapters. In Chapter 1 we review the Baum-Connes conjecture and its relation with representation theory

In Chapter 2 we describe the equivariant K-theory of the real semisimple Lie group which acts on the (complex) flag variety of its complexification group. We construct an assembly map in the framework of KK-theory. Then we prove that it is an isomorphism. The prove relies on a careful study of the orbits of the real group action on the flag variety and then piecing together the orbits. This result can be considered as a special case of the Baum-Connes conjecture with coefficient.

In Chapter 3 we study the noncommutative Poisson bracket P on the classical family algebra $\mathcal{C}_\tau(\mathfrak{g})$. We show that P is the first-order deformation from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$ where the later is the quantum family algebra. We will prove that the noncommutative Poisson bracket is in fact a Hochschild 2-coboundary therefore the deformation is infinitesimally trivial.

In the appendix we talk about some further topics and open problems in this area.

Contents

Chapter 1

A Review of the Baum-Connes Conjecture and the Representations of Lie Groups

In this chapter we give a quick review of the Baum-Connes conjecture. Most of the materials in this chapter can be found in the book by A. Valette [46] or in the survey paper by N. Higson [22].

1.1 Group C^*-Algebra

Let G be a second countable, locally compact group. We denote by $L^1(G)$ the convolution algebra of complex value, integrable functions on G, where the multiplication is given by:

$$f * g(t) := \int_G f(s)g(s^{-1}t)\mathrm{d}s$$

here ds is the left invariant measure on G. Moreover, there is a natural involution on G

$$f^*(t) := \overline{f(t^{-1})}\Delta(t^{-1}),$$

which make it a Banach $*$-algebra. Here Δ is the modular function on G.

Definition 1.1.1. The *reduced* C^*-algebra of G is defined to be the completion of $L^1(G)$ via the regular representation on $L^2(G)$, and we denote the reduced C^*-algebra by $C_r^*(G)$.

It is well-known that $C_r^*(G)$ reflects the tempered unitary dual when G is a reductive Lie group, see [9], 4.1. Moreover, Vincent Lafforgue shows that $\mathrm{K}_*(C_r^*(G))$ is closely related to the discrete series of G, see [36].

1.2 Group C^*-Algebra and Index Theory

Suppose that M is a smooth, closed manifold with D a elliptic differential operator on it. D is a Fredholm operator and we can define the index of D

$$\mathrm{Ind}(\mathrm{D}) := \dim\ker\mathrm{D} - \dim\mathrm{coker}\mathrm{D}.$$

Now assume that G is a discrete group, if we have a homomorphism from $\pi_1(M)$ to G then we can define a more refined index valued in $\mathrm{K}^0(C_r^*(G))$.

In fact, let $\widetilde{M}$ be the universal covering space of M, we consider the quotient of $\widetilde{M} \times C_r^*(G)$ by the diagonal action of $\pi_1(M)$ and denote it by M_G. Clearly M_G is a flat bundle over M whose fibers are finitely generated projective modules over $C_r^*(G)$. Moreover, let D_G denote the natural lifting of D onto the sections of the bundle M_G. In good cases $\ker(\mathrm{D}_G)$ and $\mathrm{coker}(\mathrm{D}_G)$ are finitely generated projective $C_r^*(G)$-modules and we define

$$\mathrm{Ind}_G(\mathrm{D}) := [\ker(\mathrm{D}_G)] - [\mathrm{coker}(\mathrm{D}_G)] \in \mathrm{K}^0(C_r^*(G)). \tag{1.1}$$

In general cases we can perturbe D_G so that $\ker(\mathrm{D}_G)$ and $\mathrm{coker}(\mathrm{D}_G)$ become finitely generated and projective, and define $\mathrm{Ind}_G(\mathrm{D})$ accordingly.

1.3 A Geometric Construction of The Assembly Map and Baum-Connes Conjecture

Let X be a topological space. We consider the *K-homology* $\mathrm{K}_*(X)$, which is the dual of $\mathrm{K}^*(X)$. Baum and Douglas in 1982 ([6]) realized $\mathrm{K}_n(X)$ geometrically as equivalent classes of triples (M, E, f), where M is a spin^c, n dimensional closed manifold, E is an Hermitian vector bundle on M and $f : M \to X$ is a continuous map, for more details see [7].

Remark 1. We should fix the notations here. Following the standard notation, when X is a topological space, we use $\mathrm{K}^*(X)$ denote the K-theory of X and $\mathrm{K}_*(X)$ denote the K-homology of X. On the other hand, when A is a C^*-algebra, we use $\mathrm{K}_*(A)$ denote the K-theory of A and $\mathrm{K}^*(A)$ denote the K-homology of A.

If $X = BG$ the classifying space of G then since $\pi_1(BG) = G$, we have a map $f_* : \pi_1(M) \to G$. Now assume $\dim M$ is even, then the spin^c structure on M gives a Dirac operator on M with coefficient on E. Therefore according to last section, we obtain an index in $\mathrm{K}_0(C_r^*(G))$.

Definition 1.3.1. The above construction defines a map

$$\mu_{\mathrm{red}} : \mathrm{K}_*(BG) \to \mathrm{K}_*(C_r^*(G)) \tag{1.2}$$

and we call it the *assembly map*.

Remark 2. Roughly speaking, $\mathrm{K}_*(C_r^*(G))$ detects the connected components of $\hat{G}$, the dual of G, and the assembly map gives us a component of $\hat{G}$ from certain geometric data.

Now we can state the first form of the Baum-Connes conjecture

Conjecture 1 (Baum-Connes Conjecture for torsion free groups)**.** *If G is a discrete , torsion-free group, then the map $\mu_{red} : K_*(BG) \to K_*(C_r^*(G))$ is an isomorphism.*

Remark 3. If G has torsion, then the assembly map μ_{red} is not surjective, see [22].

Given the above remark, we need to modify the assembly map for general second-countable locally compact group G.

1.4 The Analytic K-homology

First we make the following definitions

Definition 1.4.1 ([4], Definition 1.3)**.** The action of G on X is *proper* if for every $x \in X$, there existsa triple (U, H, ρ) such that

1. U is an open neighborhood of x in X and $GU \subset U$,
2. H is a compact subgroup of G,
3. $\rho : U \to G/H$ is a G-map from U to the homogeneous space G/H.

For example, if G itself is compact then any G-space X is proper.

Among the proper G-spaces there is a universal one

Definition 1.4.2 ([4], Definition 1.6)**.** A proper G-space $\mathcal{E}G$ is called *universal* if for any proper G-space X, there exists a G-map $f : X \to \mathcal{E}G$, and any two such maps are G-homotopic.

Remark 4. If G is compact, then it is obvious that pt, considered as a G-space, is universal proper. For more about universal proper G-space for other groups G, see [4], Section 2.

Next we introduct equivariant K-homology

Definition 1.4.3. Let X be a proper G-space. A *generalized elliptic G-operator* over X is a triple (ϕ, π, F) where

- ϕ is a unitary representation of G on some Hilbert space $\mathcal{H}$,
- π is a $*$-representation of $C_0(X)$ by bounded operators on $\mathcal{H}$ which is compatible with ϕ,
- F is a bounded, self-adjoint operator on $\mathcal{H}$, which is G-invariant and such that the operators $\pi(f)(F^2 - 1)$ and $[\pi(f), F]$ are compact for all $f \in C_0(X)$.

The triple (ϕ, π, F) is also called a *cycle* over X.

Such a cycle (ϕ, π, F) is even if the Hilbert space $\mathcal{H}$ is $\mathbb{Z}_2$-graded, i.e. $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, and in this decomposition

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix}, \ \pi = \begin{pmatrix} \pi_0 & 0 \\ 0 & \pi_1 \end{pmatrix}, \ F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}.$$

A cycle is odd if $\mathcal{H}$ is ungraded.

Remark 5. A cycle encodes the Dirac operator in terms of operator algebras. In fact, if X is a spin^c-manifold with spinor bundle $\mathcal{S}$; D is a Dirac operator associated to a vector bundle E over X. Then we can take $\mathcal{H} = L^2(X, \mathcal{S} \otimes \mathcal{E})$. Obviously D is unbounded, but we can define

$$F = \frac{\mathrm{D}}{\sqrt{1+\mathrm{D}^2}}$$

using spectral theory and F has the properties in Definition 1.4.3.

We will use the cycles to build the K-homology.

Definition 1.4.4. 1. Let X be a proper G-space. A cycle over X is call degenerate if for any $f \in C_0(X)$ we have $[\pi(f), F] = 0$ and $\pi(f)(F^2 - 1) = 0$,

2. Two cycles $\alpha_0 = (\phi_0, \pi_0, F_0)$ and $\alpha_1 = (\phi_1, \pi_1, F_1)$ are said to be homotopic if $\phi_0 = \phi_1$, $\pi_0 = \pi_1$ and there exists a norm continuous path F_t connection F_0 and F_1 such that for each $t \in [0, 1]$, the triple (ϕ_0, π_0, F_t) is a cycle of the same parity,

3. Two cycles α_0 and α_1 are said to be equivalent (denoted $\alpha_0 \sim \alpha_1$) if there exists two degenerate cycles β_0 and β_1 such that $\alpha_0 \oplus \beta_0$ is homotopic to $\alpha_1 \oplus \beta_1$ up to unitary equivalence,

4. We define $\mathrm{K}_0^G(X)$ to be the set of equivalence classes of even cycles over X and $\mathrm{K}_1^G(X)$ to be the set of equivalence classes of odd cycles over X. In fact, $\mathrm{K}_0^G(X)$ and $\mathrm{K}_1^G(X)$ can be made into abelian groups.

Remark 6. For the equivalence of the geometric K-homology and the analytic K-homology, see [7].

Remark 7. Let α be an even cycle, if we consider it as an odd cycle by forgetting the $\mathbb{Z}_2$-grading, then it is not difficult to see that α is homotopic to a degenerate cycle, hence it is zero in $\mathrm{K}_1^G(X)$.

Remark 8. If G is discrete and torsion-free, then the universal proper G-space $\mathcal{E}G$ is the universal bundle EG and we have

$$\mathrm{K}_*^G(\mathcal{E}G) \cong \mathrm{K}_*(BG).$$

If G has torsion, it is not true. For example, when G is finite, $\mathcal{E}G$ is a point and is not G-homotopic to EG.

1.5 The Assembly Map

We can define a modified assembly map

$$\mu_{\text{red}} : \mathrm{K}_*^G(\mathcal{E}G) \to \mathrm{K}_*(C_r^*(G)). \tag{1.3}$$

The idea of the construction is similar to that in Section 1.3 but it is technically more complicated. For more details see [4] or [46], or Section 2.3 below.

Concerning the assembly map, we have the following conjecture:

Conjecture 2 (Baum-Connes Conjecture)**.** *The map $\mu_{red} : K_*^G(\mathcal{E}G) \to K_*(C_r^*(G))$ is an isomorphism.*

1.6 The Baum-Connes Conjecture with Coefficients

Let G be a locally compact topologicial group. For any G-C^*-algebras A and B we can define the equivariant KK-theory $\mathrm{KK}^G(A, B)$ as in [29] 2.4, which is a simultaneous generalization of both K-theory and K-homology.

Definition 1.6.1. Let A and B be two G-C^*-algebras. A *cycle* over (A, B) is a triple

$(\phi, \pi, \mathcal{F})$ where

- ϕ is a unitary representation of G on some Hilbert B-module $\mathcal{E}$, unitary in the sense that

$$(\phi(g)\xi|\phi(g)\eta) = g \cdot (\xi|\eta) \in B,$$

- π is a $*$-representation $A \to \mathcal{L}_B(\mathcal{E})$ which is compatible with ϕ,

- $\mathcal{F}$ is a self-adjoint operator in $\mathcal{L}_B(\mathcal{E})$, which is G-invariant and such that the operators $\pi(a)(\mathcal{F}^2 - 1)$, $[\pi(a), \mathcal{F}]$ and $[\phi(g), \mathcal{F}]$ are compact for all $a \in A$ and $g \in G$.

We also have even and odd cycles depending on whether or not it is $\mathbb{Z}_2$-graded.

Definition 1.6.2. 1. A cycle $(\phi, \pi, \mathcal{F})$ over (A, B) is call *degenerate* if $\pi(a)(\mathcal{F}^2 - 1)$, $[\pi(a), \mathcal{F}]$ and $[\phi(g), \mathcal{F}]$ are zero for all $a \in A$ and $g \in G$,

2. Two cycles $\alpha_0 = (\phi_0, \pi_0, \mathcal{F}_0)$ and $\alpha_1 = (\phi_1, \pi_1, \mathcal{F}_1)$ are said to be homotopic if $\phi_0 = \phi_1$, $\pi_0 = \pi_1$ and there exists a norm continuous path $\mathcal{F}_t$ connection $\mathcal{F}_0$ and $\mathcal{F}_1$ such that for each $t \in [0, 1]$, the triple $(\phi_0, \pi_0, \mathcal{F}_t)$ is a cycle of the same parity,

3. Two cycles α_0 and α_1 are said to be equivalent (denoted $\alpha_0 \sim \alpha_1$) if there exists two degenerate cycles β_0 and β_1 such that $\alpha_0 \oplus \beta_0$ is homotopic to $\alpha_1 \oplus \beta_1$ up to unitary equivalence,

4. We define $\mathrm{KK}_0^G(A, B)$ to be the set of equivalence classes of even cycles over (A, B) and $\mathrm{KK}_1^G(A, B)$ to be the set of equivalence classes of odd cycles over (A, B). In fact, $\mathrm{KK}_0^G(A, B)$ and $\mathrm{KK}_1^G(A, B)$ can be made into abelian groups.

Remark 9. If is easy to see that the K-theory $\mathrm{K}_G^*(A) \cong \mathrm{KK}_*^G(\mathbb{C}, A)$ and the K-homology $\mathrm{K}_*^G(A) \cong \mathrm{KK}_*^G(A, \mathbb{C})$.

Remark 10. If X and Y are two G-spaces, then we also denote $\mathrm{KK}_*^G(C_0(X), C_0(Y))$ simply by $\mathrm{KK}_*^G(X, Y)$.

Remark 11. An essential feature of KK-theory is the *Kasparov product*, which means we have a bi-linear map

$$\mathrm{KK}_i^G(A, B) \otimes \mathrm{KK}_j^G(B, C) \to \mathrm{KK}_{i+j}^G(A, C). \tag{1.4}$$

More generally we have

$$\mathrm{KK}_i^G(A, B\hat{\otimes}E) \otimes \mathrm{KK}_j^G(E\hat{\otimes}C, D) \to \mathrm{KK}_{i+j}^G(A\hat{\otimes}C, B\hat{\otimes}D). \tag{1.5}$$

Roughly speaking, the Kasparov product is defined by taking index, for the details of the construction, see [29] or [21].

In the framework of KK-theory, Baum-Connes conjecture can be restated as:

$$\mu_{\mathrm{red}} : \mathrm{KK}^G(\mathcal{E}G, \mathrm{pt}) \to \mathrm{K}_*(C_r^*(G))$$

is an isomorphism.

Now for a G-C^*-aglebra A, we can define a generalized assembly map (see [4] or [46], or Section 2.3 below)

$$\mu_{\mathrm{red},A} : \mathrm{KK}^G(S, A) \to \mathrm{K}_G(A) \tag{1.6}$$

Conjecture 3 (Baum-Connes Conjecture with coefficients in A)**.** *The map $\mu_{red,A}$ is an isomorphism.*

1.7 Baum-Connes Conjecture and the Representation of Lie Groups

Now let G be a connected Lie group and U be its maximal compact subgroup. Then it is easy to see that the homogeneous space G/U is the universal proper G-space $\mathcal{E}G$.

If we assume G/U is even-dimensional and has a G-equivariant spin structure (it always exists if we go to the double covering $\widetilde{G}$ of G), then we have the following map

$$\widetilde{\mu}_{\text{red}} : R(U) \to \mathrm{K}_*(C^*_r(G)). \tag{1.7}$$

where $R(G)$ is the representation ring of U. Each representation $[V] \in R(U)$ maps to the G-index of the twisted Dirac operator D_V on G/U. This is the *Connes-Kasparov Conjecture* for G, see [14].

In Chapter 2 below we will consider the case that G is a real semisimple Lie group and the Baum-Connes conjecture with coefficients in the *flag variety* of G. In Chapter 3 we will proceed from Baum-Connes conjecture to *Mackey's Analogue*, and then to the deformation of the *family algebras.*

Chapter 2

Flag Varieties and Equivariant K-Theory

2.1 Introduction

Let G be a locally compact topologicial group and A is a C^*-algebra equipped with a continuous action of G by C^*-algebra automorphisms. Following [9] section 4, we define the equivariant K-theory of A to be the K-theory of the reduced crossed product algebra:

$$\mathrm{K}_G^*(A) := \mathrm{K}_*(C_r^*(G, A)).$$

The equivariant K-theory defined in this way has a useful connection to Baum-Connes conjecture and representation theory. In particular, $\mathrm{K}_G^*(pt) = \mathrm{K}_*(C_r^*(G))$.

Be aware that this is not the same as Kasparov's definition [29].

When G is compact, this definition coincides with the usual equivariant K-theoy, see [27].

Back to general G. For any A and B We can define the equivariant KK-theory $\mathrm{KK}^G(A, B)$ as in [29] 2.4. Now we can state the Baum-Connes conjecture. Let U be its maximal compact subgroup and $S := G/U$ be the quotient space. We have the *assembly map* [22]

$$\mu_{\text{red}} : \mathrm{KK}^G(S, \mathrm{pt}) \to \mathrm{K}_G(\mathrm{pt}).$$

The *Baum-Connes conjecture* claims that the assembly map μ_{red} is an isomorphism. We notice that in 2003, J. Chabert, S. Echterhoff, R. Nest [13] proved this conjecture for almost connected and for linear p-adic group G.

More generally, we have the Baum-Connes conjecture conjecture with coefficient in A, which claims that the map

$$\mu_{\mathrm{red},A} : \mathrm{KK}^G(S,A) \to \mathrm{K}_G(A).$$

is an isomorphism. This is still an open problem for general G and A and, in fact there are counter examples for some certain G and A, see [25].

When G is a real semisimple Lie group, we can also consider the Baum-Connes conjecture with coefficient on flag varieties: Let $G_{\mathbb{C}}$ be the complexification of G, We have the flag variety $\mathcal{B}$ of $G_{\mathbb{C}}$. The group $G_{\mathbb{C}}$ (hence G and U) acts on $\mathcal{B}$, so we also have the assembly map

$$\mu_{\mathrm{red},\mathcal{B}} : \mathrm{KK}^G(S,\mathcal{B}) \to \mathrm{K}_G(\mathcal{B}). \tag{2.1}$$

The main result of this chapter is the following theorem:

Theorem 2.1.1. *For any real semisimple Lie group G, the assembly map*

$$\mu_{red,\mathcal{B}} : KK^G(S,\mathcal{B}) \to K_G(\mathcal{B})$$

is an isomorphism.

The proof of Theorem 2.1.1 in this chapter relies on a careful study of the orbits of the real group action on the flag variety: We first proof the isomorphism on one single orbit of the G-action and then piecing together the orbits. The proof does not require the hard techniques in KK-theory and representation thoery therefore it can be treat as an *elementary proof*.

This chapter is organized as follows: In Section 2.2 and 2.3 we construct the assembly map. In Section 2.4 we study the assembly map on one of the G-orbits of the flag variety. In Section 2.5 we study the G-orbits on $\mathcal{B}$ and in Section 2.6 we prove the Theorem 2.1.1. In Section 2.7 we give an example to illustrate the idea of this chapter.

This work is inspired by the study of equivariant K-theory in [9] the *Matsuki correspondence* in [39]. Hopefully it will be useful in the representation theory of real semisimple Lie groups.

2.2 Notations and First Constructions

We will use the following notations in this chapter

1. Let G be a connected real semisimple Lie group, U be the identity component of a maximal compact subgroup of G. In the sequel we fix such a U and call it *the* maximal compact subgroup of G.

2. We denote the space G/U by S.

3. Let $G_{\mathbb{C}}$ be the complexification of G, $B_{\mathbb{C}}$ be the Borel subgroup of $G_{\mathbb{C}}$ and $\mathcal{B}$ be the flag variety. We know that $\mathcal{B} \cong G_{\mathbb{C}}/B_{\mathbb{C}}$.

4. Let X be any space with continuous G-action. Let $C_0(X)$ be the space of

continuous functions on X which vanishes at infinity. If X is compact, then $C_0(X) = C(X)$ is the space of all continuous functions on X. We define

$$\mathrm{K}_G^*(X) := \mathrm{K}^*(C_r^*(G, C_0(X))). \tag{2.2}$$

$\mathrm{K}_U^*(X)$ can be defined in the same way. In particular we can define $\mathrm{K}_U^*(\mathcal{B})$ and $\mathrm{K}_G^*(\mathcal{B})$.

Obviously G acts on the flag variety $\mathcal{B}$. Unlike $G_{\mathbb{C}}$, the G-action is not transitive, see [39] and Section 2.5 of this chapter.

2.3 The Dirac-Dual Dirac Method and the Assembly Map

In this section we will construct the assembly map

$$\mu_{\mathrm{red},T} : \mathrm{KK}^G(S, T) \to \mathrm{K}_G(T) \tag{2.3}$$

for any G-space T. We work in the framework of Kasparov as in [29].

2.3.1 Poincare Duality in KK-theory

We have the *Poincare duality* isomorphism in KK-theory.

Theorem 2.3.1. *[[29] Theorem 4.10, see also [9] Section 4.3] For a G-manifold X, let $C_\tau(X)$ denote the algebra of continuous sections of the Clifford bundle over X*

vanishing at infinity. Then we have the following isomorphism

$$KK^G(X,T) \cong K_G(C_0(T) \otimes C_\tau(X)). \tag{2.4}$$

□

Let $X = S$. Under the Poincare duality, to get the assembly map, it is sufficient to construct a map

$$\mathrm{K}_G(C_0(T) \otimes C_\tau(S)) \to \mathrm{K}_G(T).$$

2.3.2 The Dirac Element and the assembly map

For G, X and $C_\tau(X)$ as in Theorem 2.3.1, Kasparov defines the *Dirac element* [29] 4.2:

$$d_{G,X} \in \mathrm{KK}^G(C_\tau(X), \mathbb{C}) \tag{2.5}$$

Definition 2.3.1 ([29] 4.2)**.** Let $\mathcal{H} = L^2(\wedge^*(X))$ be the Hilbert space of L^2-forms on X with the usual decomposition $\wedge^* = \wedge^{\mathrm{even}} \oplus \wedge^{\mathrm{odd}}$. Let ϕ be the natural action of G on $\mathcal{H}$. There is a homomorphism $\pi : C_\tau(X) \to L(\mathcal{H})$ defined on 1-forms by

$$\pi(\omega) = \epsilon_\omega + \epsilon_\omega^*$$

where ϵ_ω is the exterior multiplication by ω and ϵ_ω^* is its adjoint.

Moreover, let $\mathrm{D} = \mathrm{d} + \mathrm{d}^*$ and

$$F = \frac{\mathrm{D}}{\sqrt{1 + \mathrm{D}^2}}.$$

Then (ϕ, π, F) is a cycle over X and gives the element $d_{G,X} \in \mathrm{KK}^G(C_\tau(X), \mathbb{C})$.

Remark 12. We do not require that X is spin in the definition of $d_{G,X}$.

Now we want to find the relation between equivariant KK-theory and the K-theory of crossed-product algebras.

First remember we have the map

$$\sigma_T : \mathrm{KK}^G(A, B) \longrightarrow \mathrm{KK}^G(A \otimes C_0(T), B \otimes C_0(T)).$$

Apply this to $d_S \in \mathrm{KK}^G(C_\tau(S), \mathbb{C})$ we get

$$\sigma_T(d_S) \in \mathrm{KK}^G(C_\tau(S) \otimes C_0(T), C_0(T)).$$

Next, we know that Kasparov gives the following definition-theorem (Theorem 3.11 in [29])

Theorem 2.3.2. *There is a natural homomorphism*

$$j_r^G : KK^G(A, B) \longrightarrow KK(C_r^*(G, A), C_r^*(G, B))$$

which is compatible with the Kasparov product. Moreover, for $1_A \in KK^G(A, A)$

$$j_r^G(1_A) = 1_{C_r^*(G,A)} \in KK(C_r^*(G, A), C_r^*(G, A)).$$

□

Apply the map j_r^G to $\sigma_T(d_S)$ which is in $\mathrm{KK}^G(C_\tau(S) \otimes C_0(T), C_0(T))$ we get

$$j_r^G(\sigma_T(d_S)) \in \mathrm{KK}(C_r^*(G, C_\tau(S) \otimes C_0(T)), C_r^*(G, C_0(T))).$$

We denote $j_r^G(\sigma_T(d_S))$ by $\mathrm{D}_{G,S}$ or simply by D.

Definition 2.3.2 (The assembly map)**.** Let $S = G/U$, for any T, the Poincare duality and the Kasparov product in KK-theory gives us the desired map

$$\cdot \otimes \mathrm{D} : \mathrm{KK}^G(S,T) \cong \mathrm{K}_G(C_0(T) \otimes C_\tau(S)) \to \mathrm{K}_G(T). \tag{2.6}$$

Remark 13. As pointed out in remark 12, we do not require S to be spin to define the assembly map.

2.3.3 The Spin Case

Let us look at the spin case and get some intuition.

When S is spin and of even dimension, it is well known that $C_\tau(S))$ is strongly Morita equivalent to $C_0(S)$. Hence the Poincare duality gives us

$$\mathrm{KK}^G(S,T) \cong \mathrm{K}_G(T \times S). \tag{2.7}$$

In this case, the Dirac element $d_{G,S}$ is exactly the index map of the Dirac operator ([3]) and this justified the name "Dirac element". Therefore the assembly map is given by the index map

$$\mathrm{D} : \mathrm{K}_G(T \times S) \to \mathrm{K}_G(T) \tag{2.8}$$

We can look at $\mathrm{K}_G(T \times S)$ from another viewpoint. Remember that $S = G/U$. In fact we have the following general result

Lemma 2.3.3. *Let T be a G-space in the above setting. Then $G \times_H T$ is G-isomorphic* *to $G/H \times T$, where G acts on $G/H \times T$ by the diagonal action. Hence*

$$K_G^*(G/H \times T) \cong K_G^*(G \times_H T).$$

Proof: See [9], Section 2.5. In fact, both sides are quotient spaces of $G \times T$ by right actions of H. The point is that the actions are different.

For $G \times_H T$

$$(g, t) \cdot h := (gh, h^{-1}t).$$

For $G/H \times T$

$$(g, t) \circ h := (gh, t).$$

Then we see that the following map

$$G \times_H T \to G/H \times T$$
$$(g, t) \mapsto (g, gt)$$

intertwines the action $\cdot$ and $\circ$. Moreover it is equivariant under the left G-action. □

The following isomorphism is very natural, see [41]

Lemma 2.3.4 (The induction map)**.** *For any group G, $H \subset G$ a closed subgroup, and an H-space T, there is an induction map*

$$K_H^*(T) \to K_G^*(G \times_H T)$$

which is an natural isomorphism.

Here the G-action on $G \times_H T$ is the left multiplication on the first component.

Proof: Just notice that $C_r^*(H, C_0(T))$ and $C_r^*(G, C_0(G \times_H T))$ are Strongly Morita equivalent. $\square$

Remark 14. The sprit of Proposition 2.3.3 and 2.3.4 will appear later in Lemma 2.4.2.

Now let H be U, the maximal compact subgroup. According to Proposition 2.3.3 and 2.3.4, the assembly map in Definition 2.3.2 has the following form

$$\mathrm{D} : \mathrm{K}_U(T) \to \mathrm{K}_G(T). \tag{2.9}$$

The *Connes-Kasparov conjecture* claims that the above map is an isomorphism.

Remark 15. In the statement of the Connes-Kasparov conjecture we do not require G/U to be spin, see [40].

2.3.4 The Dual Dirac Element

We are looking for a inverse element of $d_{G,X}$. For this purpose Kasparov introduce the concept of *G-special manifold* in [29] 5.1. For G-special manifold X, there exists an element, called the *dual Dirac element*

$$\eta_{G,X} \in \mathrm{KK}^G(\mathbb{C}, C_\tau(X))$$

such that the Kasparov product

$$\mathrm{KK}^G(C_\tau(X), \mathbb{C}) \times \mathrm{KK}^G(\mathbb{C}, C_\tau(X)) \longrightarrow \mathrm{KK}^G(C_\tau(X), C_\tau(X))$$

gives

$$d_{G,X} \otimes_{\mathbb{C}} \eta_{G,X} = 1_{C_\tau(X)}.$$

When the group G is obvious, we will denote them by d_X and η_X.

Kasparov also showed that for the maximal compact subgroup U of G, the homogenous space G/U is a G-special manifold.

Remark 16. Although $d_{G,X} \otimes_{\mathbb{C}} \eta_{G,X} = 1_{C_\tau(X)}$ for special manifold, the Kasparov product in the other way

$$\eta_X \otimes_{C_\tau(X)} d_X$$

need not to be 1.

We denote $\eta_X \otimes_{C_\tau(X)} d_X$ by γ_X.

Remark 17. If $\gamma_X = 1$, then d_X and η_X are invertible elements under the Kasparov product.

2.3.5 When is the Dirac Element Invertible?

Kasparov proved that $\gamma_X = 1$ in some special cases, which is sufficient for our purpose. To state the result in full generality we need to introduce the concept of restriction homomorphism

Let $f : G_1 \to G_2$ be a homomorphism between groups, the restriction homomorphism

$$r^{G_2,G_1} : \mathrm{KK}^{G_1}(A, B) \longrightarrow \mathrm{KK}^{G_2}(A, B)$$

Proposition 2.3.5. *If X is a G_2 manifold and $f : G_1 \to G_2$ as above. Then under* *the map* r^{G_2,G_1} *we have*

$$\begin{aligned} r^{G_2,G_1}(d_{G_1,X}) =& d_{G_2,X} \\ r^{G_2,G_1}(\eta_{G_1,X}) =& \eta_{G_2,X} \\ r^{G_2,G_1}(\gamma_{G_1,X}) =& \gamma_{G_2,X}. \end{aligned}$$

□

If the manifold X is G/U, where U is the maximal compact subgroup of G, we denote $d_{G/U}$ by $d_{(G)}$, $\eta_{G/U}$ by $\eta_{(G)}$ and $\gamma_{G/U}$ by $\gamma_{(G)}$

Theorem 2.3.6 ([29], 5.9)**.** *Let* $f : G_1 \to G_2$ *be a homomorphism between* ***almost connected*** *groups with the kernel* $\ker f$ ***amenable*** *and the image closed. Then the restriction homomorphism gives us*

$$r^{G_2,G_1}(\gamma_{(G_2)}) = \gamma_{(G_1)}. \tag{2.10}$$

□

Corollary 2.3.7. *For any* G*, let* $H < G$ *be a closed subgroup. Then we have*

$$r^{G,H}(\gamma_{(G)}) = \gamma_{(H)}. \tag{2.11}$$

□

Corollary 2.3.8. $\gamma_{(G)} = 1$ *for every* ***amenable almost connected group*** G*.*

Proof: In Theorem 2.3.6, let $G_1 = G$ and G_2 be the trivial group, we know $\gamma_{(G_2)} = 1$ therefore $\gamma_{(G)} = 1$. □

Remark 18. Remember Remark 17, we know that if G is an amenable almost connected group, then $d_{G/U} = d_{(G)}$ and $\eta_{G/U} = \eta_{(G)}$ are invertible elements in the KK-groups.

Now we can immediately get a isomorphic result in the almost connected amenable case. The following result is implicitly given in [29],5. 10.

Theorem 2.3.9. *If P is an almost connected amenable group, L is the maximal compact subgroup of P, T is an P-space, then the assembly map*

$$\mu_{P,T} : KK^P(P/L, T) \to K_P(T) \tag{2.12}$$

is an isomorphism.

Proof: By Theorem 2.3.8 we know that $\gamma_{(P)} = 1$ hence $d_{(P)}$ is invertible. Since D in the definition of the assembly map (Definition 2.3.2) is obtained from $d_{(P)}$, and remember Theorem 2.3.2, invertible elements go to invertible elements. So $\mu_{P,T}$ is an isomorphism. □

2.4 The Assembly Map on a Single G-Orbit of the Flag Variety

According to [39], there are finitely many G-orbits on $\mathcal{B}$. Let us denote α^+ to be one of them. Let H be the isotropy group of G at a critical point (see [39] for the definition of critical points) $x \in \alpha^+$.

Remark 19. This notation will be justified in Section 2.5.

We want to prove

Proposition 2.4.1. *The assembly map*

$$\mu_{red,\alpha^+} : KK^G(S,\alpha^+) \to K_G(\alpha^+). \tag{2.13}$$

is an isomorphism.

Remark 20. Proposition 2.4.1 is the building block of the main theorem of this chapter-Theorem 2.1.1. We will piece together the blocks in Section 2.6.

The proof of Proposition 2.4.1 consists of several steps. First we prove the lemma

Lemma 2.4.2 (Interchange subgroups)**.** *There is an isomorphism:*

$$KK^G(S,\alpha^+) \xrightarrow{\sim} KK^H(S,pt).$$

Proof: First by Poincare duality

$$\mathrm{KK}^G(S,\alpha^+) \cong \mathrm{K}_G(C_0(\alpha^+) \otimes C_\tau(S)). \tag{2.14}$$

Then notice that α^+ can by identified with G/H. By a strong Morita equivalence argument similar to Lemma 2.3.4 we have

$$\mathrm{K}_G(C_0(\alpha^+) \otimes C_\tau(S)) \cong \mathrm{K}_H(C_\tau(S)). \tag{2.15}$$

Finally by Poincare duality again we have

$$\mathrm{K}_H(C_\tau(S)) \cong \mathrm{KK}^H(S,\mathrm{pt}). \tag{2.16}$$

We get our result. □

Now we are ready to obtain the following result

Proposition 2.4.3. *We have the following commuting diagram:*

$$\begin{array}{ccc} KK^G(S,\alpha^+) & \xrightarrow{\sim} & KK^H(S,pt) \\ \Big\downarrow{\scriptstyle \mu_{red,\alpha^+}} & & \Big\downarrow{\scriptstyle \mu_{red,pt}} \\ K_G(\alpha^+) & \xrightarrow{\sim} & K_H(pt) \end{array} \tag{2.17}$$

where the vertical maps are the assembly maps and the horizontal isomorphisms are given in Lemma 2.3.4 and Proposition 2.4.3.

Proof: To prove the proposition we need to investigate the maps. First we look at the right vertical map. At the beginning we have the Dirac element

$$d_{G,S} \in \mathrm{KK}^G(C_\tau(S),\mathbb{C})$$

apply the restriction homomorphism $r^{G,H}$ we get

$$r^{G,H}(d_{G,S}) \in \mathrm{KK}^H(C_\tau(S),\mathbb{C}).$$

Nevertheless we have the Dirac element

$$d_{H,S} \in \mathrm{KK}^H(C_\tau(S),\mathbb{C}).$$

In fact from the definition it is easy to see that they are equal:

$$r^{G,H}(d_{G,S}) = d_{H,S}.$$

Then we apply the map

$$j_r^H : \mathrm{KK}^H(C_\tau(S), \mathbb{C}) \longrightarrow \mathrm{KK}(C_r^*(H, C_\tau(S)), C_r^*(H)).$$

we get

$$j_r^H(d_{H,S}) \in \mathrm{KK}(C_r^*(H, C_\tau(S)), C_r^*(H))$$

and we denote it by D_H. Right multiplication of D_H gives the vertical map on the right in the diagram

$$\mathrm{KK}^H(S, \mathrm{pt}) \overset{\mu_{\mathrm{red},\mathrm{pt}}}{\longrightarrow} \mathrm{K}_H(\mathrm{pt}).$$

On the other hand we have the map

$$\sigma_{\alpha^+} : \mathrm{KK}^G(C_\tau(S), \mathbb{C}) \longrightarrow \mathrm{KK}^G(C_\tau(S) \otimes C_0(\alpha^+), C_0(\alpha^+))$$

so we get

$$\sigma_{\alpha^+}(d_{G,S}) \in \mathrm{KK}^G(C_\tau(S) \otimes C_0(\alpha^+), C_0(\alpha^+))$$

then via j_r^G we get

$$j_r^G(\sigma_{\alpha^+}(d_{G,S})) \in \mathrm{KK}(C_r^*(G, C_\tau(S) \otimes C_0(\alpha^+)), C_r^*(G, C_0(\alpha^+)))$$

which we denote by D_{G,α^+}. Right multiplication of D_{G,α^+} gives the other vertical map

$$\mathrm{KK}^G(S, \alpha^+) \overset{\mu_{\mathrm{red},\alpha^+}}{\longrightarrow} \mathrm{K}_G(\alpha^+).$$

The horizontal maps in the diagram are given by Strongly Morita equivalence. Now, under the Strongly Morita equivalence, $D_{G,\alpha^+} \cong D_H$, so the diagram commutes. $\square$

According to Propostion 2.4.3, in order to prove Proposition 2.4.1, i.e.

$$\mu_{\mathrm{red},\alpha^+} : \mathrm{KK}^G(S,\alpha^+) \to \mathrm{K}_G(\alpha^+)$$

is an isomorphism, it is sufficient to prove the following proposition

Proposition 2.4.4.

$$\mu_{red,pt} : KK^H(S,pt) \to K_H(pt) \tag{2.18}$$

is an isomorphism.

Proof: It is sufficient to prove

$$D_H = j_r^H(d_{H,S}) \in \mathrm{KK}(C_r^*(H, C_\tau(S)), C_r^*(H))$$

is invertible. In fact, we can prove that $d_{H,S} \in \mathrm{KK}^H(C_\tau(S),\mathbb{C})$ is invertible. This follows from the fact that H is almost connected amenable together with some formal arguments.

As in the construction in Section 3, we have the dual Dirac element

$$\eta_{H,S} \in \mathrm{KK}^H(\mathbb{C}, C_\tau(S))$$

and

$$d_{H,S}\otimes_{\mathbb{C}}\eta_{H,S} = 1 \in \mathrm{KK}^H(C_\tau(S), C_\tau(S)),$$

$$\eta_{H,S}\otimes_{C_\tau(H)}d_{H,S} = \gamma_{H,S} \in \mathrm{KK}^H(\mathbb{C},\mathbb{C}).$$

We want to prove $\gamma_{H,S} = 1$. Remember that H is an almost connected amenable group and we have Theorem 2.3.8, which claims that

$$\gamma_{(H)} = 1.$$

where by definition $\gamma_{(H)} = \gamma_{H,H/U\cap H}$. We need to prove $\gamma_{(H)}$ is equal to $\gamma_{H,S}$.

Notice that $\gamma_{H,S}$ is nothing but the image of the element $\gamma_{G,S}$ under the restriction homomorphism

$$r^{G,H} : \mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \longrightarrow \mathrm{KK}^H(\mathbb{C}, \mathbb{C}). \tag{2.19}$$

i.e.

$$r^{G,H}(\gamma_{G,S}) = \gamma_{H,S}. \tag{2.20}$$

Other other hand, in the notation of Theorem 2.3.8, $\gamma_{G,S}$ is nothing but $\gamma_{(G)}$, and again by Theorem 2.3.8 we have

$$r^{G,H}(\gamma_{(G)}) = \gamma_{(H)}$$

Compare the last two identity we get

$$\gamma_{H,S} = \gamma_{(H)} \tag{2.21}$$

so

$$\gamma_{H,S} = 1. \tag{2.22}$$

Now we proved that $d_{H,S}$ hence D_H, is invertible, as a result

$$\mu_{\mathrm{red,pt}} : \mathrm{KK}^H(S, \mathrm{pt}) \to \mathrm{K}_H(\mathrm{pt})$$

is an isomorphism. □

Proof of Proposition 2.4.1: Combine Proposition 2.4.3 and 2.4.4 we know get

$$\mu_{\mathrm{red},\alpha^+} : \mathrm{KK}^G(S, \alpha^+) \to \mathrm{K}_G(\alpha^+)$$

is an isomorphism, which finishes the prove of Proposition 2.4.1.□

2.5 The G-orbits on the Flag Variety

We have proved the isomorphism on one orbit of G. Now we need to study the G=orbits on $\mathcal{B}$ and in the next section we will "piece together orbits".

The result on the G-orbits in [39] is important to our purpose, so we summarize their result here

Theorem 2.5.1 ([39] 1.2, 3.8)**.** *On the flag variety $\mathcal{B}$ there exists a real value function f such that*

1. *f is a Morse-Bott function on $\mathcal{B}$.*
2. *f is U invariant, hence the gradient flow $\phi : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ is also U invariant.*
3. *The critical point set $\mathcal{C}$ consists of finitely many U-orbits α. The flow preserves the orbits of G.*

4. *The limits* $\lim_{t\to\pm\infty}\phi_t(x) := \pi^{\pm}(x)$ *exist for any* $x \in \mathcal{B}$*. For* α *a critical* U*-orbit, the stable set*

$$\alpha^+ = (\pi^+)^{-1}(\alpha)$$

is an G*-orbit, and the unstable set*

$$\alpha^- = (\pi^-)^{-1}(\alpha)$$

is an $U_{\mathbb{C}}$*-orbit, where* $U_{\mathbb{C}}$ *is the complexification of* U *in* $G_{\mathbb{C}}$*.*

5. $\alpha^+ \cap \alpha^- = \alpha$.

□

We will use the following corollary in [39]:

Corollary 2.5.2 ([39] 1.4)**.** *Let* α *and* β *be two critical* U*-orbits. Then the closure* $\overline{\alpha^+} \supset \beta^+$ *if and only if*

$$\alpha^+ \cap \beta^- \neq \emptyset.$$

□

From this we can get

Corollary 2.5.3. *Let* α *and* β *be two different critical* U*-orbits, i.e.* $\alpha \neq \beta$*. Then* $\overline{\alpha^+} \supset \beta^+$ *implies that the Morse-Bott function* f *has values*

$$f(\alpha) > f(\beta)$$

Proof: By the previous corollary,

$$\alpha^+ \cap \beta^- \neq \emptyset.$$

so there exists an $x \in \alpha^+ \cap \beta^-$.

Since $\lim_{t\to+\infty} \phi_t(x) \in \alpha$, we have

$$f(\alpha) \geqslant f(x),$$

similarly

$$f(x) \geqslant f(\beta).$$

On the other hand since $\alpha \neq \beta$ we get $\alpha \not\subset \beta^-$ and $\beta \not\subset \alpha^+$. So

$$x \notin \alpha, x \notin \beta$$

so

$$f(x) \neq f(\alpha), f(x) \neq f(\beta).$$

So we have

$$f(\alpha) > f(\beta).$$

□

We can now give a partial order on the set of G-orbits of $\mathcal{B}$.

Definition 2.5.1. If $f(\alpha) > f(\beta)$, we say that $\alpha + > \beta +$.

If $f(\alpha) = f(\beta)$, we choose an arbitary partial order on them.

Now let us list all G-orbits in $\mathcal{B}$ in ascending order, keep in mind that there are finitely many of them:

$$\alpha_1^+ < \alpha_2^+ < \dots \alpha_k^+. \tag{2.23}$$

From the definition we can easily get

Corollary 2.5.4. *For any G-orbits α_i^+, the union*

$$Z_i := \bigcup_{\alpha_j^+ \leqslant \alpha_i^+} \alpha_j^+$$

is a closed subset of $\mathcal{B}$. Notice that $\alpha_i^+ \subset Z_i$

Proof: It is sufficient to prove that Z_i contains all its limit points, which is a direct corollary of Definition 2.5.1 and Corollary 2.5.3.

□

Remark 21. Corollary 2.5.3, Definition 2.5.1 and Corollary 2.5.4 are not given in [39].

2.6 The Baum-Connes Conjecture on Flag Varieties

With the construction in the last section, we can piece together the orbits

Proposition 2.6.1. *For $1 \leqslant i \leqslant k-1$ we have a short exact sequence of crossed product algebras:*

$$0 \to C_r^*(G, C_0(\alpha_{i+1}^+)) \to C_r^*(G, C(Z_{i+1})) \to C_r^*(G, C(Z_i)) \to 0.$$

Proof: From the construction we also get

$$Z_i \subset Z_{i+1},\ \alpha_{i+1}^{+} \subset Z_{i+1},$$
$$Z_i \cup \alpha_{i+1}^{+} = Z_{i+1},\ Z_i \cap \alpha_{i+1}^{+} = \emptyset,$$

and Z_i is closed in Z_{i+1}, α_{i+1}^{+} is open in Z_{i+1}.

Since $\mathcal{B}$ is a compact manifold, we get that Z_i and Z_{i+1} are both compact.

The inclusion gives a short exact sequence:

$$0 \to C_0(\alpha_{i+1}^{+}) \to C(Z_{i+1}) \to C(Z_i) \to 0. \tag{2.24}$$

Now we need to go to the reduced crossed-product C^*-algebras. The following technique result will help us:

Theorem 2.6.2 ([30] Theorem 6.8)**.** *Let G be a locally compact group and*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of G-C^* algebra. Then we have a short exact sequence:

$$0 \to C_r^*(G, A) \to C_r^*(G, B) \to C_r^*(G, C) \to 0.$$

□

Since we have

$$0 \to C_0(\alpha_{i+1}^{+}) \to C(Z_{i+1}) \to C(Z_i) \to 0.$$

exact, Theorem 2.6.2 gives the short exact sequence

$$0 \to C_r^*(G, C_0(\alpha_{i+1}^+)) \to C_r^*(G, C(Z_{i+1})) \to C_r^*(G, C(Z_i)) \to 0. \tag{2.25}$$

This finishes the proof of Proposition 2.6.1 □

From Proposition 2.6.1 we have the well-known six-term long exact sequence

$$\begin{array}{ccccc} \mathrm{K}^*(C_r^*(G, C_0(\alpha_{i+1}^+))) & \longrightarrow & \mathrm{K}^*(C_r^*(G, C(Z_{i+1}))) & \longrightarrow & \mathrm{K}^*(C_r^*(G, C(Z_i))) \\ \uparrow & & & & \downarrow \\ \mathrm{K}^{*+1}(C_r^*(G, C(Z_i))) & \longleftarrow & \mathrm{K}^{*+1}(C_r^*(G, C(Z_{i+1}))) & \longleftarrow & \mathrm{K}^{*+1}(C_r^*(G, C_0(\alpha_{i+1}^+))). \end{array}$$

i.e.

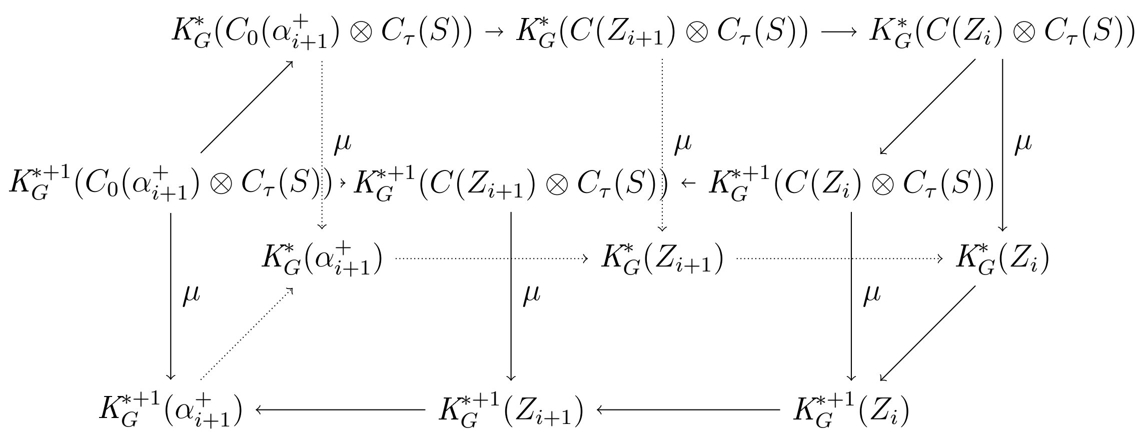
$$\begin{array}{ccccc} \mathrm{K}_G^*(\alpha_{i+1}^+) & \longrightarrow & \mathrm{K}_G^*(Z_{i+1}) & \longrightarrow & \mathrm{K}_G^*(Z_i) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_G^{*+1}(Z_i) & \longleftarrow & \mathrm{K}_G^{*+1}(Z_{i+1}) & \longleftarrow & \mathrm{K}_G^{*+1}(\alpha_{i+1}^+). \end{array} \tag{2.26}$$

Similarly we have

$$\begin{array}{ccccc} \mathrm{K}_G^*(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)) & \longrightarrow & \mathrm{K}_G^*(C(Z_{i+1}) \otimes C_\tau(S)) & \longrightarrow & \mathrm{K}_G^*(C(Z_i) \otimes C_\tau(S)) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_G^{*+1}(C(Z_i) \otimes C_\tau(S)) & \longleftarrow & \mathrm{K}_G^{*+1}(C(Z_{i+1}) \otimes C_\tau(S)) & \longleftarrow & \mathrm{K}_G^{*+1}(C_0(\alpha_{i+1}^+) \otimes C_\tau(S)). \end{array} \tag{2.27}$$

The fact is that Formula 2.26 and 2.27 together form a commuting diagram.

Proposition 2.6.3. *We have the following commuting diagram:*



(2.28)

where the top and bottom are the six-term exact sequences and the vertical arrows are assembly maps μ*.*

Proof: The diagram commutes because all the vertical maps μ come from the same element

$$d_{G,S} \in \mathrm{KK}^G(C_\tau(S), \mathbb{C})$$

as in Section 2.4. □

After all these work we can prove Theorem 2.1.1.

Proof of Theorem 2.1.1: We use induction on the Z_i's. First, for $Z_1 = \alpha_1^+$, by Proposition 2.4.1,

$$\mathrm{K}_G^*(C(Z_1) \otimes C_\tau(S)) \xrightarrow{\mu} \mathrm{K}_G^*(Z_1) \tag{2.29}$$

is an isomorphism.

Assume that for Z_i,

$$\mathrm{K}^*_G(C(Z_i) \otimes C_\tau(S)) \xrightarrow{\mu} \mathrm{K}^*_G(Z_i) \tag{2.30}$$

is an isomorphism.

By Proposition 2.4.1, the vertical maps on the left face of Commuting Diagram 2.28 are isomorphisms. Moreover by induction we can get that the vertical maps on the right face are isomorphism too, hence by a 5-lemma-argument we get the middle vertical maps are also isomorphisms, i.e. for Z_{i+1},

$$\mathrm{K}^*_G(C(Z_{i+1}) \otimes C_\tau(S)) \xrightarrow{\mu} \mathrm{K}^*_G(Z_{i+1}) \tag{2.31}$$

is an isomorphism.

There are finitely many orbits and let α_k^+ be the largest orbit, it follows that

$$Z_k = \bigcup_{\text{all orbits}} \alpha_i^+ = \mathcal{B} \tag{2.32}$$

hence

$$\mu_{\mathrm{red},\mathcal{B}} : \mathrm{KK}^G(S, \mathcal{B}) \to \mathrm{K}_G(\mathcal{B}) \tag{2.33}$$

is an isomorphism. we finished the proof Theorem 2.1.1. □

2.7 An Example

We look at the case when $G = \mathrm{SL}(2,\mathbb{R})$ and $G_{\mathbb{C}} = \mathrm{SL}(2,\mathbb{C})$. Hence

$$B_{\mathbb{C}} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{C}^*, B \in \mathbb{C} \right\}$$

and

$$\mathcal{B} = G_{\mathbb{C}}/B_{\mathbb{C}} = \mathbb{C}P^1 \cong S^2.$$

It is well-known that the $G_{\mathbb{C}}$ (hence) G acts on $\mathcal{B} = \mathbb{C}P^1$ by fractional linear transform, i.e. using projective coordinate

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}.$$

Or let $z = u/v$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}. \tag{2.34}$$

From Formula 2.34 we can see that the action of G on $\mathcal{B}$ is not transitive. In fact, it has three orbits

$$\begin{aligned} \alpha_1^+ =& \mathbb{R} \cup \infty \cong S^1 \text{ the equator,} \\ \alpha_2^+ =& \{x + iy | y > 0\} \cong \mathbb{C} \text{ the upper hemisphere,} \\ \alpha_2^+ =& \{x + iy | y < 0\} \cong \mathbb{C} \text{ the lower hemisphere.} \end{aligned}$$

α_1^+ is a closed orbit with dimension 1; α_2^+ and α_3^+ are open orbits with dimension 2.

Let's look at α_1^+ first. Take the point $1 \in \alpha_1^+$. The isotropy group at 1 is the upper triangular group B in $\mathrm{SL}(2,\mathbb{R})$. So

$$\mathrm{K}_G^*(\alpha_1^+) = \mathrm{K}_B^*(pt).$$

B is solvable hence amenable and $\mathbb{Z}/2\mathbb{Z}$ is the maximal compact group of B. By Theorem 2.3.9

$$\mathrm{K}_B^0(pt) = R(\mathbb{Z}/2\mathbb{Z})$$

is the representation ring of the group with two elements and

$$\mathrm{K}_B^1(pt) = 0.$$

So

$$\mathrm{K}_G^0(\alpha_1^+) = R(\mathbb{Z}/2\mathbb{Z})$$

and

$$\mathrm{K}_G^1(\alpha_1^+) = 0.$$

For α_2^+ and α_3^+, the isotropy groups are both

$$T = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$$

hence by the similar reason to α_1^+ we have

$$\mathrm{K}_G^0(\alpha_2^+) = \mathrm{K}_G^0(\alpha_3^+) = \mathrm{K}_T^0(pt) = R(T)$$

is the representation ring of T and

$$\mathrm{K}^1_G(\alpha_2^+) = \mathrm{K}^1_G(\alpha_3^+) = \mathrm{K}^1_T(pt) = 0.$$

Now $\alpha_2^+ \cup \alpha_3^+$ is open in $\mathcal{B}$ so as in the last section we have the short exact sequence

$$0 \longrightarrow C_0(\alpha_2^+ \cup \alpha_3^+) \longrightarrow C(\mathcal{B}) \longrightarrow C(\alpha_1^+) \longrightarrow 0 \tag{2.35}$$

and further

$$0 \longrightarrow C_r^*(G, \alpha_2^+ \cup \alpha_3^+) \longrightarrow C_r^*(G, \mathcal{B}) \longrightarrow C_r^*(G, \alpha_1^+) \longrightarrow 0.$$

i.e.

$$0 \longrightarrow C_r^*(G, \alpha_2^+) \oplus C_r^*(G, \alpha_3^+) \longrightarrow C_r^*(G, \mathcal{B}) \longrightarrow C_r^*(G, \alpha_1^+) \longrightarrow 0. \tag{2.36}$$

We get the six-term exact sequence

$$\begin{array}{ccccc} \mathrm{K}^0_G(\alpha_2^+) \oplus \mathrm{K}^0_G(\alpha_3^+) & \longrightarrow & \mathrm{K}^0_G(\mathcal{B}) & \longrightarrow & \mathrm{K}^0_G(\alpha_1^+) \\ \uparrow & & & & \downarrow \\ \mathrm{K}^1_G(\alpha_1^+) & \longleftarrow & \mathrm{K}^1_G(\mathcal{B}) & \longleftarrow & \mathrm{K}^1_G(\alpha_2^+) \oplus \mathrm{K}^1_G(\alpha_3^+). \end{array} \tag{2.37}$$

Combine with the previous calculation we get

$$0 \longrightarrow R(T) \oplus R(T) \longrightarrow \mathrm{K}^0_G(\mathcal{B}) \longrightarrow R(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \tag{2.38}$$

and $\mathrm{K}^1_G(\mathcal{B}) = 0$.

In conclusion we have

$$\begin{aligned} \mathrm{K}^0_G(\mathcal{B}) \approx & R(T) \oplus R(T) \oplus R(\mathbb{Z}/2\mathbb{Z}), \\ \mathrm{K}^1_G(\mathcal{B}) = & 0. \end{aligned} \tag{2.39}$$

Then we look at $\mathrm{K}^0_U(\mathcal{B})$, and $\mathrm{K}^1_U(\mathcal{B})$. We know that for $G = \mathrm{SL}(2, \mathbb{R})$ the maximal compact subgroup $U = T$. By Bott periodicity we have

$$\mathrm{K}^0_U(\alpha_2^+) = \mathrm{K}^0_U(\alpha_3^+) \cong \mathrm{K}^0_U(\mathbb{C}) \cong \mathrm{K}^0_U(pt) = R(U) = R(T) \tag{2.40}$$

and

$$\mathrm{K}^1_U(\alpha_2^+) = \mathrm{K}^1_U(\alpha_3^+) \cong \mathrm{K}^1_U(\mathbb{C}) \cong \mathrm{K}^1_U(pt) = 0. \tag{2.41}$$

As for α_1^+, we notice that U acts on $\alpha_1^+ \cong S^1$ by "square", so the isotropy group is $\mathbb{Z}/2\mathbb{Z}$. Hence

$$\mathrm{K}^0_U(\alpha_1^+) = R(\mathbb{Z}/2\mathbb{Z})$$

and

$$\mathrm{K}^1_U(\alpha_1^+) = 0.$$

By the six-term long exact sequence we have

$$\begin{aligned} \mathrm{K}^0_U(\mathcal{B}) \approx & R(T) \oplus R(T) \oplus R(\mathbb{Z}/2\mathbb{Z}), \\ \mathrm{K}^1_U(\mathcal{B}) = & 0. \end{aligned} \tag{2.42}$$

Compare 2.39 and 2.42 we have

$$\mathrm{K}^*_U(\mathcal{B}) \approx \mathrm{K}^*_G(\mathcal{B}).$$

On the other hand, for $G = \mathrm{SL}(2, \mathbb{R})$, $U = T$ and $S = G/U$, we have S is spin and $\dim S = 2$, then by Baum-Connes conjecture (in fact, Connes-Kasparov conjecture as in Section 2.3.3) we know that the above $\approx$ is an isomorphism, i.e

$$\mathrm{K}_U^*(\mathcal{B}) \cong \mathrm{K}_G^*(\mathcal{B}). \tag{2.43}$$

Remark 22. Using Bott periodicity theorem we can obtain precisely the algebra structure of $\mathrm{K}_U^*(\mathcal{B})$ as in [45]. Therefore Baum-Connes conjecture will be a powerful tool to investigate $\mathrm{K}_G(\mathcal{B})$ and to study the representation theory of G.

Chapter 3

The Noncommutative Poisson Bracket and the Deformation of the Family Algebras

3.1 Introduction

3.1.1 Baum-Connes Conjecture and Mackey's Analogue

The representation theory of semisimple Lie group G has another interesting constituent. Let

$$G = K \exp \mathfrak{p} \tag{3.1}$$

be the Cartan decomposition of G and

$$G_c = K \rtimes \mathfrak{p} \tag{3.2}$$

be the Cartan motion group associated to G.

The Baum-Connes conjecture (more precisely, the Connes-Kasparov conjecture) implies that $R(K) \cong \mathrm{K}_*(C_r^*(G))$. Since K is also the maximal compact subgroup of G_c, we have $R(K) \cong \mathrm{K}_*(C^*_r(G_c))$. As a result,

$$\mathrm{K}_*(C^*_r(G)) \cong \mathrm{K}_*(C^*_r(G_c)). \tag{3.3}$$

For more details see [23].

The Mackey's analogue ([38]) is to find an identification (in various senses, not only in the level of K-theory) of the representations of G and those of G_c.

To study Mackey's analogue of the admissible representations, in [24] N. Higson introduced the *spherical Hecke algebras* $\mathcal{R}(\mathfrak{g},\tau)$ and $\mathcal{R}(\mathfrak{g}_c,\tau)$ respectively, where τ is a irreducible representation of K. These algebras have the importance that the irreducible $\mathcal{R}(\mathfrak{g},\tau)$ modules are 1-1 correspondent to irreducible $(\mathfrak{g},K)$-modules of G with nonzero τ-isotypical component, and the similar result holds for $\mathcal{R}(\mathfrak{g}_c,\tau)$, see [24].

For the structures of the spherical Hecke algebras, we have the following proposition:
Proposition 3.1.1 ([24] Propostion 2.13)**.** *For complex semisimple Lie group G, we have the following isomorphisms as algebras:*

$$\begin{aligned} \mathcal{R}(\mathfrak{g},\tau) &\cong [U(\mathfrak{g}) \otimes End(V_\tau)^{op}]^K \\ \mathcal{R}(\mathfrak{g}_c,\tau) &\cong [S(\mathfrak{g}) \otimes End(V_\tau)^{op}]^K. \end{aligned} \tag{3.4}$$

The right hand sides are the *quantum family algebra* $\mathcal{Q}_\tau(\mathfrak{g})$ and the *classical family algebra* $\mathcal{C}_\tau(\mathfrak{g})$, introduced by A. A. Kirillov in [31].

In [24] Higson constructed the *generalized Harish-Chandra homomorphisms*:

$$\begin{aligned} \mathrm{GHC}_\tau &: \mathcal{R}(\mathfrak{g}, \tau) \to U(\mathfrak{h}) \\ \mathrm{GHC}_{\tau,c} &: \mathcal{R}(\mathfrak{g}_c, \tau) \to S(\mathfrak{h}) \end{aligned} \tag{3.5}$$

and relates them to the admissible duals of G and G_c with minimal K-type τ.

The Mackey's analogue for admissible dual has the following form:

Theorem 3.1.2 ([24], Section 8)**.** *Under the identification $U(\mathfrak{h}) \cong S(\mathfrak{h})$, the two homomorphisms GHC_τ and $GHC_{\tau,c}$ has the same image.*

In this chapter we will study further the relations between $\mathcal{R}(\mathfrak{g}, \tau)$ and $\mathcal{R}(\mathfrak{g}_c, \tau)$, i.e. the two family algebras $\mathcal{Q}_\tau(\mathfrak{g})$ and $\mathcal{C}_\tau(\mathfrak{g})$.

3.1.2

The *family algebras* are introduced by A. A. Kirillov in the year 2000 in [31] and [32] as follows:

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, $S(\mathfrak{g})$ and $U(\mathfrak{g})$ be the symmetric algebra and the universal enveloping algebra of $\mathfrak{g}$ respectively. Let G be a connected and simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$. G has adjoint actions ad on $S(\mathfrak{g})$ and $U(\mathfrak{g})$.

On the other hand, let V_τ be a finite dimensional complex representation of $\mathfrak{g}$. Then τ gives rise to a representation of G. Therefore G has a natural action on $\mathrm{End}_{\mathbb{C}} V_\tau$:

$$\forall A \in \mathrm{End}_{\mathbb{C}} V_\tau, g \in G, \ \ g \cdot A := \tau(g) \circ A \circ \tau(g)^{-1}.$$

As a result, G has natural diagonal actions on $\mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} S(\mathfrak{g})$ and $\mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} U(\mathfrak{g})$:

for any $g \in G$ and for any $A_i \otimes a^i \in \mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} S(\mathfrak{g})$, $B_i \otimes b^i \in \mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} U(\mathfrak{g})$,

$$g \cdot (A_i \otimes a^i) := g \cdot A_i \otimes (\mathrm{ad} g)\ a^i = \tau(g) \circ A_i \circ \tau(g)^{-1} \otimes (\mathrm{ad} g)\ a^i,$$
$$g \cdot (B_i \otimes b^i) := g \cdot B_i \otimes (\mathrm{ad} g)\ b^i = \tau(g) \circ B_i \circ \tau(g)^{-1} \otimes (\mathrm{ad} g)\ b^i.$$

Now we come to the definition of the family algebras, see [31] and [32]:

Definition 3.1.1 (The family algebras)**.** The *classical family algebra* is defined to be:

$$\mathcal{C}_\tau(\mathfrak{g}) := (\mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} S(\mathfrak{g}))^G. \tag{3.6}$$

The *quantum family algebra* is defined to be:

$$\mathcal{Q}_\tau(\mathfrak{g}) := (\mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} U(\mathfrak{g}))^G. \tag{3.7}$$

Kirillov proved that $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$ are algebras, that is, they are closed under multiplications. He ([31], [32]) and Rozhkovskaya ([43]) have found various the relation between family algebras and the representations of $\mathfrak{g}$. On the other hand, in 2011, N. Higson relates family algebras with the admissible representations of complex semisimple Lie groups in [24]

In this chapter, we study family algebras in another approach. It is well-known that we have a *Poisson bracket* on $S(\mathfrak{g})$ (see [33]): Let X_i be a basis of $\mathfrak{g}$ and c_{ij}^k be the structure constant with respect to the basis X_i, then for any $a, b \in S(\mathfrak{g})$, the Poisson bracket is defined to be

$$\{\, a, b\} := c_{ij}^k X_k \cdot \partial^i a \cdot \partial^j b.$$

Now we can define the *noncommutative Poisson bracket* on the classical family algebra:

Definition 3.1.2 (The noncommutative Poisson bracket on $\mathcal{C}_\tau(\mathfrak{g})$)**.** Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}_\tau(\mathfrak{g})$, $\mathcal{A} = A_i \otimes a^i$, $\mathcal{B} = B_j \otimes b^j$. We define the noncommutative Poisson bracket P as follows:

$$\{\, \mathcal{A}, \mathcal{B}\} := A_i B_j \otimes \{\, a^i, b^j\}. \tag{3.8}$$

In this chapter we will study the properties of the noncommutative Poisson bracket (for short, Poisson bracket) on $\mathcal{C}_\tau(\mathfrak{g})$. The following are two important results we get:

- The Poisson bracket on $\mathcal{C}_\tau(\mathfrak{g})$ characterize the first-order deformations from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$, just as the Poisson bracket on $S(\mathfrak{g})$ characterize the first-order deformations from $S(\mathfrak{g})$ to $U(\mathfrak{g})$, see Proposition 3.5.6.

- In the Hochschild cochain complex of $\mathcal{C}_\tau(\mathfrak{g})$, the Poisson bracket is a 2-coboundary. In fact we can explicitly find a Hochschild 1-cochain ∇ which maps to the Poisson bracket under the Hochschild differential, see Theorem 3.6.4.

It is expected that this result can help us find a quantization map $\mathcal{C}_\tau(\mathfrak{g}) \to \mathcal{Q}_\tau(\mathfrak{g})$, as proposed by Higson in [24].

This chapter is organized as follows: In Section 3.2 we review the family algebras, in Section 3.3 we study the first properties of the noncommutative Poisson bracket P, in Section 3.4 we summarize the results on Hochschild cohomology, Gerstenhaber bracket and their relation to the deformation theory. In Section 3.5 we give the relation between P and the deformation from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$, in Section 3.6 we prove that the noncommutative Poisson bracket P is a Hochschild 2-coboundary and therefore the deformation is infinitesimally trivial.

Remark 23. Although Kirillov and Higson in [31], [32] and [24] require the Lie algebra $\mathfrak{g}$ to be semisimple and the representation τ to be irreducible, in this chapter we do not need this restriction.

3.2 A brief introduction to the Family algebras

Most of the materials in this section can be found in [31] and [32].

First of all, we use the following notation-definition

Definition 3.2.1.

$$\begin{aligned} \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) &:= \mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} S(\mathfrak{g}), \\ \widetilde{\mathcal{Q}_\tau}(\mathfrak{g}) &:= \mathrm{End}_{\mathbb{C}} V_\tau \otimes_{\mathbb{C}} U(\mathfrak{g}). \end{aligned} \tag{3.9}$$

$\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ and $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$ consist of matrices with entries in $S(\mathfrak{g})$ and $U(\mathfrak{g})$, respectively. Therefore they are algebras in a natural way: for any $A_i \otimes a^i$, $B_j \otimes b^j \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$(or $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$), their product is given by the following formula:

$$(A_i \otimes a^i) \cdot (B_j \otimes b^j) := A_i B_j \otimes a^i b^j. \tag{3.10}$$

Remark 24. A_i and B_j do not commute in $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ and $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$. Moreover, in the $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$ case a^i and b^j do not commute either.

The following simple result will be frequently used:

Proposition 3.2.1. *In both $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ and $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$, the matrix component and the $S(\mathfrak{g})$ component always commute. In more detail, for any $A_i \otimes a^i$, $B_j \otimes b^j \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$(or $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$),* *we have*

$$
\begin{aligned}
(A_i \otimes a^i) \cdot (B_j \otimes b^j) =& A_i B_j \otimes a^i b^j \\
=& (Id \otimes a^i) \cdot (A_i B_j \otimes b^j) \\
=& (A_i B_j \otimes a^i) \cdot (Id \otimes b^j).
\end{aligned}
\tag{3.11}
$$

Proof: It is obvious. □

By Definition 3.1.1 we know

$$\mathcal{C}_\tau(\mathfrak{g}) = \widetilde{\mathcal{C}_\tau}(\mathfrak{g})^G \text{ and } \mathcal{Q}_\tau(\mathfrak{g}) := \widetilde{\mathcal{Q}_\tau}(\mathfrak{g}))^G.$$

Now we show that the Lie group action can be reduced to the Lie algebra action.

Proposition 3.2.2 (The criterion for classical family algebra, [31] Section 1)**.** *Let* $A_i \otimes a^i \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$*, then* $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$ *if and only if*

$$\forall X \in \mathfrak{g},\ \ [\tau(X), A_i] \otimes a^i + A_i \otimes \{X,\ a^i\} = 0, \tag{3.12}$$

or in other words,

$$\forall X \in \mathfrak{g},\ \ [\tau(X), A_i] \otimes a^i = A_i \otimes \{a^i,\ X\}. \tag{3.13}$$

Proof: By definition 3.1.1, we know that $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$ if and only if:

$$\tau(g) \circ A_i \circ \tau(g)^{-1} \otimes (\mathrm{ad} g)\ a^i = A_i \otimes a^i.$$

It is well-known that the adjoint action of $\mathfrak{g}$ on $S(\mathfrak{g})$ is exactly the Poisson bracket. As a result, Equation 3.12 and Equation 3.13 are infinitesimal versions of the above equation. Since G is connected and simply connected, they are equivalent. □

Similarly we have

Proposition 3.2.3 (The criterion for quantum family algebra, [31] Section 1)**.** *Let* $A_i \otimes a^i \in \widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$*, then* $A_i \otimes a^i \in \mathcal{Q}_\tau(\mathfrak{g})$ *if and only if*

$$\forall X \in \mathfrak{g},\ [\tau(X), A_i] \otimes a^i + A_i \otimes [X,\ a^i] = 0,$$

or in other words,

$$\forall X \in \mathfrak{g},\ [\tau(X), A_i] \otimes a^i = A_i \otimes [a^i,\ X].$$

Proof: Similar to the proof of Proposition 3.2.3. □

Then we can prove the following result:

Corollary 3.2.4 (see also [31] and [32])**.** $\mathcal{C}_\tau(\mathfrak{g})$ *and* $\mathcal{Q}_\tau(\mathfrak{g})$ *are subalgebras of* $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ *and* $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$) *respectively.*

Proof: Let $A_i \otimes a^i$ and $B_j \otimes b^j$ be two elements in $\mathcal{C}_\tau(\mathfrak{g})$. Their product

$$(A_i \otimes a^i) \cdot (B_j \otimes b^j) = A_i B_j \otimes a^i b^j.$$

Now $\forall X \in \mathfrak{g}$,

$$
\begin{aligned}
[\tau(X), A_iB_j] \otimes a^ib^j =& [\tau(X), A_i]B_j \otimes a^ib^j + A_i[\tau(X), B_j] \otimes a^ib^j \\
=& ([\tau(X), A_i] \otimes a^i) \cdot (B_j \otimes b^j) + (A_i \otimes a^i) \cdot ([\tau(X), B_j] \otimes b^j).
\end{aligned}
$$

The second equality is because of Proposition 3.2.1: the matrix component always **commutes** with the $S(\mathfrak{g})$ component.

Now by Proposition 3.2.2, we know

$$
\begin{aligned}
\text{the above } =& (A_i \otimes \{a^i, X\}) \cdot (B_j \otimes b^j) + (A_i \otimes a^i) \cdot B_j \otimes \{b^j, X\} \\
=& A_iB_j \otimes \{a^i, X\}b^j + A_iB_j \otimes a^i\{b^j, X\} \text{ (Proposition 3.2.2)} \\
=& A_iB_j \otimes \{a^ib^j, X\}.
\end{aligned}
$$

Hence we get

$$A_iB_j \otimes a^ib^j \in \mathcal{C}_\tau(\mathfrak{g}).$$

In the same way we can show that if $A_i \otimes a^i$ and $B_j \otimes b^j$ are in $\mathcal{Q}_\tau(\mathfrak{g})$, then

$$A_iB_j \otimes a^ib^j \in \mathcal{Q}_\tau(\mathfrak{g}). \ \square$$

It is not difficult to see that the family algebras are nontrivial. In fact, let $I(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$ be the invariant subalgebra of $S(\mathfrak{g})$ and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. We have

Proposition 3.2.5 ([32]). *$I(\mathfrak{g})$ embeds into $\mathcal{C}_\tau(\mathfrak{g})$ as scalar matrices*

$$
\begin{aligned}
I(\mathfrak{g}) &\hookrightarrow \mathcal{C}_\tau(\mathfrak{g}) \\
a &\mapsto Id \otimes a.
\end{aligned}
\tag{3.14}
$$

Similarly $Z(\mathfrak{g})$ embeds into $\mathcal{Q}_\tau(\mathfrak{g})$ as scalar matrices too.

Proof: It is obvious that $I(\mathfrak{g})$ embeds into $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$ as scalar matrices. Now by Proposition 3.2.2, it is easy to see that the image is contained in $\mathcal{C}_\tau(\mathfrak{g})$.

The proof for $Z(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$ is the same. □

Example 1. For any $\mathfrak{g}$, when the representation τ is the trivial representation, we see that $I(\mathfrak{g}) = \mathcal{C}_\tau(\mathfrak{g})$ and $Z(\mathfrak{g}) = \mathcal{Q}_\tau(\mathfrak{g})$.

Example 2. For $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$ and $\{e, f, h\}$ be the standard basis of $\mathrm{sl}(2,\mathbb{C})$ which satisfies the commutation relation

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{3.15}$$

Let τ be the 2-dimensional standard representation, we can find an element $M \in \mathcal{C}_\tau(\mathfrak{g})$ which is not in $I(\mathfrak{g})$. In fact

$$M = \begin{pmatrix} \frac{h}{2} & f \\ e & -\frac{h}{2} \end{pmatrix} \tag{3.16}$$

We can also find an element in $\mathcal{Q}_\tau(\mathfrak{g})$ with the same expression of M, see [31] and [32].

Remark 25. When τ is nontrivial irreducible and $\mathfrak{g}$ is semisimple, $I(\mathfrak{g})$ is not equal to $\mathcal{C}_\tau(\mathfrak{g})$ and $Z(\mathfrak{g})$ is not equal to $\mathcal{Q}_\tau(\mathfrak{g})$ either, see Corollary 3.6.3 below or [31].

3.3 The noncommutative Poisson bracket on $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$

The noncommutative Poisson bracket on $\mathcal{C}_\tau(\mathfrak{g})$ in Definition 3.1.2 can be easily extended to $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$:

Definition 3.3.1. Let $\mathcal{A}, \mathcal{B} \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, $\mathcal{A} = A_i \otimes a^i$, $\mathcal{B} = B_j \otimes b^j$. We define the noncommutative Poisson bracket as follows:

$$\{\mathcal{A}, \mathcal{B}\} := A_i B_j \otimes \{a^i, b^j\}. \tag{3.17}$$

We will also denote the noncommutative Poisson bracket by P.

Remark 26. P. Xu gives a similar construction in [52] Example 2.2. Nevertheless, our construction has different purpose than his.

Remark 27. The Poisson bracket on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ is not anti-symmetric and does not satisfies the Leibniz rule and the Jacobi identity.

Nevertheless, J. Block and E. Getzler in 1992 give a definition of Poisson bracket on noncommutative algebra in [8] and we can prove that our noncommutative Poisson bracket P satisfies the requirement of Poisson bracket in that sense:

Definition 3.3.2 ([8] Definition 1.1)**.** A *Poisson bracket* on a (posssibly noncommutative) algebra A is a Hochschild 2-cocycle $P \in Z^2(A, A)$ such that $P \circ P \in C^3(A, A)$ is a 3-coboundary. In other words

$$P \circ P \in B^3(A, A) \subset Z^3(A, A) \subset C^3(A, A). \tag{3.18}$$

For Hochschild cohomology see Section 3.4.1 and for the definition of $P \circ P$ see Proposition 3.4.2.

Remark 28. In Definition 3.3.2, we can consider the condition $P \in Z^2(A, A)$ as a noncommutative Leibniz rule and $P \circ P \in B^3(A, A)$ as a noncommutative Jacobi identity. They together implies that P can be lift to an associative product on A up to order 3, see Corollary 3.4.8.

For our algebra $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$ and the Poisson bracket P in Definition 3.3.1, first we can prove that P is a 2-cocycle. We have the following proposition:

Proposition 3.3.1. *For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \widetilde{\mathcal{C}}_\tau(\mathfrak{g})$, we have*

$$\mathcal{A}\{\mathcal{B}, \mathcal{C}\} - \{\mathcal{A}\mathcal{B}, \mathcal{C}\} + \{\mathcal{A}, \mathcal{B}\mathcal{C}\} - \{\mathcal{A}, \mathcal{B}\}\mathcal{C} = 0. \tag{3.19}$$

In other words, we have $d_H P = 0$ where d_H is the Hochschild differential. Therefore

$$P \in Z^2(\widetilde{\mathcal{C}}_\tau(\mathfrak{g}), \widetilde{\mathcal{C}}_\tau(\mathfrak{g})). \tag{3.20}$$

Proof: Let $\mathcal{A} = A_i \otimes a^i$, $\mathcal{B} = B_j \otimes b^j$ and $\mathcal{C} = C_k \otimes c^k$. Remember Proposition 3.2.1: the matrix component and the $S(\mathfrak{g})$ component always commute. Then by the definition of the Poisson bracket we get

$$\begin{aligned} &\mathcal{A}\{\mathcal{B}, \mathcal{C}\} - \{\mathcal{A}\mathcal{B}, \mathcal{C}\} + \{\mathcal{A}, \mathcal{B}\mathcal{C}\} - \{\mathcal{A}, \mathcal{B}\}\mathcal{C} \\ =&A_i B_j C_k \otimes (a_i\{\, b_j, c_k\} - \{\, a_i b_j, c_k\} + \{\, a_i, b_j c_k\} - \{\, a_i, b_j\}c_k). \end{aligned}$$

By the Leibniz rule of the (ordinary) Poisson bracket on $S(\mathfrak{g})$ we know that

$$a_i\{\, b_j, c_k\} - \{\, a_i b_j, c_k\} + \{\, a_i, b_j c_k\} - \{\, a_i, b_j\}c_k = 0. \ \square$$

Proposition 3.3.2. *$P \circ P$ is a 3-coboundary. In other words, $P \circ P \in B^3(\widetilde{\mathcal{C}}_\tau(\mathfrak{g}), \widetilde{\mathcal{C}}_\tau(\mathfrak{g}))$. In fact, we can define a 2-cochain $\Phi \in C^2(\widetilde{\mathcal{C}}_\tau(\mathfrak{g}), \widetilde{\mathcal{C}}_\tau(\mathfrak{g}))$ as follows: Let $\mathcal{A} = A \otimes a$*

and $\mathcal{B} = B \otimes b$ (to simplify the notation we omit the super and sub-indices)

$$\begin{aligned}\Phi(\mathcal{A},\mathcal{B}) :=& AB \otimes \frac{1}{2} c_{ij}^{s} c_{kl}^{t} X_s X_t \partial^i \partial^k a \partial^j \partial^l b \\ &+ AB \otimes \frac{1}{3} c_{ks}^{t} c_{ji}^{s} X_t (\partial^k \partial^j a \partial^i b + \partial^i a \partial^k \partial^j b).\end{aligned} \tag{3.21}$$

Then we have

$$P \circ P + d_H \Phi = 0. \tag{3.22}$$

Proof: For any $\mathcal{A},\mathcal{B},\mathcal{C} \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, by Definition

$$P \circ P(\mathcal{A},\mathcal{B},\mathcal{C}) = \{\mathcal{A},\{\mathcal{B},\mathcal{C}\}\} - \{\{\mathcal{A},\mathcal{B}\},\mathcal{C}\}. \tag{3.23}$$

Let $\mathcal{A} = A \otimes a$, $\mathcal{B} = B \otimes b$ and $\mathcal{C} = C \otimes c$, then

$$P \circ P(\mathcal{A},\mathcal{B},\mathcal{C}) = ABC \otimes (\{a,\{b,c\}\} - \{\{a,b\},c\}).$$

Now the problem reduces to $S(\mathfrak{g})$. We have the following lemma:

Lemma 3.3.3. *We can define a 2-cochain $\phi \in C^2(S(\mathfrak{g}), S(\mathfrak{g}))$ as follows: for any $a, b \in S(\mathfrak{g})$*

$$\phi(a,b) := \frac{1}{2} c_{ij}^{s} c_{kl}^{t} X_s X_t \partial^i \partial^k a \partial^j \partial^l b + \frac{1}{3} c_{ks}^{t} c_{ji}^{s} X_t (\partial^k \partial^j a \partial^i b + \partial^i a \partial^k \partial^j b). \tag{3.24}$$

Then for any $a, b, c \in S(\mathfrak{g})$ we have

$$\{a,\{b,c\}\} - \{\{a,b\},c\} + (d_H \phi)(a,b,c) = 0. \tag{3.25}$$

Assume we have Lemma 3.3.3, by abusing the notations we have $\Phi = \mathrm{Id} \otimes \phi$, then we immediately get $P \circ P + \mathrm{d_H}\Phi = 0$. □

Proof of Lemma 3.3.3: We can check it by hand using Jacobi identity. Another approach involves the star-porduct on $S(\mathfrak{g})$ and the general result of deformation theory and we defer it to Proposition 3.5.2. □

By Proposition 3.3.1 and 3.3.2 we know that the noncommutative Poisson bracket in Definition 3.3.1 is indeed a Poisson bracket in the sense of Definition 3.3.2.

Before we move on, we need to prove that the Poisson bracket indeed maps $\mathcal{C}_\tau(\mathfrak{g}) \otimes \mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{C}_\tau(\mathfrak{g})$. That is the following proposition:

Proposition 3.3.4. *For any $\mathcal{A}, \mathcal{B} \in \mathcal{C}_\tau(\mathfrak{g})$, we have that $\{\mathcal{A}, \mathcal{B}\}$ is still in $\mathcal{C}_\tau(\mathfrak{g})$. In other words, the noncommutative Poisson bracket in Definition 3.1.2 is well-defined.*

Proof: We can proof this proposition by computation using Proposition 3.2.2 and the definition of the noncommutative Poisson bracket P. In Section 3.6 we will give another proof using a different construction of P. See Corollary 3.6.7. □

3.4 Generalities of Deformation Theory of Algebras

3.4.1 Hochschild cohomology

Let us review the theory of Hochschild cohomology, see [50] or [12] Section 2 for references.

Let A be an associative $\mathbb{C}$-algebra. The associated *Hochschild complex* $C^\bullet(A, A)$ is defined as follows:

$$C^n(A,A) := \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A),\ n \geqslant 0. \tag{3.26}$$

The differential $\mathrm{d_H}$ is defined on homogeneous elements $f \in C^n(A,A)$ by the formula

$$\begin{aligned}(\mathrm{d_H}(f))(a_0, a_1, \ldots, a_n) :=& a_0 f(a_1, \ldots, a_n) + \sum_{k=1}^{n} (-1)^k f(a_0, \ldots, a_{k-1}a_k, \ldots, a_n) \\ &+ (-1)^{n+1} f(a_0, \ldots, a_{n-1}) a_n.\end{aligned} \tag{3.27}$$

We see that $\mathrm{d_H} f \in C^{n+1}(A,A)$. We can prove $\mathrm{d_H} \circ \mathrm{d_H} = 0$ therefore $C^\bullet(A,A)$ is a cochain complex.

The *Hochschild cohomology* of A is defined as the cohomology group of the cochain complex $C^\bullet(A,A)$, and we denote it by $\mathrm{HH}^\bullet(A,A)$ or for short $\mathrm{HH}^\bullet(A)$:

$$\mathrm{HH}^n(A) := \mathrm{H}^n(C^\bullet(A,A)). \tag{3.28}$$

Now let us look at the case $n = 2$. The following observation is easy to get:

Proposition 3.4.1. *Let $f \in C^2(A,A) = Hom_{\mathbb{C}}(A \otimes A, A)$. Then f is a 2-coboundary if and only if there exists a $g \in C^1(A,A) = Hom_{\mathbb{C}}(A,A)$ such that for any $a, b \in A$*

$$f(a,b) = ag(b) - g(ab) + g(g)b. \tag{3.29}$$

Moreover, f is a 2-cocycle if and only if for any $a, b, \in A$

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0. \tag{3.30}$$

Proof: Direct check by definition. □

3.4.2 The Gerstenhaber bracket on Hochschild cochains and cohomologies

In this section we give a quick review of the *Gerstenhaber bracket.* For more details and proofs see [17] or [8] Section 1. For further topics see the survey [15].

First, we define an operation $\circ : C^k(A,A) \otimes C^l(A,A) \to C^{k+l-1}(A,A)$. Let $f_1 \in C^k(A,A)$ and $f_2 \in C^l(A,A)$,

$$\begin{aligned}(f_1 \circ f_2)(a_1, \ldots, a_{k+l-1}) &:= \\ &= \sum_{i=0}^{k-1} (-1)^{(k-i-1)(l-1)} f_1(a_1 \ldots, a_i, f_2(a_{i+1}, \ldots, a_{i+l}), a_{i+l+1}, \ldots, a_{k+l-1}).\end{aligned} \tag{3.31}$$

In particular, for 2-cochains we have

Proposition 3.4.2. *Let $f_1, f_2 \in C^2(A,A)$, then $f_1 \circ f_2 \in C^3(A,A)$ and is given by*

$$(f_1 \circ f_2)(a_1, a_2, a_3) = f_1(f_2(a_1, a_2), a_3) - f_1(a_1, f_2(a_2, a_3)). \tag{3.32}$$

In particular, for $f \in C^2(A,A)$ we have

$$(f \circ f)(a_1, a_2, a_3) = f(f(a_1, a_2), a_3) - f(a_1, f(a_2, a_3)). \tag{3.33}$$

Proof: This is just the definition. □

The Gerstenhaber bracket is defined to be

$$[f_1, f_2]_{\mathrm{G}} := f_1 \circ f_2 - (-1)^{(k-1)(l-1)} f_2 \circ f_1. \tag{3.34}$$

The Gerstenhaber bracket is a Lie bracket. In fact we have the following

Theorem 3.4.3. *The operation "$\circ$" gives a pre-Lie algebra structure on $C^{\bullet-1}(A, A)$. Therefore we obtain that $(C^{\bullet-1}(A, A), [\,,\,]_G)$ is a graded Lie algebra.*

Proof: See [17]. □

Proposition 3.4.4. *Let $f \in C^2(A, A)$, then*

$$[f, f]_G = 2f \circ f. \tag{3.35}$$

Proof: We get this directly from the definitions. □

The Gerstenhaber bracket is compatible with the Hochschild differential $\mathrm{d_H}$. In fact $\mathrm{d_H}$ is inner in the Gerstenhaber bracket. More precisely, let $\mu : A \otimes A \to A$ denote the multiplication map in A. Then $\mu \in C^2(A, A)$. We have the following

Proposition 3.4.5. *For any $f \in C^k(A, A)$, we have*

$$d_H f = [\mu, f]_G \in C^{k+1}(A, A). \tag{3.36}$$

We also have $[\mu, \mu]_G = 0$.

Proof: Compare the definition of $\mathrm{d_H}$ in Equation 3.27 and the definition of the Gerstenhaber bracket in Equation 3.31 and Equation 3.34. The fact that $[\mu, \mu]_{\mathrm{G}} = 0$ is exactly the associativity of μ. □

As a result, we have the following theorem:

Theorem 3.4.6. *The Gerstenhaber bracket is compatible with the Hochschild differ-*

ential d_H. In other words, for any $f_1 \in C^k(A,A)$ and $f_2 \in C^l(A,A)$, we have

$$d_H([f_1, f_2]_G) = [d_H f_1, f_2]_G + (-1)^{k-1}[f_1, d_H f_2]_G. \qquad (3.37)$$

Therefore the Gerstenhaber bracket reduces to the Hochschild cohomology $HH^{\bullet-1}(A)$.

Proof: Since $\mathrm{d_H}$ is an inner derivation according to Propostion 3.4.5, Equation 3.37 is a consequence of the super-Jacobi identity of the graded Lie algebra $(C^{\bullet-1}(A,A), [\,,\,]_{\mathrm{G}})$.
□

3.4.3 $\mathbf{HH}^\bullet(A)$ and the deformations of A

The Hochschild cohomology plays an important role in the deformation theory, see [18] or [12] Section 2.

Let A be an associative $\mathbb{C}$ algebra (infact we can replace $\mathbb{C}$ by any field). A *deformation* of the algebra structure of A means that we fix A as a $\mathbb{C}$-vector space and change the multiplication operation on A. More precisely let $\mathbb{C}[[t]]$ be the formal power series of t and we define

$$A[[t]] := A \otimes_{\mathbb{C}} \mathbb{C}[[t]]. \qquad (3.38)$$

$A[[t]]$ is obviously a $\mathbb{C}[[t]]$-module.

A deformation of the algebra structure on A is given by a map

$$m : A[[t]] \otimes A[[t]] \longrightarrow A[[t]] \qquad (3.39)$$

where m is required to be $\mathbb{C}[[t]]$-bilinear. So we only need to know the value of m on $A \otimes A$.

For any $a, b \in A$, we can write $m(a, b)$ as

$$m(a,b) = ab + \sum_{k=0}^{\infty} t^k m_k(a,b). \tag{3.40}$$

We see that each m_k belongs to $C^2(A, A)$.

Remark 29. The element t is called the deformation parameter. If we evaluate at $t = 0$ we get the original multiplication on A. On the other hand if we evaluate at $t \neq 0$, omit the convergence problem, we get a new binary operation $A \otimes A \to A$.

Being a multiplication, m needs to satisfy the *associativity law.*

Theorem 3.4.7 (Formal deformation, see [18] Chapter I.1)**.** *Let $m(a, b) = ab + \sum_{k=0}^{\infty} t^k m_k(a, b)$ as in Equation 3.40. Then m satisfies the associativity law if and only if for each $k \geqslant 1$, we have*

$$d_H m_k + \frac{1}{2}\sum_{i=1}^{k-1}[m_i, m_{k-i}] = 0. \tag{3.41}$$

If this holds, we say that m gives a formal deformation of A.

Proof: The associativity law means that for any $a, b, c \in A$, we have

$$m(a, m(b,c)) - m(m(a,b),c) = 0. \tag{3.42}$$

Now consider m as an element in $C^2(A[[t]], A[[t]])$, then Equation 3.42 is exactly

$$[m, m]_{\mathrm{G}} = 0. \tag{3.43}$$

We write $m = \mu + \sum_{k=1}^{\infty} t^k m_k$ where μ is the original multiplication on A. Then because we know $[\mu, f]_{\mathrm{G}} = \mathrm{d_H} f$ and $[\mu, \mu]_{\mathrm{G}} = 0$ in Proposition 3.4.5, Equation 3.43 becomes the *Maurer-Cartan Equation*

$$\mathrm{d_H}(\sum_{k=1}^{\infty} t^k m_k) + \frac{1}{2}[\sum_{k=1}^{\infty} t^k m_k, \sum_{k=1}^{\infty} t^k m_k]_{\mathrm{G}} = 0. \tag{3.44}$$

In the expansion of Equation 3.44, we take the t^k term and get Equation 3.41. □

Corollary 3.4.8 (Infinitesimal deformation). *m satisfies the associativity law* $\mod t^2$ *if and only if $d_H m_1 = 0$, i.e. for any $a, b, c \in A$, we have*

$$am_1(b, c) - m_1 ab, c + m_1(a, bc) - m_1(a, b)c = 0. \tag{3.45}$$

If this holds, we say that m gives an infinitesimal deformation of A.

Moreover, m satisfies the associativity law $\mod t^3$ *if and only if $d_H m_1 = 0$ together with*

$$d_H m_2 + \frac{1}{2}[m_1, m_1]_G = 0. \tag{3.46}$$

The above equation is equivalent to

$$d_H m_2 + m_1 \circ m_1 = 0 \tag{3.47}$$

since in Proposition 3.4.4 we know that $[m_1, m_1]_G = 2m_1 \circ m_1$

Proof: This is an direct corollary of Theorem 3.4.7. □

On the other hand, we need to study the problem that when the deformation m is trivial. In other words, wether or not we can find an algebraic isomorphism

$$\theta: \ (A[[t]], \mu) \longrightarrow (A[[t]], m) \tag{3.48}$$

where θ is $\mathbb{C}[[t]]$-linear and is given by

$$\theta(a) = a + \sum_{k=1}^{\infty} t^k \theta_k(a). \tag{3.49}$$

The requirement for θ is for any $a, b \in A$

$$\theta(ab) = m(\theta(a), \theta(b)). \tag{3.50}$$

The existence of θ is a complicated problem. First we have:

Proposition 3.4.9 (Infinitesimally trivial deformation)**.** *There exists a $\theta_1 \in C^1(A, A)$ such that $\theta = id + t\theta_1$ satisfyies Equation 3.50 $\mod t^2$ if and only if $m_1 \in B^2(A, A)$. If this holds, we say that m is an infinitesimally trivial deformation of A.*

Proof: We expand both sides of Equation 3.50 and look at the t term we get

$$\theta_1(ab) = \theta_1(a)b + a\theta_1(b) + m_1(a, b) \tag{3.51}$$

In other words

$$m_1 + \mathrm{d_H}\theta_1 = 0. \ \square \tag{3.52}$$

Further discussion of the triviality of deformations involves the concept of *gauge equivalence* of Maurer-Cartan elements, see [35] Section 1 or [37] Chapter 13.

3.5 The noncommutative Poisson bracket and the deformation of $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$

We will see in this section that the Poisson bracket plays an essential role in the deformation of $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$.

3.5.1 A quick review of the deformation from $S(\mathfrak{g})$ to $U(\mathfrak{g})$ and the Poisson bracket

Before studying the deformation of $\widetilde{\mathcal{C}}_\tau(\mathfrak{g})$, let us first review the corresponding theory of $S(\mathfrak{g})$ and $U(\mathfrak{g})$.

It is well-known that

$$S(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/(X \otimes Y - Y \otimes X) \tag{3.53}$$

and

$$U(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/(X \otimes Y - Y \otimes X - [X, Y]) \tag{3.54}$$

where $\mathrm{T}(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$.

Moreover, we consider the algebra

$$U_t(\mathfrak{g}) = \mathrm{T}(\mathfrak{g})/(X \otimes Y - Y \otimes X - t[X, Y]). \tag{3.55}$$

For $t \neq 0$ all the algebras $U_t(\mathfrak{g})$ are isomorphic to $U(\mathfrak{g})$, and when $t = 0$, $U_0(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$. t is called the *deformation parameter*.

We have the *Poincaré-Birkhoff-Witt* map ($\mathrm{I}_{\mathrm{PBW}}$) from $S(\mathfrak{g})$ to $U_t(\mathfrak{g})$ given by:

$$\begin{aligned} \mathrm{I}_{\mathrm{PBW}}: \quad S(\mathfrak{g}) &\longrightarrow U_t(\mathfrak{g}) \\ X_1 X_2 \dots X_k &\longmapsto \sum_{\sigma \in S_k} \frac{1}{k!} X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(k)}. \end{aligned} \tag{3.56}$$

The Poincaré-Birkhoff-Witt theorem (see [34]) tells us that the above map $\mathrm{I}_{\mathrm{PBW}}$ is an isomorphism between **$\mathfrak{g}$-vector spaces**.

Remark 30. The map $\mathrm{I}_{\mathrm{PBW}}$ is not an algebraic isomorphism unless $\mathfrak{g}$ is an abelian Lie algebra or $t = 0$.

Therefore we have the following definition

Definition 3.5.1. The map $\mathrm{I}_{\mathrm{PBW}}$ pulls back the multiplication of $U_t(\mathfrak{g})$ to $S(\mathfrak{g})$ and we call it the **star-product** on $S(\mathfrak{g})$, denoted by $*_t$. For any $a, b \in S(\mathfrak{g})$

$$a *_t b := \mathrm{I}_{\mathrm{PBW}}^{-1} \left(\mathrm{I}_{\mathrm{PBW}}(a) \cdot \mathrm{I}_{\mathrm{PBW}}(b)\right). \tag{3.57}$$

In particular we denote $*_1$ simply as $*$. When $t = 0$ the star-product reduces to the original production on $S(\mathfrak{g})$. Obviously $*_t$ satisfies the associativity law because the multiplication on $U_t(\mathfrak{g})$ is associative.

Now by definition, the map $\mathrm{I}_{\mathrm{PBW}}$ gives an *algebraic isomorphism*

$$\mathrm{I}_{\mathrm{PBW}} : (S(\mathfrak{g}), *_t) \xrightarrow{\sim} (U_t(\mathfrak{g}), \cdot). \tag{3.58}$$

Therefore we can identify $U_t(\mathfrak{g})$ with $(S(\mathfrak{g}), *_t)$, especially we can identify $U(\mathfrak{g})$ with $(S(\mathfrak{g}), *)$.

Remark 31. Our star-product $*_t$ is not exactly the same as the sart-product constructed by Kontsevich in [35] Section 8. Nevertheless, they give isomorphic algebra structures on $S(\mathfrak{g})$.

The star-product $*_t$ depends on the deformation parameter t. In fact we can write the first few terms of $*_t$.

Proposition 3.5.1 ([20] Section 3)**.** *We can write $*_t$ as*

$$a *_t b = ab + \frac{t}{2}\{a, b\} + O(t^2). \tag{3.59}$$

□

In other words, the Poisson bracket on $S(\mathfrak{g})$ is exactly the *first-order deformation* from $S(\mathfrak{g})$ to $U(\mathfrak{g})$.

Remark 32. In fact we can find the expressing of the t^2 term in the star-product. According to [19] Remark 4.7, for any $a, b \in S(\mathfrak{g})$, the t^2 term is

$$m_2(a, b) := \frac{1}{8} c_{ij}^s c_{kl}^t X_s X_t \partial^i \partial^k a \partial^j \partial^l b + \frac{1}{12} c_{ks}^t c_{ji}^s X_t (\partial^k \partial^j a \partial^i b + \partial^i a \partial^k \partial^j b). \tag{3.60}$$

Now we can give another proof of Lemma 3.3.3

Proposition 3.5.2 (Lemma 3.3.3)**.** *We can define a 2-cochain $\phi \in C^2(S(\mathfrak{g}), S(\mathfrak{g}))$ as follows: for any $a, b \in S(\mathfrak{g})$*

$$\phi(a, b) := \frac{1}{2} c_{ij}^s c_{kl}^t X_s X_t \partial^i \partial^k a \partial^j \partial^l b + \frac{1}{3} c_{ks}^t c_{ji}^s X_t (\partial^k \partial^j a \partial^i b + \partial^i a \partial^k \partial^j b). \tag{3.61}$$

Then for any $a, b, c \in S(\mathfrak{g})$ we have

$$\{a, \{b, c\}\} - \{\{a, b\}, c\} + (d_H \phi)(a, b, c) = 0. \tag{3.62}$$

Proof: In the framework of deformation theory (see Section 3.4.3). Let $m = *_t$ be the star-product. Compare Proposition 3.5.1, Equation 3.60 and Equation 3.40 we get

$$P = 2m_1 \text{ and } \phi = 4m_2.$$

where we denote the Poisson bracket on $S(\mathfrak{g})$ by P also.

Since we know from the definition that the star-product is associative, by Proposition 3.4.8, especially Equation 3.47 we get

$$m_1 \circ m_1 + \mathrm{d_H} m_2 = 0$$

hence

$$P \circ P + \mathrm{d_H} \phi = 0$$

and this is exactly Equation 3.62. □

If we restrict ourselves to the invariant subalgebra $I(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$, then we have the following well-known result about the Poisson bracket:

Proposition 3.5.3 ([33]). *The Poisson bracket vanishes on $I(\mathfrak{g})$. In other words, for any $a, b \in I(\mathfrak{g})$, we have*

$$\{a, b\} = 0. \tag{3.63}$$

Proof: This is almost the definition of $I(\mathfrak{g})$. In fact

$$\begin{aligned}\{a, b\} &= c_{ij}^k X_k (\partial^i a)(\partial^j b) \\ &= (c_{ij}^k X_k \partial^i a)(\partial^j b) \\ &= (\mathrm{ad} X_j\,(a))(\partial^j b).\end{aligned}$$

Since $a \in I(\mathfrak{g})$, we know that $\mathrm{ad} X_j\,(a) = 0$ for any X_j, as a result, $\{a, b\} = 0$. □

On the other hand, we can find the image of $\mathrm{I}_{\mathrm{PBW}}$ restricted on $I(\mathfrak{g})$.

Proposition 3.5.4 ([34])**.** *The image of $I(\mathfrak{g})$ under the Poincaré-Birkhoff-Witt map I_{PBW} is exactly $Z_t(\mathfrak{g})$, the center of $U_t(\mathfrak{g})$. In other words,*

$$I_{PBW} : I(\mathfrak{g}) \to Z_t(\mathfrak{g}) \tag{3.64}$$

is an isomorphism between vector spaces.

Proof: Just remember that $\mathrm{I}_{\mathrm{PBW}} :\ S(\mathfrak{g}) \to U_t(\mathfrak{g})$ is an isomorphism between $\mathfrak{g}$-vector spaces,i.e. it is compatible with the $\mathfrak{g}$-actions. □

Remark 33. Proposition 3.5.3 and Proposition 3.5.4 tell us that the first-order deformation from $I(\mathfrak{g})$ to $Z(\mathfrak{g})$ is zero.

In fact we have the much deeper *Duflo's isomorphism theorem*:

Theorem 3.5.5 ([16], [35] Section 8, [1] and [12])**.** *There exists an **algebraic isomorphism**:*

$$Duf : I(\mathfrak{g}) \to Z(\mathfrak{g}) \tag{3.65}$$

□

Remark 34. In general, the map Duf will be different from the Poincaré-Birkhoff-Witt map $\mathrm{I_{PBW}}$ in Proposition 3.5.4, although they have the same domain and image.

Remark 35. Since $Z(\mathfrak{g})$ is isomorphic to $Z_t(\mathfrak{g})$ as algebras, the map Duf can be easily generalized to the map $\mathrm{Duf}_t : I(\mathfrak{g}) \to Z_t(\mathfrak{g})$ for any t.

3.5.2 The deformation from $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ to $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$ and the noncommutative Poisson bracket

Parallel to the constructions of $S(\mathfrak{g})$, let us make the following definition

Definition 3.5.2. We define the algebra $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$ as

$$\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g}) := \mathrm{End}V_\tau \otimes U_t(\mathfrak{g}). \tag{3.66}$$

Moreover, we define

$$\mathcal{Q}^t_\tau(\mathfrak{g}) := (\mathrm{End}V_\tau \otimes U_t(\mathfrak{g}))^G. \tag{3.67}$$

By definition, we have $\widetilde{\mathcal{Q}^0_\tau}(\mathfrak{g}) = \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, $\mathcal{Q}^0_\tau(\mathfrak{g}) = \mathcal{C}_\tau(\mathfrak{g})$ and for any $t \neq 0$ we have $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g}) \cong \widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$, $\mathcal{Q}^t_\tau(\mathfrak{g}) \cong \mathcal{Q}_\tau(\mathfrak{g})$.

We also have the Poincaré-Birkhoff-Witt map on the family algebras:

Definition 3.5.3. The Poincaré-Birkhoff-Witt map $\mathrm{F_{PBW}}$ on family algebras is defined to be $\mathrm{Id} \otimes \mathrm{I_{PBW}}$. In other words:

$$\begin{aligned} \mathrm{F_{PBW}} : \quad \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) &\longrightarrow \widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g}) \\ A_i \otimes a^i &\longmapsto A_i \otimes \mathrm{I_{PBW}}(a^i). \end{aligned} \tag{3.68}$$

$\mathrm{F_{PBW}}$ is an isomorphism between $\mathfrak{g}$-vector spaces.

As $\mathrm{I_{PBW}}$, $\mathrm{F_{PBW}}$ is not an algebraic isomorphism either. Nevertheless it can also pull back the product on $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$ to $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$:

Definition 3.5.4. The star-product $*_t$ on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ is defined to the pull-back of the product on $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$ via the map $\mathrm{F}_{\mathrm{PBW}}$. In other words, for any $\mathcal{A}, \mathcal{B} \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$

$$\mathcal{A} *_t \mathcal{B} := \mathrm{F}_{\mathrm{PBW}}^{-1} \left(\mathrm{F}_{\mathrm{PBW}}(\mathcal{A}) \cdot \mathrm{F}_{\mathrm{PBW}}(\mathcal{B})\right). \tag{3.69}$$

Moreover, if we write $\mathcal{A} = A_i \otimes a^i$ and $\mathcal{B} = B_j \otimes b^j$, then

$$(A_i \otimes a^i) *_t (B_j \otimes b^j) = A_i B_j \otimes (a^i *_t b^j). \tag{3.70}$$

Therefore the map $\mathrm{F}_{\mathrm{PBW}}$ gives an *algebraic isomorphism*

$$\mathrm{F}_{\mathrm{PBW}} : (\widetilde{\mathcal{C}_\tau}(\mathfrak{g}), *_t) \xrightarrow{\sim} (\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g}), \cdot). \tag{3.71}$$

Therefore we can identify $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$ with $(\widetilde{\mathcal{C}_\tau}(\mathfrak{g}), *_t)$, especially we can identify $\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$ with $(\widetilde{\mathcal{C}_\tau}(\mathfrak{g}), *)$.

For the star-product on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, we also have

Proposition 3.5.6. *We can write the star-product $*_t$ on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ as*

$$\mathcal{A} *_t \mathcal{B} = \mathcal{A}\mathcal{B} + \frac{t}{2}\{\mathcal{A}, \mathcal{B}\} + O(t^2). \tag{3.72}$$

In other words, the Poisson bracket on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ is exactly the first-order deformation from $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ to$\widetilde{\mathcal{Q}_\tau}(\mathfrak{g})$.

Proof: This is just a combination of the definition of star-product (Definition 3.5.4), the definition of noncommutative Poisson bracket (Definition 3.3.1) and Proposition 3.5.1. □

Remark 36. By now, the results in this subsection exemplified the slogan "the deformation theory of an algebra A is the same as that of the matrix algebra $\mathrm{Mat}_{n\times n}(A)$." However, when restrict to the invariant subalgebras, these two become different.

If we restrict ourselves to the family algebra $\mathcal{C}_\tau(\mathfrak{g})$, i.e. the invariant subalgebra of $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, we get the follow proposition which is similar to Proposition 3.5.4

Proposition 3.5.7. *The image of $\mathcal{C}_\tau(\mathfrak{g})$ under the Poincaré-Birkhoff-Witt map F_{PBW} is exactly $\mathcal{Q}^t_\tau(\mathfrak{g})$, the invariant subalgebra of $\widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$. In other words,*

$$F_{PBW} : \mathcal{C}_\tau(\mathfrak{g}) \to \mathcal{Q}^t_\tau(\mathfrak{g}) \tag{3.73}$$

is an isomorphism between vector spaces.

Proof: Just remember that $\mathrm{F_{PBW}}$: $\widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \to \widetilde{\mathcal{Q}^t_\tau}(\mathfrak{g})$ is an isomorphism between $\mathfrak{g}$-vector spaces,i.e. it is compatible with the $\mathfrak{g}$-actions. □

Now it is natural to ask for the corresponding result of Proposition 3.5.3 and the Duflo's isomorphism theorem 3.5.5 on family algebras.

In fact, in Theorem 3.6.4 of this chapter we will prove that the noncommutative Poisson bracket vanishes in the *Hochschild cohomology.* The generalization of Duflo's isomorphism theorem to family algebras is still an open problem, see Section A.2.

3.6 The vanishing of the noncommutative Poisson bracket in $\mathrm{HH}^2(\mathcal{C}_\tau(\mathfrak{g}))$

3.6.1 The map ∇

In this section we focus on the classical family algebra $\mathcal{C}_\tau(\mathfrak{g})$ and the matrix algebra $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$.

Definition 3.6.1 (The definition of ∇)**.** We define the map $\nabla : \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \to \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ as follows: Fix a basis X_k of $\mathfrak{g}$. Let $A_i \otimes a^i \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$,

$$\nabla(A_i \otimes a^i) := A_i \tau(X_k) \otimes \partial^k(a^i). \tag{3.74}$$

Notice that $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ is nothing but a matrix algebra with entries in $S(\mathfrak{g})$. In the form of matrices,

$$\nabla(\mathcal{A}) = \partial^k(\mathcal{A})\tau(X_k). \tag{3.75}$$

Hence ∇ is a first-order differential operator on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$.

From Equation 3.75 it is not difficult to see that the map ∇ dose not depend on the expression of $\mathcal{A} \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ as $A_i \otimes a^i$.

To show ∇ is a well-defined map, it is now sufficient to prove the following proposition:

Proposition 3.6.1. *The map $\nabla : \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \to \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ is independent of the choice of the basis of $\mathfrak{g}$.*

Proof: We need to do some computation. Let $\tilde{X}_j$ be another basis of $\mathfrak{g}$. Then

$$\tilde{X}_j = T_j^k X_k$$

where T_j^k is the transition matrix. Then, let $\tilde{\partial}^j$ be the partial derivation with respect to $\tilde{X}_j$, we have

$$\tilde{\partial}^j = (T^{-1})^j_k \partial^k.$$

Let $\tilde{\nabla}$ be the ∇ map under the basis $\tilde{X}_j$, for $A_i \otimes a^i \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, we have

$$\begin{aligned}\tilde{\nabla}(A_i \otimes a^i) =& A_i \tau(\tilde{X}_j) \otimes \tilde{\partial}^j(a^i) \\ =& A_i \tau(T_j^k X_k) \otimes (T^{-1})^j_l \partial^l(a^i).\end{aligned}$$

The constant $(T^{-1})^j_l$ can be moved to the first component, hence

$$\begin{aligned}\text{the above} =& T_j^k (T^{-1})^j_l A_i \tau(X_k) \otimes \partial^l(a^i) \\ =& \delta_l^k A_i \tau(X_k) \otimes \partial^l(a^i) \\ =& A_i \tau(X_k) \otimes \partial^k(a^i) \\ =& \nabla(A_i \otimes a^i)\end{aligned}$$

So ∇ is invariant under the change of basis of $\mathfrak{g}$. □

The map ∇ is obviously $\mathbb{C}$-linear, moreover it has the following important property:

Proposition 3.6.2. *The image under ∇ of the subalgebra $\mathcal{C}_\tau(\mathfrak{g})$ is contained in $\mathcal{C}_\tau(\mathfrak{g})$ itself.*

Proof: The proof requires some careful computation.

Let $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$, then

$$\nabla(A_i \otimes a^i) = A_i \tau(X_k) \otimes \partial^k(a^i).$$

By Proposition 3.2.2, we only need to show that, for X_j which is one of the basis of $\mathfrak{g}$, we have

$$[\tau(X_j), A_i\tau(X_k)] \otimes \partial^k a^i = A_i\tau(X_k) \otimes \{\ \partial^k a^i, X_j\}. \tag{3.76}$$

In fact,

the left hand side

$$\begin{aligned}
=&[\tau(X_j), A_i]\tau(X_k) \otimes \partial^k a^i + A_i[\tau(X_j), \tau(X_k)] \otimes \partial^k a^i \\
=&\nabla([\tau(X_j), A_i] \otimes a^i) + A_i[\tau(X_j), \tau(X_k)] \otimes \partial^k a^i.
\end{aligned}$$

To make the following computation more clear, let us denote:

$$\begin{aligned}
\mathrm{I} :=&\nabla([\tau(X_j), A_i] \otimes a^i), \\
\mathrm{II} :=&A_i[\tau(X_j), \tau(X_k)] \otimes \partial^k a^i.
\end{aligned}$$

For I, since $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$, by Proposition 3.2.2 we have:

$$\mathrm{I} = \nabla([\tau(X_j), A_i] \otimes a^i) = \nabla(A_i \otimes \{a^i, X_j\})$$

From the definition of the Poisson bracket on $S(\mathfrak{g})$, we know

$$\{a^i, X_j\} = c_{sl}^r X_r \partial^s a^i \partial^l X_j = c_{sl}^r X_r \partial^s a^i \delta_j^l = c_{sj}^r X_r \partial^s a^i.$$

Therefore

$$
\begin{aligned}
\mathrm{I} =& \nabla(A_i \otimes c^r_{sj} X_r \partial^s a^i) \\
=& A_i \tau(X_l) \otimes \partial^l (c^r_{sj} X_r \partial^s a^i) \\
=& A_i \tau(X_l) \otimes c^r_{sj} (\partial^l (X_r) \partial^s a^i + X_r \partial^l \partial^s a^i) \\
=& A_i \tau(X_l) \otimes c^r_{sj} \delta^l_r \partial^s a^i + A_i \tau(X_l) \otimes c^r_{sj} X_r \partial^l \partial^s a^i \\
=& A_i \tau(X_r) \otimes c^r_{sj} \partial^s a^i + A_i \tau(X_l) \otimes c^r_{sj} X_r \partial^s \partial^l a^i.
\end{aligned}
$$

Nevertheless, we have

$$
c^r_{sj} X_r \partial^s \partial^l a^i = \{\partial^l a^i, X_j\}
$$

As a result

$$
\mathrm{I} = A_i \tau(X_r) \otimes c^r_{sj} \partial^s a^i + A_i \tau(X_l) \otimes \{\partial^l a^i, X_j\} \tag{3.77}
$$

As for II, we know

$$
\begin{aligned}
\mathrm{II} =& A_i [\tau(X_j), \tau(X_k)] \otimes \partial^k a^i \\
=& A_i \tau([X_j, X_k]) \otimes \partial^k a^i.
\end{aligned}
$$

We know that $[X_j, X_k] = c^r_{jk} X_r$ hence

$$
\tau([X_j, X_k]) = \tau(c^r_{jk} X_r) = c^r_{jk} \tau(X_r) = -c^r_{kj} \tau(X_r).
$$

As a result

$$\begin{aligned} \mathrm{II} =& A_i \tau([X_j, X_k]) \otimes \partial^k a^i \\ =& - A_i c_{kj}^r \tau(X_r) \otimes \partial^k a^i \\ =& - A_i \tau(X_r) \otimes c_{kj}^r \partial^k a^i. \end{aligned} \tag{3.78}$$

Combine Equation 3.77 and 3.78 we get

$$\begin{aligned} & \text{the left hand side of 3.76} \\ =& \mathrm{I} + \mathrm{II} \\ =& A_i \tau(X_r) \otimes c_{sj}^r \partial^s a^i + A_i \tau(X_l) \otimes \{\partial^l a^i, X_j\} - A_i \tau(X_r) \otimes c_{kj}^r \partial^k a^i \\ =& A_i \tau(X_l) \otimes \{\partial^l a^i, X_j\} \\ =& \text{ the right hand side of 3.76.} \end{aligned}$$

Therefore we finishes the proof. □

Now according to Proposition 3.6.2, we can say that ∇ is a $\mathbb{C}$-linear map from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{C}_\tau(\mathfrak{g})$. In other words, ∇ belongs to the Hochschild 1-cochain $C^1(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g}))$. see Section 3.4.1 for a review of Hochschild cohomology.

Before moving on to the next section, we give a direct application of the map ∇.

Corollary 3.6.3 ([31] Section 1)**.** *When the Lie algebra $\mathfrak{g}$ is semisimple and τ is a nontrivial irreducible representation, the classical family algebra $\mathcal{C}_\tau(\mathfrak{g}))$ is more than $I(\mathfrak{g})$, i.e. $I(\mathfrak{g}) \subsetneq \mathcal{C}_\tau(\mathfrak{g}))$, and we also have $Z(\mathfrak{g}) \subsetneq \mathcal{Q}_\tau(\mathfrak{g}))$.*

Proof: Let Cas be the quadratic Casimir element in $I(\mathfrak{g})$, $\deg \mathrm{Cas} = 2$. Then by Proposition 3.6.2, we know that $\nabla(\mathrm{Cas}) \in \mathcal{C}_\tau(\mathfrak{g}))$ but $\deg \nabla(\mathrm{Cas}) = 1$. Since τ is nontrivial we know that $\nabla(\mathrm{Cas}) \neq 0$. On the other hand, since $\mathfrak{g}$ is semisimple, there is no nonzero degree-1 element in $I(\mathfrak{g})$, therefore $\nabla(\mathrm{Cas}) \notin I(\mathfrak{g})$ hence $I(\mathfrak{g}) \subsetneq \mathcal{C}_\tau(\mathfrak{g})$).

Since there is a PBW map $\mathrm{F}_{\mathrm{PBW}} : \mathcal{C}_\tau(\mathfrak{g})) \to \mathcal{Q}_\tau(\mathfrak{g}))$ which maps $I(\mathfrak{g})$ to $Z(\mathfrak{g})$, we know that $Z(\mathfrak{g}) \subsetneq \mathcal{Q}_\tau(\mathfrak{g}))$. □

Remark 37. In fact, in Example 2, the element M is obtained in the same way as $\nabla\mathrm{Cas}$ in the above corollary.

Remark 38. Then map ∇ is motivated by the element M_P defined in Section 1 of [31]. Nevertheless in that chapter M_P is defined only for $P \in I(\mathfrak{g})$ and here we extend the domain to all $\mathcal{C}_\tau(\mathfrak{g})$.

3.6.2 ∇ and the Poisson bracket

In this subsection, we build up the relation between ∇ and the Poisson bracket P.

First we review some notations of Hochshchild cohomology. Notice that $\nabla : \mathcal{C}_\tau(\mathfrak{g}) \to \mathcal{C}_\tau(\mathfrak{g})$ is a Hochshchild 1-cochain, i.e.

$$\nabla \in C^1(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g})).$$

Let

$$\mathrm{d_H} : C^1(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g})) \to C^2(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g}))$$

be the differential map in the Hochschild complex.

Let $\mathcal{A}, \mathcal{B} \in \mathcal{C}_\tau(\mathfrak{g})$. Then by the definition of $\mathrm{d_H}$, we have

$$(\mathrm{d_H}\nabla)(\mathcal{A}, \mathcal{B}) = \mathcal{A}\nabla(\mathcal{B}) - \nabla(\mathcal{A}\mathcal{B}) + \nabla(\mathcal{A})\mathcal{B}. \tag{3.79}$$

The following theorem is the main result of this chapter.

Theorem 3.6.4. *For any* $\mathcal{A} = A_i \otimes a^i,\ \mathcal{B} = B_j \otimes b^j \in \mathcal{C}_\tau(\mathfrak{g})$, *we have*

$$\{A, B\} = -\mathcal{A}\nabla(\mathcal{B}) + \nabla(\mathcal{A}\mathcal{B}) - \nabla(\mathcal{A})\mathcal{B}. \tag{3.80}$$

In other words

$$P + d_H\nabla = 0 \tag{3.81}$$

as elements in the Hochschild 2-cochain $C^2(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g}))$. *Therefore the Poisson bracket is a coboundary in* $C^2(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g}))$.

Proof: First let us see what is $\nabla(\mathcal{A}\mathcal{B})$:

$$\begin{aligned}\nabla(\mathcal{A}\mathcal{B}) =& \nabla(A_iB_j \otimes a^ib^j) \\ =& A_iB_j\tau(X_k) \otimes \partial^k(a^ib^j) \\ =& A_iB_j\tau(X_k) \otimes (\partial^k a^i)b^j + A_iB_j\tau(X_k) \otimes a^i(\partial^k b^j)\end{aligned} \tag{3.82}$$

To make the computation more clear, let us denote:

$$\begin{aligned}\mathrm{I} :=& A_iB_j\tau(X_k) \otimes (\partial^k a^i)b^j, \\ \mathrm{II} :=& A_iB_j\tau(X_k) \otimes a^i(\partial^k b^j).\end{aligned}$$

Then

$$\nabla(AB) = \mathrm{I} + \mathrm{II}. \tag{3.83}$$

It is easy to see that $\mathrm{II} = \mathcal{A}\nabla(\mathcal{B})$. In fact

$$\mathrm{II} = A_iB_j\tau(X_k) \otimes a^i(\partial^k b^j) = (A_i \otimes a^i) \cdot (B_j\tau(X_k) \otimes \partial^k b^j) = \mathcal{A}\nabla(\mathcal{B}). \tag{3.84}$$

Unfortunately, $\mathrm{I} \neq (\nabla \mathcal{A})\mathcal{B}$ in general. We know that

$$\mathrm{I} = A_i B_j \tau(X_k) \otimes (\partial^k a^i) b^j$$

and

$$\begin{aligned}(\nabla \mathcal{A})\mathcal{B} =& (A_i \tau(X_k) \otimes \partial^k a^i) \cdot (B_j \otimes b^j) \\ =& A_i \tau(X_k) B_j \otimes (\partial^k a^i) b^j.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{I} - (\nabla \mathcal{A})\mathcal{B} =& (A_i B_j \tau(X_k) - A_i \tau(X_k) B_j) \otimes (\partial^k a^i) b^j \\ =& A_i [B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j.\end{aligned} \tag{3.85}$$

We need to further simplify $A_i[B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j$. In fact we have the following lemma

Lemma 3.6.5. *any* $\mathcal{A} = A_i \otimes a^i, \ \mathcal{B} = B_j \otimes b^j \in \mathcal{C}_\tau(\mathfrak{g})$, *we have*

$$A_i[B_j, \tau(X_k)] \otimes (\partial^k a^i) b^j = A_i B_j \otimes \{\, a^i, b^j\} = \{\, \mathcal{A}, \mathcal{B}\}. \tag{3.86}$$

Assuming Lemma 3.6.5, then by Equation 3.85 we have

$$\mathrm{I} = \{\, \mathcal{A}, \mathcal{B}\} + (\nabla \mathcal{A})\mathcal{B}. \tag{3.87}$$

Put equations 3.83, 3.84 and 3.87 together, we have:

$$\begin{aligned}
&\nabla(\mathcal{A}\mathcal{B}) - \mathcal{A}\nabla(\mathcal{B}) - \nabla(\mathcal{A})\mathcal{B} \\
=&\mathrm{I} + \mathrm{II} - \mathcal{A}\nabla(\mathcal{B}) - \nabla(\mathcal{A})\mathcal{B} \\
=&\{\mathcal{A},\mathcal{B}\} + (\nabla\mathcal{A})\mathcal{B} + \mathcal{A}\nabla(\mathcal{B}) - \mathcal{A}\nabla(\mathcal{B}) - (\nabla\mathcal{A})\mathcal{B} \\
=&\{\mathcal{A},\mathcal{B}\}.
\end{aligned} \tag{3.88}$$

This proves Theorem 3.6.4. □

Proof of Lemma 3.6.5: First by Proposition 3.2.1 we have

$$A_i[B_j,\tau(X_k)] \otimes (\partial^k a^i)b^j = (A_i \otimes \partial^k a^i)\cdot([B_j,\tau(X_k)] \otimes b^j).$$

Since $\mathcal{B} = B_j \otimes b^j$ **is contained in** $\mathcal{C}_\tau(\mathfrak{g})$, by Proposition 3.2.2 we know that

$$\begin{aligned}
&(A_i \otimes \partial^k a^i)\cdot(B_j \otimes \{X_k, b^j\}) \\
=&A_iB_j \otimes (\partial^k a^i)\{X_k, b^j\} \\
=&A_iB_j \otimes \{a^i, b^j\} \\
=&\{\mathcal{A},\mathcal{B}\}.
\end{aligned}$$

This proves Lemma 3.6.5. □

Remark 39. Although both the map ∇ and the Poisson bracket P can be defined on the larger algebra $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, we do **not** have the relation

$$\{\mathcal{A},\mathcal{B}\} = -\mathrm{d_H}\nabla(\mathcal{A},\mathcal{B})$$

for any $\mathcal{A}, \mathcal{B} \in \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$. In fact, in the proof of Lemma 3.6.5 we see that we need $\mathcal{B} \in \mathcal{C}_\tau(\mathfrak{g})$.

From the view point of Proposition 3.4.9, we have the following

Corollary 3.6.6. *The deformation from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$ is infinitesimally trivial.*

Proof: We know in Proposition 3.5.6 that the first order deformation m_1 is $\frac{1}{2}P$, therefore this corollary is just a direct consequence of Theorem 3.6.4. □

We can also give an alternative proof of Proposition 3.3.4 using Theorem 3.6.4:

Corollary 3.6.7 (Proposition 3.3.4)**.** *For any $\mathcal{A}, \mathcal{B} \in \mathcal{C}_\tau(\mathfrak{g})$, we have that $\{\mathcal{A}, \mathcal{B}\}$ is still in $\mathcal{C}_\tau(\mathfrak{g})$.*

Proof: In the proof of Theorem 3.6.4, we do not require a priori that $\{\mathcal{A}, \mathcal{B}\} \in \mathcal{C}_\tau(\mathfrak{g})$. Now by Proposition 3.6.2 we know that ∇ maps $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{C}_\tau(\mathfrak{g})$, hence from Theorem 3.6.4 we get the result we want. □

3.6.3 A digression to an alternative of ∇.

We can define a map $\nabla' : \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \to \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ to be

$$\nabla'(A_i \otimes a^i) := \tau(X_k) A_i \otimes \partial^k a^i. \tag{3.89}$$

Similar to Proposition 3.6.2, we can check that ∇' also maps $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{C}_\tau(\mathfrak{g})$.

Remark 40. The difference between ∇ and ∇' is: in the definition of ∇, the matrix $\tau(X_k)$ is multiplied from the right; while in the definition of ∇', the matrix $\tau(X_k)$ is multiplied from the left.

In general $\nabla'(\mathcal{A}) \neq \nabla(\mathcal{A})$, we want to compute their difference. First we define the *first Chern class* on $\widetilde{\mathcal{C}_\tau}(\mathfrak{g})$ following [12] Section 1.1.

Definition 3.6.2. The first Chern class c_1 is a map $\widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \to \widetilde{\mathcal{C}_\tau}(\mathfrak{g})$, $c_1 := \mathrm{tr}(\mathrm{ad})$. More precisely

$$\begin{aligned} c_1 : \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) &\longrightarrow \widetilde{\mathcal{C}_\tau}(\mathfrak{g}) \\ A \otimes a &\mapsto A \otimes c_{ij}^j \partial^i a. \end{aligned} \tag{3.90}$$

It is easy to check that c_1 is $\mathfrak{g}$-invariant hence c_1 maps $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{C}_\tau(\mathfrak{g})$. Moreover, it is also easy to check that the first Chern class is closed in the Hochschild cochain, in other words, $c_1 \in Z^1(\widetilde{\mathcal{C}_\tau}(\mathfrak{g}), \widetilde{\mathcal{C}_\tau}(\mathfrak{g}))$ and $c_1 \in Z^1(\mathcal{C}_\tau(\mathfrak{g}), \mathcal{C}_\tau(\mathfrak{g}))$.

Having the first Chern class, we can express the difference between ∇ and ∇' in $\mathcal{C}_\tau(\mathfrak{g})$:

Proposition 3.6.8. *In $\mathcal{C}_\tau(\mathfrak{g})$ we have $\nabla - \nabla' = -c_1$.*

Proof: For any $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$

$$\begin{aligned} \nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) =& A_i \tau(X_k) \otimes \partial^k a^i - \tau(X_k) A_i \otimes \partial^k a^i \\ =& [A_i, \tau(X_k)] \otimes \partial^k a^i \\ =& \partial^k([A_i, \tau(X_k)] \otimes a^i) \text{ (We can move the partial derivative out).} \end{aligned}$$

Since $A_i \otimes a^i \in \mathcal{C}_\tau(\mathfrak{g})$, we have

$$[A_i, \tau(X_k)] \otimes a^i = A_i \otimes \{X_k, a^i\}.$$

Therefore

$$\begin{aligned}
\nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) =& \partial^k(A_i \otimes \{X_k, a^i\}) \\
=& A_i \otimes \partial^k(\{X_k, a^i\}) \\
=& A_i \otimes \partial^k(c_{kj}^l X_l \partial^j a^i) \\
=& A_i \otimes (c_{kj}^k \partial^j a^i + c_{kj}^l X_l \partial^k \partial^j a^i) \\
=& A_i \otimes c_{kj}^k \partial^j a^i + A_i \otimes c_{kj}^l X_l \partial^k \partial^j a^i \\
=& -c_1(A_i \otimes a^i) + A_i \otimes c_{kj}^l X_l \partial^k \partial^j a^i
\end{aligned}$$

Since c_{kj}^l is anti symmetric with respect to k, j, it is easy to see that

$$A_i \otimes c_{kj}^l X_l \partial^k \partial^j a^i = 0$$

Hence we get

$$\nabla(A_i \otimes a^i) - \nabla'(A_i \otimes a^i) = -c_1(A_i \otimes a^i) \ \square$$

Corollary 3.6.9. *In $\mathcal{C}_\tau(\mathfrak{g})$ we have $d_H\nabla' = d_H\nabla = P$ the Poisson bracket. Therefore we can replace ∇ by ∇' in Theorem 3.6.4.*

Proof: We know that $\mathrm{d_H}\nabla = P$ and $\nabla - \nabla' = -c_1$. In Definition 3.6.2 we also know that c_1 is closed, i.e. $\mathrm{d_H}\, c_1 = 0$. $\square$

Moreover, we have the following result

Proposition 3.6.10. *When $\mathfrak{g}$ is a semisimple Lie algebra, we have $\nabla' = \nabla$ in $\mathcal{C}_\tau(\mathfrak{g})$.*

Proof: We know that for semisimle Lie algebra, the adjoint representation is traceless, in other words

$$c_{ij}^{i} = 0 \text{ for any } j.$$

Therefore $c_1 = 0$ for semisimple $\mathfrak{g}$. $\square$

Remark 41. I am still not clear about the significance of the result in this subsection,especially its relation with the quantization problem, see Section A.2.

Appendix: Open Problems

A.1 Flag Variety and Representations of Semisimple Lie Groups

For a real semisimple Lie group G, it is well-known that the G action on $\mathcal{B}$ is closely related to the representation theory of G. For example, in geometric representation theory, people realize the admissible representations of G as G-equivariant constructible twisted sheaves on $\mathcal{B}$. see [28] Section 3.

We notice that $\mathrm{K}^*_G(\mathcal{B})$ also reveals information about the G-action on $\mathcal{B}$, so it is worthwhile to compute it explicitly, and find its significance in representation theory.

A.2 The Quantization Problem of Family Algebras

In this section we restrict to the case that $\mathfrak{g}$ is complex semisimple and the representation τ to be semisimple.

As we have mentioned in the introduction, N. Higson find the relation between family algebras and Mackey's analogue in [24]. In the end of [24], Higson proposed the problem of constructing a *quantization map* Q between $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_\tau(\mathfrak{g}) & \overset{Q}{\dashrightarrow} & \mathcal{Q}_\tau(\mathfrak{g}) \\ \big\downarrow \mathrm{GHC}_{\tau,c} & & \big\downarrow \mathrm{GHC}_\tau \\ S(\mathfrak{h}) & \xrightarrow{\cong} & U(\mathfrak{h}) \end{array} \tag{A.1}$$

Here Q is a vector space isomorphism but need not to be an algebraic isomorphism.

Remark 42. Acording to Theorem 3.6.4 and Corollary 3.6.6, the deformation from $\mathcal{C}_\tau(\mathfrak{g})$ to $\mathcal{Q}_\tau(\mathfrak{g})$ is infinitesimally trivial, which suggests that they are very closed to each other. The quantization problem ask us to find precisely the relations between $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$.

Remark 43. On the other hand, in the 2002 Ph. D thesis [43] Chapter 6, N. Rozhkovskaya studied the family algebras for $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C})$ and for any finite dimensional irreducible representation of $\mathfrak{g}$. In fact she gave explicitly the generators and generation relations of $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$. According to her formulas, $\mathcal{C}_\tau(\mathfrak{g})$ and $\mathcal{Q}_\tau(\mathfrak{g})$ are not isomorphic as algebras unless $\tau =$ the trivial, the standard or the adjoint representations. This suggests that we cannot expect the quantization map Q to be an algebraic isomorphism.

In [1] and [2] A. Alekseev, and E. Meinrenken give a new proof of Duflo's isomorphism theorem using the quantization map of the *Weil algebras*.

In [47] Z. Wei introduced the *covariant Weil algebras* as simultaneous generalizations of Weil algebras and family algebras. It is expected that the quantization problem of family algebras can be solved in the framework of covariant Weil algebras.

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